MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **74** (2015) 449-464

ISSN 0340 - 6253

A Numerical Approach to Fokker-Planck Equation with Space- and Time-Fractional and Non Fractional Derivatives

M. A. Firoozjaee[a], **S. A. Yousefi**[a], **H. Jafari**[b,c]

[a] *Department of Mathematics, Shahid Beheshti University, G. C., Tehran, Iran.*

[b] *Department of Mathematics and Computer Science, University of Mazandaran, Babolsar, Iran*

[c] *Department of Mathematical Sciences, University of South Africa, P.O. Box 392, UNISA 0003, South Africa*

(Received March 5, 2015)

Abstract

In this paper, we consider the Fokker-Planck equation (FPE) and the Fokker-Planck equation with space- and time-fractional derivatives. The Fokker-Planck equation arises in various fields in chemistry, natural science, including astrophysics problems, biological applications, chemical physics and other fields. Transforming the Fokker-Planck equation into optimization problem and using polynomial basis functions, we obtain the system of algebraic equation. Then, we solve the system of nonlinear algebraic equation and we have the coefficients of polynomial basis functions expansion. We extensively discuss the convergence of the method. Illustrative examples are included to demonstrate the validity and applicability of the technique.

1 Introduction

The Fokker-Planck equation was introduced by Fokker and Planck to describe the Brownian motion of particles [1]. The Fokker-Planck equation arises in various fields in chemistry, natural science, including astrophysics problems, biological applications, chemical physics, economics, electron relaxation in gases, nucleation, optical bistability, polymer dynamics, quantum optics,reactive systems and numerous other applications [1].

The chemical Fokker-Planck equation is commonly used approximations of the chemical master equation. This equation is derived from an uncontrolled, second-order truncation of the Kramers-Moyal expansion of the chemical master equation and hence their accuracy remains to be clarified. The chemical Fokker-Planck equation turns out to be more accurate than the linear-noise approximation of the chemical master equation (the linear Fokker-Planck equation) which leads to mean concentration estimates accurate to order $\Omega^{-1/2}$ and variance estimates accurate to order $\Omega^{-3/2}$. This higher accuracy is particularly conspicuous for chemical systems realized in small volumes such as biochemical reactions inside cells. A formula is also obtained for the approximate size of the relative errors in the concentration and variance predictions of the chemical Fokker-Planck equation, where the relative error is defined as the difference between the predictions of the chemical Fokker-Planck equation and the master equation divided by the prediction of the master equation [2, 3].

A FPE describes the change of probability of a random function in space and time; hence it is naturally used to describe solute transport. The general FPE for the motion of a concentration field $u(x,t)$ of one space variable x at time t has the form

$$\frac{u(x,t)}{\partial t} = [-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)]u(x,t), \tag{1}$$

with the initial condition

$$u(x,0) = f(x), \quad x \in \mathbb{R}, \tag{2}$$

where $B(x) > 0$ is the diffusion coefficient and $A(x)$ is the drift coefficient. The drift and diffusion coefficients may also depend on time.

$$\frac{u(x,t)}{\partial t} = [-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)]u(x,t), \tag{3}$$

with the initial condition (2). (3) is a linear second-order partial differential equation of parabolic type. There is a more general form of FPE which is called nonlinear Fokker-Planck equation. Nonlinear FPE has important applications in various areas such as chemical physics, chemistry, plasma physics, population dynamic, biophysics, neuroscience, polymer physics, psychology and marketing. In one variable case, the nonlinear FPE is written in the following form

$$\frac{u(x,t)}{\partial t} = [-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)]u(x,t), \tag{4}$$

with the initial condition (2). The general nonlinear FPE with space- and time-fractional derivatives for the motion of a concentration field $u(x,t)$ of the one space variable x at time t has the form [4]

$$\begin{aligned} &\tfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = [-\tfrac{\partial^\beta}{\partial x^\beta} A(t,x,u) + \tfrac{\partial^{2\beta}}{\partial x^{2\beta}} B(t,x,u)]u(x,t), \\ &x \in [0,1], \quad t \in (0,T), \quad 0 < \alpha, \beta \leq 1, \end{aligned} \tag{5}$$

with the initial condition

$$u(x,0) = f(x), \qquad x \in [0,1], \tag{6}$$

where α and β are parameters describing the order of the fractional time- and space derivatives, respectively. The function $u(x,t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t < 0$ and $x < 0$. The fractional derivatives are considered in the Caputo sense [5, 6],

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{1}{(t-s)^\alpha} ds. \tag{7}$$

The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses. In the case of $\alpha = 1$ and $\beta = 1$, the fractional equation reduces to the classical nonlinear FPE (4).

