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Remarks on the Energy and the Minimum Dominating Energy of a Graph

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Abstract

Upper bounds for energy and minimum dominating energy of graph are established. Also upper and lower bounds for minimum dominating eigenvalues of a given graph are determined.

1 Introduction

Let $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, be a simple graph with n vertices, m edges, and the sequence of vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(i)$, $i = 1, 2, \ldots, n$. If i-th and j-th vertices of graph G are adjacent, we denote it as $i \sim j$. Well known properties of the sequence of vertex degrees are (see [4])

$$\sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n} d_i^2 = M_1$$

where M_1 is the first Zagreb index [10].

The adjacency matrix, $A = (a_{ij})$, of graph G is defined as

$$a_{ij} = \begin{cases} 1, & \text{if } i \sim j \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of adjacency matrix A, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, represent ordinary eigenvalues of graph G. Well known properties of these are [4]

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} d_i = 2m\,.$$

Denote by $|\lambda_1^*| \geq |\lambda_2^*| \geq \cdots \geq |\lambda_n^*|$, $\lambda_1 = |\lambda_1^*|$, a non increasing sequence of absolute values of the eigenvalues of G. The graph invariant called energy, $E(G)$, of G is defined to be the sum of the absolute values of the eigenvalues of G [7] (see also [6,8–10,16]), i.e.

$$E(G) = \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} |\lambda_i^*|\,.$$

A subset D of V is called a dominating set of G if every vertex of $V - D$ is adjacent to some vertex in D. Any dominating set with minimum cardinality is called a minimum dominating set. Let D, $|D| = k$, be a minimum dominating set of a graph G. The minimum dominating adjacency matrix of G, $A_D = (a_{ij}^D)$, is the $n \times n$ matrix defined by

$$a_{ij}^D = \begin{cases} 1, & \text{if } i \sim j \\ 1, & \text{if } i = j, i \in D \\ 0, & \text{otherwise.} \end{cases}$$

The minimum dominating eigenvalues of the graph G, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are eigenvalues of A_D. The following equalities are valid for γ_i, $i = 1, 2, \ldots, n$ [23]

$$\sum_{i=1}^{n} \gamma_i = k \qquad \text{and} \qquad \sum_{i=1}^{n} \gamma_i^2 = 2m + k\,.$$

Let $|\gamma_1^*| \geq |\gamma_2^*| \geq \cdots \geq |\gamma_n^*|$, $\gamma_1 = |\gamma_1^*|$, be a non increasing sequence of absolute values of the minimum dominating eigenvalues of G. The minimum dominating energy of graph G, $E_D(G) = E_D$, is defined as [23,24]

$$E_D = \sum_{i=1}^{n} |\gamma_i| = \sum_{i=1}^{n} |\gamma_i^*|\,.$$

In this paper we consider upper bounds for graph invariants E and E_D, as well as upper and lower bounds for the minimum dominating eigenvalues.

2 Preliminaries

In the text that follows we recall some inequalities that establish upper bounds for graph invariants E and E_D, that are of interest for our work.

In [19] the following inequality for graph energy E was proven

$$E \leq \sqrt{2mn}\,. \tag{1}$$

For E_D the following inequalities were proven in [23]

$$E_D \leq \sqrt{n(2m+k)} \tag{2}$$

and

$$E_D \leq \frac{2m+k}{n} + \sqrt{(n-1)\left[(2m+k) - \left(\frac{2m+k}{n}\right)^2\right]}\,. \tag{3}$$

3 Main results

In the following theorem we prove an inequality that is stronger than (1).

Theorem 1. *Let G be a simple graph with n, $n \geq 2$, vertices and m edges. Then*

$$E \leq \sqrt{2mn - \frac{n}{2}(|\lambda_1^*| - |\lambda_n^*|)^2}. \tag{4}$$

Equality holds if and only if $G \cong \overline{K}_n$ or $G \cong \frac{n}{2}K_2$ if n is even.

