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Normalized Laplacian Energy Change and Edge Deletion*

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Abstract

Let $E_{\mathcal{L}}(G)$ denote the normalized Laplacian energy of an isolate-free graph G, and let $E_{\mathcal{L}}(G-e)$ be the energy with edge $e = uv$ removed. In [4] it is shown that if e is not incident to a pendant vertex, then $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G-e)| \leq 1.8366$. We show that if u and v also have no common neighbors, then the change in energy is less than 1.5404. If $d_u \geq 3$ and $d_v \geq 3$, then $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G-e)| < .9916$. If $d_u \geq d$ and $d_v \geq d$ then the change is within $O(d^{-0.5})$.

1 Introduction

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. For each $v \in V$, $N(v)$ denotes the set neighbors of v. The *degree* of v, denoted d_v, is the cardinality of $N(v)$. We write $e = uv$ to represent an edge $e \in E$ between u and v. We call v a *pendant* if $d_v = 1$. We let $G - e$ denote the graph obtained by removing edge e. We say an edge e is a *bridge* if $G - e$ has more components than G.

Throughout this paper, if M is a square matrix of order n, we let $\operatorname{spec}(M)$ denote its multi-set of eigenvalues which we also denote with

$$\lambda_1(M), \ldots, \lambda_n(M).$$

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Spectral graph theory associates a matrix with a graph, and then ideally attempts to answer questions about the graph's structure using the eigenvalues of the matrix. Some matrices like the *adjacency* matrix A and the *combinatorial Laplacian* matrix L have been widely studied.

The *normalized Laplacian* matrix, introduced by Chung [6] in the 1990's to study random walks, has fewer known results. The normalized Laplacian matrix of a graph $G = (V, E)$ is denoted by $\mathcal{L}_G$, and is the matrix whose rows and columns are indexed by V, and defined by

$$\mathcal{L}_G[u,v] = \begin{cases} 1 & \text{if } u = v \text{ and } d_v \neq 0; \\ \dfrac{-1}{\sqrt{d_u d_v}} & \text{if } uv \in E; \\ 0 & \text{otherwise} \end{cases} .$$

It is well-known that $0 = \lambda_1(\mathcal{L}) \leq \lambda_2(\mathcal{L}) \leq \ldots \leq \lambda_n(\mathcal{L}) \leq 2$.

Our paper deals with the *normalized Laplacian energy* of a graph, which we also call its $\mathcal{L}$-energy. It is defined as

$$E_{\mathcal{L}}(G) = \sum_{i=1}^{n} |\lambda_i - 1| \tag{1}$$

where $\lambda_1(\mathcal{L}), \ldots, \lambda_n(\mathcal{L})$ are the eigenvalues of $\mathcal{L}_G$. Cavers shows that $2 \leq E_{\mathcal{L}}(G) \leq 2\left\lfloor \frac{n}{2} \right\rfloor$ and when G is connected $E_{\mathcal{L}}(G) < \sqrt{\frac{15}{28}}(n+1)$ (see [5]).

The energy of a matrix is intended to measure the deviation of the eigenvalues from their mean. While the matrix $\mathcal{L}_G$ is defined when G has isolates, in such a graph its average eigenvalue would no longer be 1, and the above definition of $E_{\mathcal{L}}(G)$ would not be meaningful. For this reason, we will assume all graphs are isolate-free.

The *Randić matrix* $R = [r_{ij}]$ of a graph G is defined [2, 7, 10] as

$$r_{ij} = \begin{cases} \dfrac{1}{\sqrt{d_u d_v}} & \text{if } uv \in E \\ 0 & \text{otherwise} \end{cases}$$

Historically, it is related to a descriptor for molecular graphs used by Milan Randić in 1975 [12]. The *Randić energy* $RE(G)$ of a graph G is $\sum_{i=1}^{n} |\rho_i|$ where $\operatorname{spec}(R) = \{\rho_1, \ldots, \rho_n\}$. It is interesting that $RE(G)$ and $E_{\mathcal{L}}(G)$ are equal in graphs without isolates [10]. Thus, results in this paper on normalized Laplacian energy apply also to Randić energy.