The fractional Fokker-Planck equation has been used in various areas of chemistry,chemical physics, engineering and physics. The fractional nonlinear Fokker-Planck-like equations [7, 8] have been used to analyze several physical situations that present anomalous diffusion, that usually contain a mix of nonlinear terms and fractional derivatives. In fact, nonlinear diffusion equation and fractional diffusion are successfully applied to several situations due to their wide application in chemistry and engineering such as frequency-dependent damping behavior of materials, viscoelasticity, diffusion processes etc [9–12]. Much study [13–15] has been devoted to fractional nonlinear Fokker-Planck equations. Liang and co-workers [16] have studied the solutions of a generalized anomalous diffusion equations with fractional derivatives. Deng in [17] developed a finite element method for the numerical resolution of the space and time fractional FPE and then proved that the convergence order is $O(k^{2-\alpha} + h^\mu)$, where k and h are the time step size and the space step size, respectively. An FPE of fractional order with respect to time is suggested by Jumarie in [18] by combining the maximum entropy principle and method of lines which could be related to dynamical systems subject to fractional Brownian motion. Chen et al. [19] examined the finite difference approximation and energy method to solve a class of initial-boundary value problems for the fractional FPE on a finite domain. Hashemi in [20] has applied the Lie symmetry analysis method for the nonlinear time fractional Fokker-Planck equation with Riemann–Liouville derivative.

2 Numerical approach for the fractional Fokker-Planck equation

We consider the linear and nonlinear Fokker–Planck equation with space- and time-fractional derivatives of the form:

$$\begin{aligned}&\tfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}=[-\tfrac{\partial^\beta}{\partial x^\beta}A(t,x,u)+\tfrac{\partial^{2\beta}}{\partial x^{2\beta}}B(t,x,u)]u(x,t),\\&x\in[0,1],\quad t\in(0,1),\quad 0<\alpha,\beta\leq 1,\end{aligned} \tag{8}$$

with the initial condition

$$u(x,0)=f(x),\qquad x\in[0,1]\,. \tag{9}$$

The method consists in the conversion of the Fokker-Planck equation with space- and time-fractional derivatives to an optimization problem and expanding the solution by polynomial basis functions unknown of coefficients.

We approximate $u(x,t)$ as

$$u(x,t)\cong u_{kk'}(x,t)=\sum_{i=0}^{k}\sum_{j=0}^{k'}c_{ij}t\phi_i(x)\phi_j(t)+f(x), \tag{10}$$

where $\phi_i(x)$, $\phi_j(t)$ are polynomial basis functions, coefficients of c_{ij} are unknown.
We substitute(10) in (8) and define

$$\begin{aligned}&J[c_{00},...,c_{0k'},...,c_{k0},...,c_{kk'}]=\\&\int_0^1\int_0^1\left(\tfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}-[-\tfrac{\partial^\beta}{\partial x^\beta}A(t,x,u)+\tfrac{\partial^{2\beta}}{\partial x^{2\beta}}B(t,x,u)]u(x,t)\right)^2dxdt.\end{aligned} \tag{11}$$

If c_{ij} are determined by the minimizing function J, then by (10) we achieve functions which approximate the minimum value of J in (11) and also satisfy initial condition. According to the necessary conditions of minimization for the function (11), we have

$$\frac{\partial J}{\partial c_{ij}}=0,\;\; i=0,\ldots k\;,j=0,...,k'. \tag{12}$$

Equations (12) gives a system of $(k+1)\times(k'+1)$ nonlinear equations. The method presented here is based on the Ritz method. We refer the interested reader to [21–23] for more information.