Proof. From the Lagrange's identity (see for example [22]),

$$0 \leq 2mn - E^2 = n\sum_{i=1}^{n}|\lambda_i^*|^2 - \left(\sum_{i=1}^{n}|\lambda_i^*|\right)^2 = \sum_{1\leq i<j\leq n}(|\lambda_i^*| - |\lambda_j^*|)^2$$

the following inequality can be obtained

$$0 \leq 2mn - E^2 \geq \sum_{i=2}^{n-1}\left((|\lambda_1^*| - |\lambda_i^*|)^2 + (|\lambda_i^*| - |\lambda_n^*|)^2\right) + (|\lambda_1^*| - |\lambda_n^*|)^2\,.$$

On the other hand, according to the Jennsen's inequality (see [21]), from the above inequality it follows that

$$0 \leq 2mn - E^2 \geq \frac{n-2}{2}\left(|\lambda_1^*| - |\lambda_n^*|\right)^2 + \left(|\lambda_1^*| - |\lambda_n^*|\right)^2 = \frac{n}{2}\left(|\lambda_1^*| - |\lambda_n^*|\right)^2. \tag{5}$$

After rearranging the above inequality, the inequality (4) is obtained. Equality in (5) holds if and only if $|\lambda_1^*| = |\lambda_2^*| = \cdots = |\lambda_n^*|$, so the equality in (4) holds if and only if G is an empty graph, $G \cong \overline{K}_n$, or G is a union of $\frac{n}{2}$ K_2 graphs, i.e., $G \cong \frac{n}{2}K_2$, if n is even. ∎

Remark 2. *Since* $(|\lambda_1^*| - |\lambda_n^*|) \geq 0$*, the inequality (4) is stronger than (1). Note that lower bounds analogous to (4) were proven in* [5] *and* [20].

By a similar procedure as that for the proof of Theorem 1, the following result can be proven for the invariant E_D.

Theorem 3. *Let* G *be a simple graph with* n *vertices and* m *edges and let* D, $|D| = k$ *be a minimum dominating set of* G. *Then*

$$E_D \leq \sqrt{n(2m+k) - \frac{n}{2}(|\gamma_1^*| - |\gamma_n^*|)^2}\,. \tag{6}$$

Equality holds if and only if $G \cong \overline{K}_n$ *or* $G \cong \frac{n}{2}K_2$ *if* n *is even.*

Remark 4. *Since* $|\gamma_1^*| - |\gamma_n^*| \geq 0$*, the inequality (6) is stronger than (2).*

Theorem 5. *Let* G *be a simple graph with* n, $n \geq 2$, *vertices and* m *edges. Then, for each real* R *with the property* $\lambda_1 \geq R \geq \sqrt{2mn}$, *the following inequality is valid*

$$E \leq R + \sqrt{(n-1)(2m - R^2)}\,. \tag{7}$$

Proof. In [18] a class of real polynomials $P_n(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_n$, denoted as $\mathcal{P}_n(a_1, a_2)$, where a_1 and a_2 are fixed real numbers, was considered. For the roots $x_1 \geq x_2 \geq \ldots \geq x_n$ of an arbitrary polynomial $P_n(x)$ from this class, the following values were introduced

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \text{and} \qquad \Delta = n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 . \tag{8}$$

Then upper and lower bounds for the polynomial roots, x_i, $i = 1, 2, \ldots, n$, were determined in terms of the introduced values

$$\begin{aligned}
&\bar{x} + \frac{1}{n}\sqrt{\frac{\Delta}{n-1}} \leq x_1 \leq \bar{x} + \frac{1}{n}\sqrt{(n-1)\Delta} \\
&\bar{x} - \frac{1}{n}\sqrt{\frac{i-1}{n-i+1}\Delta} \leq x_i \leq \bar{x} + \frac{1}{n}\sqrt{\frac{n-i}{i}\Delta}, \quad 2 \leq i \leq n-1 \\
&\bar{x} - \frac{1}{n}\sqrt{(n-1)\Delta} \leq x_n \leq \bar{x} - \frac{1}{n}\sqrt{\frac{\Delta}{n-1}}\,.
\end{aligned} \tag{9}$$

Consider now the polynomial

$$P_n(x) = \prod_{i=1}^{n} (x - |\lambda_i^*|) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_n\,.$$