Energy change relating to a graph's adjacency matrix has been studied in several papers. Day and So [8] study the energy change when the edges in an induced subgraph are removed. In [9] they observe that when removing a single edge, the energy can increase, decrease, or remain the same. However when deleting a bridge the energy must decrease (Theorem 4.2). In [1] the authors study energy change in graphs where parallel edges are allowed. All these papers, including ours, utilize a classic inequality involving singular values. Beyond that, our techniques appear to be new.

This note is motivated by results in [4, 5] involving the effect of edge deletion on $\mathcal{L}$-energy. It is shown that deleting an edge may either increase, decrease or leave the $\mathcal{L}$-energy unchanged. Cavers, Fallat and Kirkland show [4, Thr. 19] that if G is isolate-free and $e = uv$ not incident to a pendant vertex, then $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G - e)|$ is at most 1.8366.

We strengthen the above bound if we also assume $N(u) \cap N(v) = \emptyset$. In that case, $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G - e)| < 1.5404$. If d_u and d_v are at least 3, the energy change is strictly less than 1. More generally, if $d = \min\{d_u, d_v\}$, then $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G - e)| < \frac{2\sqrt{5}}{\sqrt{d}}$, and so $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G - e)|$ is in $O(d^{-0.5})$. As an application, we show how energies of certain trees conjectured to have largest $\mathcal{L}$-energy among connected graphs, are related when n is large.

The condition $N(u) \cap N(v) = \emptyset$ is equivalent to e not being in a cycle C_3. If $e = uv$ is a bridge, it satisfies the condition. In triangle-free graphs, such as a bipartite graphs, all edges satisfy the condition.

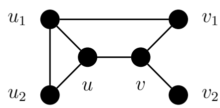

Figure 1. Edge uv with $N(u) \cap N(v) = \emptyset$.

Before proceeding, it will be useful to consider the graph G and edge $e = uv$ in Figure 1, and the matrices $\mathcal{L}_G$ and $\mathcal{L}_{G-e}$ that are given respectively below. Whenever a non-pendant edge is removed in *any* isolate-free graph, $\mathcal{L}_G$ and $\mathcal{L}_{G-e}$ must be identical except in two rows and two columns, the differing entries being precisely the entries corresponding to edges incident to u or v. Note that the condition $N(u) \cap N(v) = \emptyset$ also implies that for $w \neq u, v$ we can *not* have both $\mathcal{L}_G[u, w] \neq 0$ and $\mathcal{L}_G[v, w] \neq 0$.

$$
\begin{array}{c|cccccc}
 & u & v & u_1 & u_2 & v_1 & v_2 \\
\hline
u & & & & & & \\
v & & & & & & \\
u_1 & & & & & & \\
u_2 & & & & & & \\
v_1 & & & & & & \\
v_2 & & & & & &
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} u & v & u_1 & u_2 & v_1 & v_2 \end{array} \\
\begin{array}{c} u \\ v \\ u_1 \\ u_2 \\ v_1 \\ v_2 \end{array} &
\begin{pmatrix}
1 & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{\sqrt{6}} & 0 & 0 \\
\frac{-1}{3} & 1 & 0 & 0 & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\
\frac{-1}{3} & 0 & 1 & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & 0 \\
\frac{-1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{6}} & 1 & 0 & 0 \\
0 & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & 0 & 1 & 0 \\
0 & \frac{-1}{\sqrt{3}} & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
\quad
\begin{array}{cc}
 & \begin{array}{cccccc} u & v & u_1 & u_2 & v_1 & v_2 \end{array} \\
\begin{array}{c} u \\ v \\ u_1 \\ u_2 \\ v_1 \\ v_2 \end{array} &
\begin{pmatrix}
1 & 0 & \frac{-1}{\sqrt{6}} & \frac{-1}{2} & 0 & 0 \\
0 & 1 & 0 & 0 & \frac{-1}{2} & \frac{-1}{\sqrt{2}} \\
\frac{-1}{\sqrt{6}} & 0 & 1 & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & 0 \\
\frac{-1}{2} & 0 & \frac{-1}{\sqrt{6}} & 1 & 0 & 0 \\
0 & \frac{-1}{2} & \frac{-1}{\sqrt{6}} & 0 & 1 & 0 \\
0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

The remainder of this paper is organized as follows. In Section 2 we recall some useful facts about the singular values of a matrix. Our main results are obtained Section 3. In Section 4 we give some applications.