3 On the convergence of the method

In this section, we discuss the convergence of the method presented in Section 2. We will show that the approximate solution that has obtained by value of minimum tends to the exact solution with the increase of k, k' in (10). We show this fact in Theorem 2. Now, we define our function space and provide some needed lemmas and theorem

Consider the Banach space $\left(C^{(2,1)}(U), \|.\|_{(2,1)}\right)$ as follows

$$C^{(2,1)}\left(U\right) = \left\{ u : U \to R \quad | \quad \frac{\partial^2 u(x,t)}{\partial x^2} \in C(U), \frac{\partial u(x,t)}{\partial t} \in C(U) \right\},$$

where

$$U = [0,1] \times [0,1]\,,$$

and we define

$$\|u\|_{\infty} = sup\left|u(x,t)\right|_{(x,t)\in U}\,,$$

$$\begin{aligned}
\|u(x,t)\|_{(2,x)} &= \|u(x,t)\|_{\infty} + \left\|\frac{\partial u(x,t)}{\partial x}\right\|_{\infty} + \left\|\frac{\partial^2 u(x,t)}{\partial x^2}\right\|_{\infty}, \\
\|u(x,t)\|_{(1,t)} &= \|u(x,t)\|_{\infty} + \left\|\frac{\partial u(x,t)}{\partial t}\right\|_{\infty}, \\
\|u(x,t)\|_{(2,1)} &= \|u(x,t)\|_{(2,x)} + \|u(x,t)\|_{(1,t)}\,,
\end{aligned}$$

and

$$E\left(U\right) = \left\{u(x,t) \in C^{(2,1)}(U) | u(x,0) = f(x)\right\}. \tag{13}$$

Theorem 1 *: Suppose $\{f_n(x)\}$ is sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. IF $\{f'_n(x)\}$ converges uniformly on $[a,b]$, then $\{f_n(x)\}$ converges uniformly on $[a,b]$, to a function $f(x)$, and*

$$f'(x) = \lim_{n\to+\infty} f'_n(x), \qquad (a \le x \le b). \tag{14}$$

Proof: [24]

Now, we give a lemma which plays an important role in our study. The lemma shows that functions of the metric space $E\left(U\right)$ are dense in that space.

Lemma 1 *: Let $u(x,t) \in E\left(U\right)$. There exists a sequence of functions $\{z_l(x,t)\}_{l\in N} \subset E(U)$ such that $z_l(x,t) \to u(x,t)$ with respect to $\|.\|_{(2,1)}$.*

Proof: We have

$$u(x,t) - u(x,0) = \int_0^t \frac{\partial u(x,s)}{\partial s} ds, \tag{15}$$

whereas $\frac{\partial u(x,s)}{\partial s} \in C(U)$, according to Weierstrass theorem, there exist a sequence of polynomials $\{k_l(x,t)\}_{l\in N}$ such that $k_l \to \frac{\partial u(x,s)}{\partial s}$. We have

$$p_l(x,t) \;=\; \int_0^t k_l(x,s)ds,$$

where $p_l(x,0)=0$ and also $p_l(x,t)\to u(x,t)-u(x,0)$.
Now consider the sequence of functions $z_l(x,t)$ as follows

$$z_l(x,t)=p_l(x,t)+f(x), \tag{16}$$

where $z_l(x,0)=f(x)$.
According Theorem 1, $\frac{\partial z_l(x,t)}{\partial t}\to\frac{\partial u(x,t)}{\partial t}$, $\frac{\partial z_l(x,t)}{\partial x}\to\frac{\partial u(x,t)}{\partial x}$ and also $\frac{\partial^2 z_l(x,t)}{\partial x^2}\to\frac{\partial^2 u(x,t)}{\partial x^2}$ with norm $\|.\|_\infty$ and hence $s_l\to u$ with respect $\|.\|_{(1,t)}\, and \,\|.\|_{(2,x)}$. Therefore, we have $z_l\to u$ with respect $\|.\|_{(2,1)}$. $\square$
Consider the functional space $G_{kk'}(U)$ as follows

$$G_{kk'}(U)=E(U)\cap\left(\left\langle\{\phi_j(x)\}_{j=0}^{j=k}\right\rangle\times\left\langle\{\phi_j(t)\}_{j=0}^{j=k'}\right\rangle+f(x)\right),$$

where $\left(\left\langle\{\phi_j(x)\}_{j=0}^{j=k}\right\rangle\times\left\langle\{\phi_j(t)\}_{j=0}^{j=k'}\right\rangle\right)$ is the Banach subspace of $C^{(2,1)}(U)$ generated by the basis polynomials of degree at most k,k'. Of course $G_{kk'}(U)$ is a metric subspace of $E(U)$.