Since

$$a_1 = -\sum_{i=1}^{n} |\lambda_i^*| = -E$$

and

$$a_2 = \frac{1}{2}\left[\left(\sum_{i=1}^{n} |\lambda_i^*|\right)^2 - \sum_{i=1}^{n} |\lambda_i^*|^2\right] = \frac{1}{2}E^2 - m$$

the above polynomial belongs to a class of real polynomials $\mathcal{P}_n(-E, \frac{1}{2}E^2 - m)$. Bearing in mind (8), we have that

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} |\lambda_i^*| = \frac{E}{n} \tag{10}$$

and

$$\Delta = n\sum_{i=1}^{n} |\lambda_i^*|^2 - \left(\sum_{i=1}^{n} |\lambda_i^*|\right)^2 = 2mn - E^2\,. \tag{11}$$

For $x_1 = \lambda_1 = |\lambda_1^*|$, according to (10), (11), and the right–hand side of the first inequality in (9), we get

$$\lambda_1 \leq \frac{E}{n} + \frac{1}{n}\sqrt{(n-1)(2mn - E^2)}\,. \tag{12}$$

Now, for each real R with the property $\lambda_1 \geq R \geq \sqrt{\frac{2m}{n}}$, from (12) it follows that

$$R \leq \frac{E}{n} + \frac{1}{n}\sqrt{(n-1)(2mn - E^2)}\,.$$

After rearranging the above inequality, the inequality (7) is obtained. ■

Remark 6. *By the appropriate choice of parameter R, from the inequality (7) various inequalities from the literature for the graph energy E can be obtained (see for example* [1, 11–15, 17, 25–27]*). We will illustrate this through three examples.*

1) *Let $R = \frac{2m}{n}$. Since $\lambda_1 \geq \frac{2m}{n} \geq \sqrt{\frac{2m}{n}}$ (see* [3] *and* [14]*), the conditions of Theorem 5 are satisfied. For $R = \frac{2m}{n}$ the inequality (7) becomes*

$$E \leq \frac{2m}{n} + \frac{1}{n}\sqrt{2m(n-1)(n^2 - 2m)}\,.$$

This inequality was proven in [14] *(see also* [15]*).*

2) Let $R=\sqrt{\frac{M_1}{n}}$*. Since* $\lambda_1 \geq \sqrt{\frac{M_1}{n}} \geq \frac{2m}{n} \geq \sqrt{\frac{2m}{n}}$ *(see* [27]*), the conditions of Theorem 5 are satisfied. Now from (7), the following inequality can be obtained*

$$E \leq \sqrt{\frac{M_1}{n}} + \sqrt{(n-1)\left(2m - \frac{M_1}{n}\right)}.$$

This inequality was proven in [27].

3) Let $R=\sqrt{\frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}}$*, where* t_i *is a 2-degree of vertex* i*. Since (see* [25, 26]*)*

$$\lambda_1 \geq \sqrt{\frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}} \geq \sqrt{\frac{M_1}{n}} \geq \frac{2m}{n} \geq \sqrt{\frac{2m}{n}}$$

the conditions of Theorem 5 are satisfied. From the inequality (7) we get

$$E \leq \sqrt{\frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}} + \sqrt{(n-1)\left(2m - \frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}\right)}.$$

This inequality was proven in [27] *(see also* [1, 11–13, 17, 25]*).*

Remark 7. *Let*

$$\varphi_n(x) = \prod_{i=1}^{n}(x-\lambda_i) = x^n + a_1x^{n-1} + a_2x^{n-2} + b_3x^{n-3} + \cdots + b_n$$

be the characteristic polynomial of a graph G*. Since*

$$a_1 = -\sum_{i=1}^{n} \lambda_i = 0$$

and

$$a_2 = \frac{1}{2}\left[\left(\sum_{i=1}^{n}\lambda_i\right)^2 - \sum_{i=1}^{n}\lambda_i^2\right] = -m$$

the polynomial $\varphi_n(x)$ *belongs to a class of real polynomials* $\mathcal{P}_n(0,-m)$*. From the equalities*