2 Singular Values and Energy

The *singular values* of a rectangular matrix N with complex entries, are defined to be the square roots of the eigenvalues of the positive semi-definite matrix N^*N, where N^* is the conjugate transpose of N. From here on, we denote singular values by $\sigma_i(N), i = 1, \ldots, n$. Lemma 1 is straightforward, Lemma 2 is in [11, Cor. 3.4.3], and Lemma 3 appears in [5].

Lemma 1 *The singular values of a real symmetric matrix M are the absolute values of the eigenvalues of M.*

Lemma 2 *Let A and B be square matrices of order n. Then*

$$\sum_{i=1}^{n} \sigma_i(A+B) \leq \sum_{i=1}^{n} \sigma_i(A) + \sum_{i=1}^{n} \sigma_i(B).$$

Using (1) and Lemma 1, we can express $\mathcal{L}$-energy as:

$$E_{\mathcal{L}}(G) = \sum_{i=1}^{n} |\lambda_i - 1| = \sum_{i=1}^{n} |\lambda_i(I - \mathcal{L}_G)| = \sum_{i=1}^{n} \sigma_i(I - \mathcal{L}_G) \tag{2}$$

Lemma 3 (Cavers) *Let G and H be graphs of order n, and $M = \mathcal{L}_G - \mathcal{L}_H$. Then $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(H)| \leq \sum_{i=1}^{n} \sigma_i(M)$.*

Proof: Since $I - \mathcal{L}_H = M + (I - \mathcal{L}_G)$, from Lemma 2 we get:

$$\sum_{i=1}^{n} \sigma_i(I - \mathcal{L}_H) \leq \sum_{i=1}^{n} \sigma_i(M) + \sum_{i=1}^{n} \sigma_i(I - \mathcal{L}_H)$$

and so from (2) we have $E_{\mathcal{L}}(H) - E_{\mathcal{L}}(G) \leq \sum_{i=1}^{n} \sigma_i(M)$. To complete the proof, it suffices to show $E_{\mathcal{L}}(G) - E_{\mathcal{L}}(H) \leq \sum_{i=1}^{n} \sigma_i(M)$. Applying Lemma 2 to the equation $I - \mathcal{L}_G = (-M) + (I - \mathcal{L}_H)$ we have

$$\sum_{i=1}^{n} \sigma_i(I - \mathcal{L}_G) \leq \sum_{i=1}^{n} \sigma_i(-M) + \sum_{i=1}^{n} \sigma_i(I - \mathcal{L}_H).$$

Using (2) and $\sigma_i(M) = \sigma_i(-M)$, we have

$$E_{\mathcal{L}}(G) - E_{\mathcal{L}}(H) \leq \sum_{i=1}^{n} \sigma_i(-M) \leq \sum_{i=1}^{n} \sigma_i(M)$$

completing the proof. ■

3 Main Results

Our results give upper bounds on the change in $\mathcal{L}$-energy when we remove an edge $e = uv$, not incident to a pendant, for which $N(u) \cap N(v) = \emptyset$. Our strategy in bounding the change in normalized Laplacian energy is to use the singular values of the difference matrix $M = \mathcal{L}_G - \mathcal{L}_{G-e}$. By Lemma 3 we are guaranteed that

$$|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G - e)| \leq \sum_{i=1}^{n} \sigma_i(M). \tag{3}$$

We partition $V - \{u, v\}$ into $V_u = \{u_1, \ldots, u_{n_1-1}\}$ and $V_v = \{v_1, \ldots, v_{n_2-1}\}$ such that $N(u) - \{v\} \subseteq V_u$ and $N(v) - \{u\} \subseteq V_v$, where $n = n_1 + n_2$ is the order of G. This partition is possible because $N(u) \cap N(v) = \emptyset$.