Let $u(x,t)\in C^{(2,1)}(U)$. For the Caputo fractional derivative of order α, we have $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\in C(U)$ [25]. We also have

$$\begin{aligned}
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(x,s)}{\partial s}(t-s)^{-\alpha}ds\\
\left|\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right| &\le \frac{1}{\Gamma(1-\alpha)}\int_0^t\left|\frac{\partial u(x,s)}{\partial s}\right|(t-s)^{-\alpha}ds\\
&\le \frac{\left\|\frac{\partial u(x,t)}{\partial t}\right\|_\infty}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}ds\\
&= \frac{\left\|\frac{\partial u(x,t)}{\partial t}\right\|_\infty t^{(1-\alpha)}}{\Gamma(1-\alpha)(1-\alpha)}\\
&\le \frac{\left\|\frac{\partial u(x,t)}{\partial t}\right\|_\infty}{\Gamma(2-\alpha)},
\end{aligned}$$

so

$$\left\|\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right\|_\infty\le\frac{\left\|\frac{\partial u(x,t)}{\partial t}\right\|_\infty}{\Gamma(2-\alpha)}. \tag{17}$$

Similarly for β and 2β, we have

$$\left\|\frac{\partial^\beta u(x,t)}{\partial x^\beta}\right\|_\infty \le \frac{\left\|\frac{\partial u(x,t)}{\partial x}\right\|_\infty}{\Gamma(2-\beta)}, \tag{18}$$

$$\left\|\frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}}\right\|_\infty \le \frac{\left\|\frac{\partial^2 u(x,t)}{\partial x^2}\right\|_\infty}{\Gamma(3-2\beta)}. \tag{19}$$

Now consider the functional J in (11) as an operator $J : \left(C^{(2,1)}(U), \|.\|_{(2,1)}\right) \to R$. Lemma 2 below shows that the functional J is continuous on its domain. We use this important property later in Theorem 3 and state a theorem from real analysis which we need in the proof of Lemma 2 .

Theorem 2 : *Let f be a continuous mapping of a compact metric space X into a metric space Y, then f is uniformly continuous.*

Proof: [26]

Lemma 2 : *The functional J is continuous on the Banach space* $\left(C^{(2,1)}(U), \|.\|_{(2,1)}\right)$.

Proof: We are going to show that $J : (C^{(2,1)}(U), \|.\|_{(2,1)})$ is continuous, where

$$J(u) = \int_0^1 \int_0^1 F^2 dx dt,$$

and $F(u)$ is considered as follows

$$F(u) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - [-\frac{\partial^\beta}{\partial x^\beta} A(t,x,u) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} B(t,x,u)]u(x,t).$$

Let $u^* \in C^{(2,1)}(U)$ and $\epsilon > 0$. Consider $r > 0$ and

$$I = U \times [-L-r, L+r] \times [-L-r, L+r] \times [-L-r, L+r],$$

where

$$L = max\left\{ \|u^*\|_\infty , \left\| \frac{\partial^\alpha u^*}{\partial t^\alpha} \right\|_\infty , \left\| \frac{\partial^\beta u^*}{\partial x^\beta} \right\|_\infty , \left\| \frac{\partial^{2\beta} u^*}{\partial x^{2\beta}} \right\|_\infty \right\}.$$

Obviously for $x \in [0,1]$ and $t \in [0,1]$, we have

$$Y^* = \left(x, t, u^*, \frac{\partial^\alpha u^*}{\partial t^\alpha}, \frac{\partial^\beta u^*}{\partial x^\beta}, \frac{\partial^{2\beta} u^*}{\partial x^{2\beta}} \right) \in I.$$

Let $\delta > 0$ and $\|u - u^*\|_{(2,1)} < \delta$; hence we have $\|u - u^*\|_\infty < \delta$, $\left\|\frac{\partial u}{\partial t} - \frac{\partial u^*}{\partial t}\right\|_\infty < \delta$, $\left\|\frac{\partial u}{\partial x} - \frac{\partial u^*}{\partial x}\right\|_\infty < \delta$, $\|u_{xx} - u^*_{xx}\|_\infty < \delta$ and according to (17),(18) and (19)

$$\left\| \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^\alpha u^*(x,t)}{\partial t^\alpha} \right\|_\infty \leq \frac{1}{\Gamma(2-\alpha)} \left\| \frac{\partial u}{\partial t} - \frac{\partial u^*}{\partial t} \right\|_\infty < \frac{\delta}{\Gamma(2-\alpha)},$$