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n}\lambda_i = 0$$

and

$$\Delta = n\sum_{i=1}^{n}\lambda_i^2 - \left(\sum_{i=1}^{n}\lambda_i\right)^2 = 2mn$$

and (9), we obtain that for the eigenvalues λ_i, $i = 1, 2, \ldots, n$, the following is valid

$$0 < \sqrt{\frac{2m}{n(n-1)}} \le \lambda_1 \le \sqrt{\frac{2m(n-1)}{n}}$$

$$-\sqrt{\frac{2(i-1)m}{n(n-i+1)}} \le \lambda_i \le \sqrt{\frac{2m(n-i)}{in}}, \quad 2 \le i \le n-1$$

$$-\sqrt{\frac{2m(n-1)}{n}} \le \lambda_n \le -\sqrt{\frac{2m}{n(n-1)}} < 0\,.$$

With the exception of the left–hand side part of the first and the right–hand side part of the last inequality, the above inequalities were proven in [2].

Let

$$f_n(x) = \prod_{i=1}^{n}(x - \gamma_i) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_n$$

be the characteristic polynomial of minimum dominating adjacency matrix A_D of G. Since $a_1 = -k$ and $a_2 = \binom{k}{2} - m$, it belongs to a class of real polynomials $\mathcal{P}_n(-k, \binom{k}{2} - m)$. Based on the equalities

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n}\gamma_i = \frac{k}{n}$$

and

$$\Delta = n\sum_{i=1}^{n}\gamma_i^2 - \left(\sum_{i=1}^{n}\gamma_i\right)^2 = 2mn + k(n-k)$$

and inequality (9), we establish the following result.

Theorem 8. *Let G be a simple graph with n, $n \ge 2$, vertices and m edges and let D, $|D| = k$, be its minimum dominating set. Then*

$$\frac{k}{n} + \frac{1}{n}\sqrt{\frac{2mn + k(n-k)}{n-1}} \le \gamma_1 \le \frac{k}{n} + \frac{1}{n}\sqrt{(n-1)(2mn + k(n-k))}$$

$$\frac{k}{n} - \frac{1}{n}\sqrt{\frac{i-1}{n-i+1}(2mn + k(n-k))} \le \gamma_i \le \frac{k}{n} + \frac{1}{n}\sqrt{\frac{n-i}{i}(2mn + k(n-k))}$$

for $2 \le i \le n-1$, *and*

$$\frac{k}{n} - \frac{1}{n}\sqrt{(n-1)(2mn + k(n-k))} \le \gamma_n \le \frac{k}{n} - \frac{1}{n}\sqrt{\frac{2mn + k(n-k)}{n-1}}\,.$$

Consider the polynomial

$$\psi_n(x) = \prod_{i=1}^{n} (x - |\gamma_i^*|)^2 = x^n + a_1 x^{n-1} + a_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_n\,.$$

Since

$$a_1 = -\sum_{i=1}^{n} |\gamma_i^*| = -E_D$$

and

$$a_2 = \frac{1}{2}\left[\left(\sum_{i=1}^{n}|\gamma_i^*|\right)^2 - \sum_{i=1}^{n}|\gamma_i^*|^2\right] = \frac{1}{2}\left[E_D^2 - (2m+k)\right]$$

the polynomial $\psi_n(x)$ belongs to a class of real polynomials $\mathcal{P}_n(-E_D, \frac{1}{2}(E_D^2 - (2m+k))$. By the same procedure as with Theorem 5, the following result can be proven.

Theorem 9. *Let G be a graph with n, $n \geq 2$, vertices and m edges and let D, $|D| = k$, be its minimum dominating set. Then for each T with the property $\gamma_1 \geq T \geq \sqrt{\frac{2m+k}{n}}$, the following is valid*

$$E_D \leq T + \sqrt{(n-1)(2m+k-T^2)}\,. \tag{13}$$

Remark 10. *By an appropriate choice of the parameter T, from (13) some well known inequalities for the invariant E_D could be obtained. Thus, for example, for $T = \sqrt{\frac{2m+k}{n}}$, from (13) the inequality (2) is obtained. For $T = \frac{2m+k}{n}$, from (13) the inequality (3) is obtained.*

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