We examine the structure of M. M is symmetric with diagonal values of zero. If we order the vertices as

$$u, v, u_1, \ldots, u_{n_1-1}, v_1, \ldots, v_{n_2-1}$$

then only entries in the first two rows or first two columns of M can be nonzero:

$$M = \begin{pmatrix} 0 & x & x_1 & \ldots & x_{n_1-1} & 0 & \ldots & 0 \\ x & 0 & 0 & \ldots & 0 & y_1 & \ldots & y_{n_2-1} \\ x_1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{n_1-1} & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & y_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & y_{n_2-1} & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{4}$$

The entries of M are defined as follows. Clearly

$$M[u,v] = M[v,u] = x = \frac{-1}{\sqrt{d_u d_v}}. \tag{5}$$

In the first row,

$$M[u,u_i] = x_i = \begin{cases} 0 & \text{if } u_i \notin N(u) \\ \frac{-1}{\sqrt{d_u d_{u_i}}} + \frac{1}{\sqrt{(d_u-1)d_{u_i}}} & \text{if } u_i \in N(u) - \{v\} \end{cases} \tag{6}$$

and in the second row

$$M[v,v_i] = y_i = \begin{cases} 0 & \text{if } v_i \notin N(v) \\ \frac{-1}{\sqrt{d_v d_{v_i}}} + \frac{1}{\sqrt{(d_v-1)d_{v_i}}} & \text{if } v_i \in N(v) - \{u\} \end{cases} \tag{7}$$

Note the expressions in (6) and (7) make sense only if $d_u \geq 2$ and $d_v \geq 2$. We denote with $\mathbf{x}$ and $\mathbf{y}$ the vectors $[x_1, \ldots, x_{n_1-1}]^T$ and $[y_1, \ldots, y_{n_2-1}]^T$ respectively. Both $\mathbf{x}$ and $\mathbf{y}$ have nonzero entries, and $x \neq 0$. Our arguments depend heavily on the quantities $||\mathbf{x}||$ and $||\mathbf{y}||$, where $||\cdot||$ denotes the Euclidean norm.

Lemma 4 *For some λ_1 and λ_2, spec$(M) = \{0^{n-4}, -\lambda_2, -\lambda_1, \lambda_1, \lambda_2\}$.*

Proof: Since $\mathbf{x}$ and $\mathbf{y}$ each have a nonzero entry, it is easy to see that $\text{rank}(M) = 4$, and therefore M has exactly four nonzero eigenvalues. Now suppose $\lambda \neq 0$ is an eigenvalue of M. Let $[\alpha, \beta, \mathbf{v}, \mathbf{u}]^T$ be an eigenvector for λ, where α , $\beta \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^{n_1-1}$ and $\mathbf{u} \in \mathbb{R}^{n_2-1}$. Then

$$M \begin{bmatrix} \alpha \\ \beta \\ \mathbf{v} \\ \mathbf{u} \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \\ \mathbf{v} \\ \mathbf{u} \end{bmatrix}$$

Using (4) we see that

$$\beta x + \mathbf{x}^T\mathbf{v} = \lambda\alpha \tag{8}$$

$$\alpha x + \mathbf{y}^T\mathbf{u} = \lambda\beta \tag{9}$$

$$\alpha\mathbf{x} = \lambda\mathbf{v} \tag{10}$$

$$\beta\mathbf{y} = \lambda\mathbf{u} \tag{11}$$

Since $\lambda \neq 0$, (10) and (11) give $\mathbf{v} = \frac{\alpha}{\lambda}\mathbf{x}$ and $\mathbf{u} = \frac{\beta}{\lambda}\mathbf{y}$. Thus (8) and (9) give us:

$$\beta x + \frac{\alpha}{\lambda}||\mathbf{x}||^2 = \lambda\alpha \tag{12}$$

$$\alpha x + \frac{\beta}{\lambda}||\mathbf{y}||^2 = \lambda\beta \tag{13}$$

Isolating $\beta = \frac{1}{x}(\lambda\alpha - \frac{\alpha}{\lambda}||\mathbf{x}||^2)$ from (12) and substituting in (13) we find

$$\alpha x + \frac{||\mathbf{y}||^2}{\lambda} \cdot \frac{1}{x}(\lambda\alpha - \frac{\alpha}{\lambda}||\mathbf{x}||^2) = \frac{\lambda}{x}(\lambda\alpha - \frac{\alpha}{\lambda}||\mathbf{x}||^2).$$