$$\left\| \frac{\partial^\beta u(x,t)}{\partial x^\beta} - \frac{\partial^\beta u^*(x,t)}{\partial x^\beta} \right\|_\infty \leq \frac{1}{\Gamma(2-\beta)} \left\| \frac{\partial u}{\partial x} - \frac{\partial u^*}{\partial x} \right\|_\infty < \frac{\delta}{\Gamma(2-\beta)},$$

$$\left\| \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} - \frac{\partial^{2\beta} u^*(x,t)}{\partial x^{2\beta}} \right\|_\infty \leq \frac{1}{\Gamma(3-2\beta)} \|u_{xx} - u^*_{xx}\|_\infty < \frac{\delta}{\Gamma(3-2\beta)}.$$

So for small enough value of δ we have

$$Y = \left(x, t, u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial^\beta u}{\partial x^\beta}, \frac{\partial^{2\beta} u}{\partial x^{2\beta}}\right) \in I,$$

since F is continuous on I with respect to all its arguments and I is a compact set, according to Theorem 2, F is uniformly continuous on I. So if $\delta > 0$ be sufficiently small, then $\|Y - Y^*\| < \delta$ implies that$|F(Y) - F(Y^*)| < \epsilon$ and

$$|J(u(x,t)) - J(u^*(x,t))| < \epsilon. \qquad \square$$

Now we can show the convergence of the approximating method.

Theorem 3 *:Let $\eta_{kk'}$ be the minimum of the functional J on $G_{kk'}(U)$, then we have*

$$lim_{k,k' \to \infty} \eta_{kk'} = 0.$$

Proof: For any given $\epsilon > 0$, let $u^* \in E(U)$ such that $J(u^*) < \epsilon$ (such u^* exist by the properties of minimum). According to Lemma 2, J is continuous on $(C^{(2,1)}(U), \|.\|_{(2,1)})$ so we have

$$|J(u) - J(u^*)| < \epsilon, \tag{20}$$

provided that $\|u - u^*\| < \delta$. According to Lemma 1 for large enough value of k, k' there exist $\chi_{kk'} \in G_{kk'}(U)$ such that $\|\chi_{kk'} - u^*\|_{(2,1)} < \delta$. Moreover let $u_{kk'}$ be the element of $G_{kk'}(U)$ such that $J[u_{kk'}] = \eta_{kk'}$, then using (20) we have

$$0 \le J(u_{kk'}) \le J(\chi_{kk'}) < 2\epsilon.$$

Since the $\epsilon > 0$ is arbitrary, it follows that

$$lim_{k,k' \to \infty} \eta_{kk'} = lim_{k,k' \to \infty} J(u_{kk'}) = 0. \qquad \square$$

4 Illustrative examples

To demonstrate the effectiveness of the method, here we consider some fractional differential equations.

Example 1

Consider the linear time–fractional FPE

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < \alpha \le 1,\ 0 < x < 1,\ 0 < t < 1, \tag{21}$$

with initial conditions: $u(x,0) = x$.
For $\alpha = 1$, the exact solution is

$$u(x,t) = x + t. \tag{22}$$

A Ritz approximation to (21) is constructed as follows. The approximation $u_{kk'}(x,t)$ is sought in the form of the truncated series

$$u_{kk'}(x,t) = \sum_{i=0}^{k}\sum_{j=0}^{k'} t\phi_i(x)\phi_j(t) + x,$$

where $\phi_i(x)$ and $\phi_j(t)$ are Legendre Polynomials. We obtain

$$\begin{aligned} u(x,t) &= \left(t + 1.16\times10^{-15}t^2 - 4.996\times10^{-16}t^3 + ...\right) \\ &+ \left(1 + 1.221\times10^{-15}t - 1.776\times10^{-15}t^2 + 6.661\times10^{-16}t^3 + ...\right)x + ... \end{aligned}$$

that converge to the exact solution.
Fig. 1 shows the numerical solutions of this problem obtained by the present method with $k = 1, k' = 2$. In Fig. 1, we can see that the present method provides accurate results.

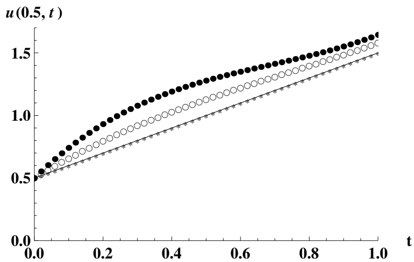

Fig.1. Exact(—) for $\alpha = 1$ and approximate solution $u(0.5,t)$ for $(\bullet\bullet\bullet)\alpha = 0.6$,$(ooo)\alpha = 0.8$,$(***)\alpha = 1$.

The following table shows the values of minimum $\eta_{kk'}$ for different values of approximations.

	$k=1, k'=1$	$k=1, k'=2$	$k=1, k'=3$
$\alpha = 1, \eta_{kk'}$	1.91×10^{-16}	-1.29×10^{-16}	-7.111×10^{-17}
$\alpha = 0.6, \eta_{kk'}$	0.024	0.012	0.007
$\alpha = 0.8, \eta_{kk'}$	0.0072	0.0036	0.0021

Example 2

Consider the linear space fractional FPE [15]

$$\frac{\partial u(x,t)}{\partial t} = [-\frac{\partial^\beta}{\partial x^\beta}x + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\frac{x^2}{2}]u(x,t), \qquad 0<\beta\le1,\ 0<x<1,\ 0<t<1, \tag{23}$$

with initial condition: $u(x,0) = x$.
For $\beta = 1$, the exact solution for this problem be

$$u(x,t) = xe^t. \tag{24}$$

We applied the method presented for different values of β and solved Equation(23). We determine

$$u_{kk'}(x,t) = \sum_{i=0}^{k}\sum_{j=0}^{k'} t\phi_i(x)\phi_j(t) + x,$$

where $\phi_i(x)$ and $\phi_j(t)$ are Legendre Polynomials. Fig. 2 shows the absolute error of this problem obtained by the present method with $k = 1, k' = 3$. Fig. 3 represents the approximate solutions of $u(0.5,t)$ for $\beta = 0.6, 0.8, 1$ with $k = 1, k' = 3$ in comparison with the exact solution $u(0.5,t)$. Numerical results are presented to demonstrate the effectiveness of the proposed method.

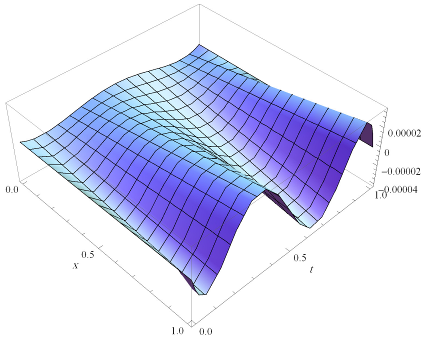

Fig.2. The absolute Error between exact and numerical solution for $\beta = 1, k = 1, k' = 3$.

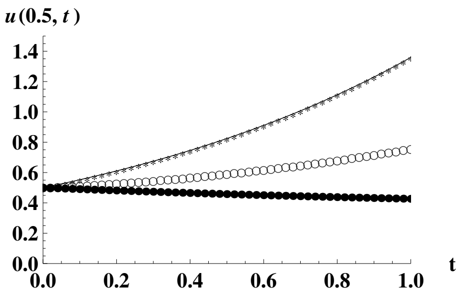

Fig.3. Exact(—) for $\beta = 1$ and approximate solution $u(0.5,t)$ for $(\bullet\bullet\bullet)\beta = 0.6$,$(ooo)\beta = 0.8$,$(***)\beta = 1$ with $k = 1, k' = 3$.

The following table shows the values of minimum $\eta_{kk'}$ for different values of approximations.

	$k=1, k'=1$	$k=1, k'=2$	$k=1, k'=3$
$\beta=1, \eta_{kk'}$	0.0012	9.18×10^{-6}	3.63×10^{-8}
$\beta=0.6, \eta_{kk'}$	0.0035	.00035	0.000131
$\beta=0.8, \eta_{kk'}$	0.00524	0.00522	0.00013

Example 3

Next, we consider the linear space- and time-fractional FPE [15]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = [-\frac{\partial^\beta}{\partial x^\beta}(\frac{x}{6}) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}(\frac{x^2}{12})]u(x,t), \qquad 0<\alpha,\ \beta\le 1,\ 0<x<1,\ 0<t<1, \quad (25)$$

with initial condition: $u(x,0)=x^2$.