Since $\alpha \neq 0$ (for otherwise, we have a zero eigenvector) we obtain

$$\lambda^4 + \lambda^2(-x^2 - ||\mathbf{y}||^2 - ||\mathbf{x}||^2) + ||\mathbf{x}||^2||\mathbf{y}||^2 = 0. \tag{14}$$

Letting $B = x^2 + ||\mathbf{y}||^2 + ||\mathbf{x}||^2$ and $C = ||\mathbf{x}||^2||\mathbf{y}||^2$, we see that the discriminant $B^2 - 4C \geq (||\mathbf{y}||^2 + ||\mathbf{x}||^2)^2 - 4C = (||\mathbf{y}||^2 - ||\mathbf{x}||^2)^2 \geq 0$ so the roots of (14) are

$$\pm\sqrt{\frac{B \pm \sqrt{B^2 - 4C}}{2}}$$

since both $\frac{B \pm \sqrt{B^2-4C}}{2}$ are positive. This completes the proof. ■

Lemma 5 $\lambda_1(M) + \lambda_2(M) = \sqrt{x^2 + ||\mathbf{y}||^2 + ||\mathbf{x}||^2 + 2||\mathbf{x}|| \cdot ||\mathbf{y}||}$

Proof: Using B and C from Lemma 4, we have

$$\lambda_1(M) + \lambda_2(M) = (\frac{B + \sqrt{B^2 - 4C}}{2})^{1/2} + (\frac{B - \sqrt{B^2 - 4C}}{2})^{1/2}. \tag{15}$$

Squaring both sides of (15) shows $(\lambda_1(M) + \lambda_2(M))^2 = B + 2\sqrt{C}$. So

$$\lambda_1(M) + \lambda_2(M) = \sqrt{B + 2\sqrt{C}} = \sqrt{x^2 + ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}|| \cdot ||\mathbf{y}||}.$$

■

Theorem 1 *Let G be an isolate-free graph, and $e = uv$ an edge with $d_u, d_v \geq 2$ and $N(u) \cap N(v) = \emptyset$. Then $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G - e)| < 1.5404$.*

Proof: By (3) and Lemma 1 we know

$$|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G - e)| \leq \sum_{i=1}^{n} \sigma_i(M) = \sum_{i=1}^{n} |\lambda_i(M)|$$

Applying Lemma 4 and Lemma 5 this sum equals

$$2(\lambda_1 + \lambda_2) = 2\sqrt{x^2 + ||Y||^2 + ||\mathbf{x}||^2 + 2||\mathbf{x}|| \cdot ||\mathbf{y}||} \tag{16}$$

We complete the proof by bounding each term on the right side of (16). From (5) we know $x^2 = \frac{1}{d_u d_v} \leq \frac{1}{4}$. From (6) and (7) we have:

$$||\mathbf{x}||^2 = \sum_{u_i \in N(u) - \{v\}} (\frac{-1}{\sqrt{d_u d_{u_i}}} + \frac{1}{\sqrt{(d_u - 1)d_{u_i}}})^2$$
$$||\mathbf{y}||^2 = \sum_{v_i \in N(v) - \{u\}} (\frac{-1}{\sqrt{d_v d_{v_i}}} + \frac{1}{\sqrt{(d_v - 1)d_{v_i}}})^2.$$