For $\alpha=1, \beta=1$ the exact solution is:

$$u(x,t) = x^2 e^{\frac{t}{2}}. \qquad (26)$$

We utilized the method presented for different values of α, β and solved Equation(25). We consider

$$u_{kk'}(x,t) = \sum_{i=0}^{k}\sum_{j=0}^{k'} t\phi_i(x)\phi_j(t) + x^2$$

where $\phi_i(x) = x^i, \phi_j(t) = t^j$. Fig. 4 shows the absolute error of this problem obtained by the present method with $\alpha=1, \beta=1$ and $k=2, k'=2$. Fig. 5 represents the approximate solutions of $u(0.5,t)$ with $k=2, k'=2$ and for $\alpha, \beta = 0.6, 0.8, 1$ in comparison with the exact solution $u(0.5,t)$. Numerical results are presented to demonstrate the effectiveness of the proposed method.

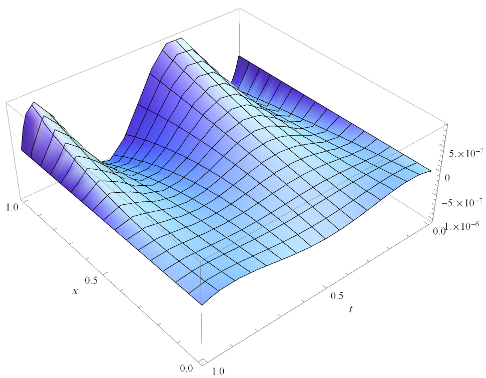

Fig.4. The absolute Error between exact and numerical solution for $\alpha=1, \beta=1$ and $k=2, k'=3$.

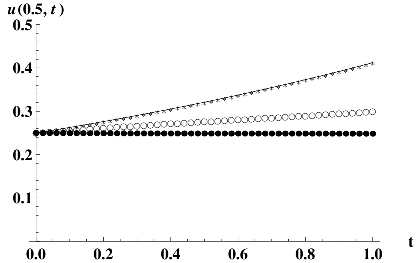

Fig.5. Exact(—) for $\alpha, \beta = 1$ and approximate solution $u(0.5, t)$ for $(\bullet\bullet\bullet)\alpha, \beta = 0.6$,$(ooo)\alpha, \beta = 0.8$,$(***)\alpha, \beta = 1$ with $k = 2, k' = 2$.

The following table shows the values of minimum $\eta_{kk'}$ for different values of approximations.

	$k = 1, k' = 1$	$k = 1, k' = 2$	$k = 2, k' = 2$
$\alpha = 1, \beta = 1, \eta_{kk'}$	0.00138	0.00138	1.26×10^{-8}
$\alpha = 0.6, \beta = 0.6, \eta_{kk'}$	0.00024	0.00023	0.00003
$\alpha = 0.8, \beta = 0.8, \eta_{kk'}$	0.0009	0.0008	0.00005

Example 4

Consider the nonlinear time–fractional FPE [15]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = [-\frac{\partial}{\partial x}(\frac{4u}{x} - \frac{x}{3}) + \frac{\partial^2}{\partial x^2}u]u(x,t), \qquad 0 < \alpha \leq 1,\ 0 < x < 1,\ 0 < t < 1, \quad (27)$$

with initial conditions: $u(x, 0) = x^2$.

For $\alpha = 1$, the exact solution is

$$u(x,t) = x^2 e^t. \quad (28)$$

We applied the method presented for different values of α and solved Equation(27). The approximation $u_{kk'}(x, t)$ is sought in the form of the truncated series

$$u_{kk'}(x,t) = \sum_{i=0}^{k}\sum_{j=0}^{k'} tx\phi_i(x)\phi_j(t) + x^2,$$

where $\phi_i(x)$ and $\phi_j(t)$ are Legendre Polynomials. Fig. 6 shows the absolute error of this problem obtained by the present method with $k = 1, k' = 4$. From Fig. 6, we can see that the present method provides accurate results. In order to show the accuracy of method, we compare numerical solutions for $\alpha = 0.6, 0.8, 1$ and $x = 0.5$. The numerical results are shown in Fig. 7.

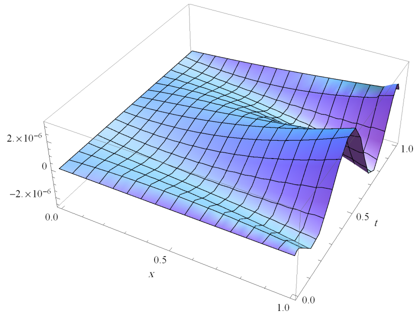

Fig.6. The absolute Error between exact and numerical solution for $\alpha = 1, k = 1, k' = 4$.