Since $\frac{-1}{\sqrt{d_u d_{u_i}}} + \frac{1}{\sqrt{(d_u-1)d_{u_i}}} = \frac{1}{\sqrt{d_{u_i}}}(\frac{-1}{\sqrt{d_u}} + \frac{1}{\sqrt{d_u-1}}) \le \frac{-1}{\sqrt{d_u}} + \frac{1}{\sqrt{d_u-1}}$, and applying a similar observation to $\frac{-1}{\sqrt{d_v d_{v_i}}} + \frac{1}{\sqrt{(d_v-1)d_{v_i}}}$, we see that

$$||\mathbf{x}||^2 \le (d_u - 1)(\frac{-1}{\sqrt{d_u}} + \frac{1}{\sqrt{d_u - 1}})^2 \tag{17}$$

$$||\mathbf{y}||^2 \le (d_v - 1)(\frac{-1}{\sqrt{d_v}} + \frac{1}{\sqrt{d_v - 1}})^2 \tag{18}$$

We claim that the right sides of (17) and (18) are each maximized by the minimum integer values $d_u = d_v = 2$. To see this, one can show

$$f(d) = (d-1)(\frac{-1}{\sqrt{d}} + \frac{1}{\sqrt{d-1}})^2 = 2 - \frac{1}{d} - 2\sqrt{1 - \frac{1}{d}} \tag{19}$$

and then observe $f'(d) < 0$, for $d > 1$. Hence $||\mathbf{x}||^2$ and $||\mathbf{y}||^2$ are each bounded above by $(1 - \frac{1}{\sqrt{2}})^2$. Thus $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G-e)|$ is at most

$$2\sqrt{x^2 + ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}|| \cdot ||\mathbf{y}||} \quad \le \quad 2\sqrt{\frac{1}{4} + 4(1 - \frac{1}{\sqrt{2}})^2} < 1.5404$$

completing the proof. ∎

Theorem 2 *Let G be an isolate-free graph, and $e = uv$ an edge with $d_u, d_v \ge 3$ and $N(u) \cap N(v) = \emptyset$. Then $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G-e)| < .9916$.*

Proof: Here $||\mathbf{x}||^2 \le 2(\frac{-1}{\sqrt{3}} + \frac{1}{\sqrt{2}})^2$, and $x^2 = \frac{1}{9}$ in the previous computation. ∎

The following theorem shows that the change in normalized Laplacian energy is in $O(d^{-1/2})$.

Theorem 3 *Let G be an isolate-free graph, and $e = uv$ an edge with $d_u, d_v \ge d$ and $N(u) \cap N(v) = \emptyset$. Then $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G-e)| < \frac{2\sqrt{5}}{\sqrt{d}}$.*

Proof: Using the right side of (19), we have $4||\mathbf{x}||^2 \le 4(2 - \frac{1}{d} - \frac{2}{d}\sqrt{d(d-1)})$, and so $2\sqrt{x^2 + 4||\mathbf{x}||^2}$ becomes

$$\begin{aligned} 2\sqrt{\frac{1}{d^2} + 8 - \frac{4}{d} - \frac{8\sqrt{d(d-1)}}{d}} &= 2\sqrt{\frac{1 + 8d^2 - 4d - 8d\sqrt{d(d-1)}}{d^2}} \\ &\le 2\sqrt{\frac{1 + 8d^2 - 4d - 8d(d-1)}{d^2}} \\ &= 2\frac{\sqrt{1+4d}}{d} < 2\frac{\sqrt{5d}}{d} = \frac{2\sqrt{5}}{\sqrt{d}} \end{aligned}$$

∎

4 Applications

Suppose we have a sequence of graphs G_k, each with an edge $e_k = u_k v_k$ and $N(u_k) \cap N(v_k) = \emptyset$. Suppose also that $\lim_{k\to\infty} \min\{d_{u_k}, d_{v_k}\} = \infty$. Then by Theorem 3,

$$\lim_{k\to\infty} (E_{\mathcal{L}}(G_k) - E_{\mathcal{L}}(G_k - e_k)) = \lim_{k\to\infty} |E_{\mathcal{L}}(G_k) - E_{\mathcal{L}}(G_k - e_k)| = 0$$

We give two illustrations involving trees. As noted earlier, all edges of bipartite graphs satisfy the disjoint neighborhood condition. In our first example, $\lim_{k\to\infty} E_{\mathcal{L}}(G_k)$ converges and in the second example it does not.