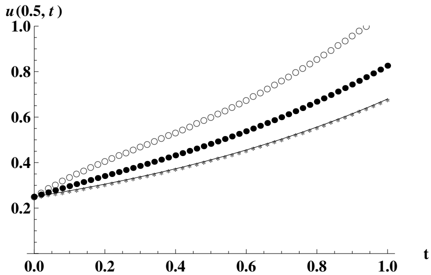

Fig.7. Exact(—) for $\alpha = 1$ and approximate solution $u(0.5, t)$ for $(ooo)\alpha = 0.6$,$(\bullet\bullet\bullet)\alpha = 0.8$,$(***)\alpha = 1$ with $k = 1, k = 3$.

The following table shows the values of minimum $\eta_{kk'}$ for different values of approximations.

	$k = 1, k' = 1$	$k = 1, k' = 2$	$k = 1, k' = 3$
$\alpha = 1, \eta_{kk'}$	0.000154	1.379×10^{-6}	6.96×10^{-9}
$\alpha = 0.6, \eta_{kk'}$	0.0028	0.0015	0.00102
$\alpha = 0.8, \eta_{kk'}$	0.001	0.0004	0.0003

Example 5

Consider the nonlinear time–fractional FPE

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = [-\frac{\partial}{\partial x}(\frac{7u}{2}) + \frac{\partial^2}{\partial x}u]u(x,t), \qquad 0 < \alpha \leq 1,\ 0 < x < 1,\ 0 < t < 1, \tag{29}$$

with initial conditions: $u(x, 0) = x$.

For $\alpha = 1$, the exact solution is

$$u(x,t) = \frac{x}{t+1}. \tag{30}$$

We applied the present method for different values of α and solved Equation(29). we consider

$$u_{kk'}(x,t) = \sum_{i=0}^{k}\sum_{i=0}^{k'} t\phi_i(x)\phi_j(t) + x,$$

where $\phi_i(x)$ and $\phi_j(t)$ are Legendre Polynomials. Fig. 8 shows the absolute error of this problem obtained by the present method with $k=1, k'=4$. From Fig. 8, we can see that the present method provides accurate results. In order to show the accuracy of method, we compare numerical solutions for $\alpha = 0.6, 0.8, 1$ and $x = 0.5$ in Fig. 9.

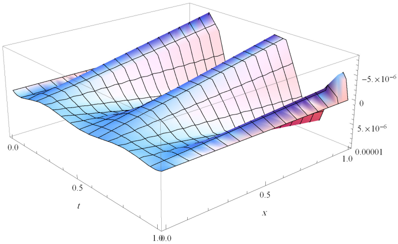

Fig.8. The absolute Error between exact and numerical solution for $\alpha = 1, k = 1, k' = 4$.

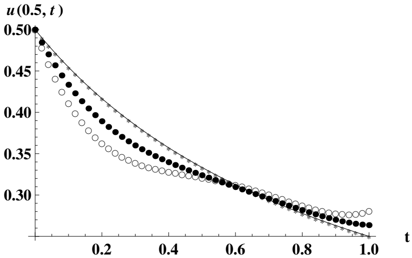

Fig.9. Exact(—) for $\alpha = 1$ and approximate solution $u(0.5, t)$ for $(ooo)\alpha = 0.6$,$(\bullet\bullet\bullet)\alpha = 0.8$,$(***)\alpha = 1$ with $k = 1, k' = 3$.

The following table shows the values of minimum $\eta_{kk'}$ for different values of approximations.

	$k=1, k'=1$	$k=1, k'=2$	$k=1, k'=3$
$\alpha = 1, \eta_{kk'}$	0.001	0.00005	2.4×10^{-6}
$\alpha = 0.6, \eta_{kk'}$	0.015	0.007	0.004
$\alpha = 0.8, \eta_{kk'}$	0.007	0.002	0.001

5 Conclusion

This paper presents a simple and effective approach to solve Fokker-Planck equation with space- and time–fractional derivatives. The desired approximate solution can be determined by solving the resulting system of equations, which can be effectively computed using symbolic computing codes on any personal computer. Illustrative examples show that this method has high accuracy and is easily implemented.

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