Diameter 3 trees: As a simple application consider an infinite sequence

$$T_1, T_2, T_3, \ldots$$

of trees, each with diameter 3. Each tree T_k of diameter 3 can be represented as two stars S_{k_1} and S_{k_2} with an edge e_k between their centers. Let $m_k = \min\{k_1, k_2\}$, and suppose $\lim_{k\to\infty} m_k = \infty$. Then we must have

$$0 = \lim_{k\to\infty} (E_{\mathcal{L}}(T_k) - E_{\mathcal{L}}(T_k - e_k))$$

It is well-known that the normalized Laplacian energy of a star is 2, and so $E_{\mathcal{L}}(T_k - e_k) = 4$. Therefore

$$\lim_{k\to\infty} E_{\mathcal{L}}(T_k) = 4$$

Interestingly, a diameter 3 tree T of order n is known to have eigenvalue 1 with multiplicity $n-4$, and therefore it will have four eigenvalues that contribute to its $\mathcal{L}$-energy (see [3, Thr. 3.2]).

Suns and double suns: For a second application we refer to the work of Gutman, Furtula and Bozkurt [10] on the energy of the Randić matrix, which for isolate-free graphs, equals $\mathcal{L}$-energy, as noted above. For each $p \geq 0$, the *p-sun*, which we denote with S^p, is the tree of order $n = 2p+1$ formed by taking the star on $p+1$ vertices and subdividing each edge. For $p, q \geq 0$ the (p,q)-*double sun*, denoted $D^{p,q}$, is the tree of order $n = 2(p+q+1)$ obtained by connecting the centers of S^p and S^q with an edge. Without loss of generality we assume $p \geq q$. When $p - q \leq 1$ the double sun is called *balanced*. Figure 2 depicts the balanced double sun $D^{2,2}$, the sun S^5, and the balanced double sun $D^{3,2}$. The authors [10] conjecture that among connected graphs of order n, the graph with largest $\mathcal{L}$-energy is the sun when n is odd, and balanced double sun when n is even. Note that when $n \equiv 2 \bmod 4$ the balanced double sun is constructed with two p-suns, and when $n \equiv 0 \bmod 4$, it uses a $p+1$-sun and a p-sun. In either case, $p = \left\lfloor \frac{n-2}{4} \right\rfloor$. Consider the balanced double sun $D^{p,p}$

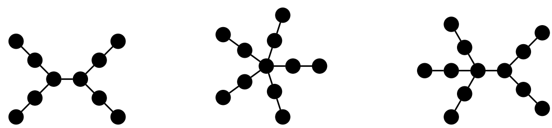

Figure 2. double sun $D^{2,2}$, sun S^5, and double sun $D^{3,2}$.

whose order is $4p+2$. It has an edge $e = uv$ for which $d_u = d_v = p+1$. Also, $D^{p,p} - e$ is two disjoint suns S^p. By Theorem 3,

$$|E_{\mathcal{L}}(D^{p,p}) - 2E_{\mathcal{L}}(S^p)| < \frac{2\sqrt{5}}{\sqrt{p+1}}.$$

Now consider the balanced double sun $D^{p+1,p}$ of order $4p+4$. It has an edge $e = uv$ for which $d_u = p+2$ and $d_v = p+1$. Also, $D^{p+1,p} - e$ is disjoint suns S^{p+1} and S^p. Therefore,

$$|E_{\mathcal{L}}(D^{p+1,p}) - E_{\mathcal{L}}(S^{p+1}) - E_{\mathcal{L}}(S^p)| < \frac{2\sqrt{5}}{\sqrt{p+1}}.$$

These observations show that for large n, the $\mathcal{L}$-energy of the balanced double suns can be approximated using smaller suns of about half the size.

5 Final Remarks

Our computational experiments suggest that the bound in Theorem 1 may be too high. In fact, we could find no example where the change in energy was more than 1. It is interesting that if $G = P_4$, the path on four vertices, and e is the middle edge, then $|E_{\mathcal{L}}(G) - E_{\mathcal{L}}(G-e)| = 1$. Can these results be improved by assuming all graphs are triangle-free? Finally, we note that Lemmas 4 and 5 depend only the pattern of the entries of M, not on the values, so we wonder if they might have other uses.

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