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Relation between Energy and (Signless) Laplacian Energy of Graphs

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Abstract

Let $G = (V, E)$ be a simple graph of order n with m edges. Also let $E(G)$, $LE(G)$ and $LE^+(G)$ be energy, Laplacian energy and signless Laplacian energy of graph G, respectively. In this paper, we obtain a relation between $E(G)$, $LE(G)$ and $LE^+(G)$ of graph G. Moreover, we present an upper bound on $E(G)$ of graph G in terms of m and rank r.

1 Introduction

Let $G = (V, E)$ be a graph of order n with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, $|E(G)| = m$. Let d_i be the degree of the vertex v_i for $i = 1, 2, \ldots, n$. The diameter of a graph is the maximum distance between any two vertices of G. Let d be the diameter of G.

Let $\mathbf{A}(G) = (a_{ij})$ be the $(0, 1)$-adjacency matrix of G such that $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Let $\mathbf{D}(G)$ be the diagonal matrix of order n whose (i, i)-entry is the degree d_i of the vertex $v_i \in V(G)$. Then the matrices $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$ and $\mathbf{L}^+(G) = \mathbf{D}(G) + \mathbf{A}(G)$ are the Laplacian and the signless Laplacian matrices, respectively.

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the adjacency matrix $A(G)$ of the graph G are also called the eigenvalues of G. The energy of the graph G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \,. \tag{1}$$

Details on the properties of graph energy can be found in [6, 7, 10, 11, 13, 14, 15, 16].

Let $\mu_1, \mu_2, \ldots, \mu_n$ and $q_1, q_2, \ldots, q_n$ be the eigenvalues of the matrices $L(G)$ and $L^+(G)$, respectively. Then the Laplacian energy of the graph G is defined as [12]

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| . \tag{2}$$

and the signless Laplacian energy of the graph G is defined as [1]

$$LE^+ = LE^+(G) = \sum_{i=1}^{n} \left| q_i - \frac{2m}{n} \right| . \tag{3}$$

Details on the properties of Laplacian energy and signless Laplacian energy can be found in [5, 8, 12, 17, 19, 21, 22] and [1], respectively.

As usual, K_n and S_n denote respectively the complete graph and the star on n vertices. Denote by $K_{p,q}$ $(p+q=n)$ a complete bipartite graph of order n. For $n \geq 3$, let S_n^+ be an unicyclic graph of order n obtained by adding an edge to S_n .The rank of the adjacency matrix of graph G is denoted by $r = \text{rank}(A)$ and is defined as the number of nonzero eigenvalues of G.

This paper is organized as follows. In Section 2, we give a list of some necessary lemmas and known results. In Section 3, we give a relation between $LE(G)$, $LE^+(G)$ and $E(G)$. In Section 4, we obtain an upper bound on $E(G)$ of graph G.

2 Preliminaries

In this section, we shall list some previously known results that will be needed in the next two sections.

Lemma 2.1. [18] *Let A and B be $n \times n$ Hermitian matrices and $C = A + B$. Then*

$$\lambda_i(C) \leq \lambda_j(A) + \lambda_{i-j+1}(B) \quad (n \geq i \geq j \geq 1),$$
$$\lambda_i(C) \geq \lambda_j(A) + \lambda_{i-j+n}(B) \quad (n \geq j \geq i \geq 1),$$

where $\lambda_i(M)$ is the ith largest eigenvalue of matrix M ($M = A, B, C$).

The following result is obtained from [8]:

Lemma 2.2. [8] *Let G be a graph of order n with m edges. Then*

$$LE(G) = \max_{1\le k\le n} \left\{ 2\mathcal{S}_k(G) - \frac{4m\,k}{n} \right\}, \tag{4}$$

where

$$\mathcal{S}_k(G) = \sum_{i=1}^{k} \mu_i.$$

Similarly Lemma 2.2, from (3), we can get the following result for signless Laplacian energy.

Lemma 2.3. *Let G be a graph of order n with m edges. Then*

$$LE^+(G) = \max_{1\le k\le n} \left\{ 2\mathcal{S}_k^+(G) - \frac{4m\,k}{n} \right\}, \tag{5}$$

where

$$\mathcal{S}_k^+(G) = \sum_{i=1}^{k} q_i.$$

Lemma 2.4. [4] *A graph G is bipartite if and only if $\lambda_1 = -\lambda_n$.*

Lemma 2.5. [9] *Let G be a connected graph. Then $q_1(G) \ge 2\,\lambda_1(G)$ with equality holding if and only if G is regular.*

3 Relation between $LE(G)$, $LE^+(G)$ and $E(G)$

In this section we give a relation between $E(G)$, $LE(G)$ and $LE^+(G)$ of graph G. Recently, Abreu et al. [1] obtained the following relation between $E(G)$, $LE(G)$ and $LE^+(G)$ of graph G:

$$LE^+(G) + LE(G) \ge \max\left\{ 2E(G),\ 2\sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right| \right\}. \tag{6}$$

Theorem 3.1. *Let G be a graph of order n with m edges and rank r. Then*

$$LE^+(G) + LE(G) \ge 4\,E(G) - \frac{4m\,r}{n} \tag{7}$$

with equality holding if and only if $G \cong n\,K_1$ or $G \cong K_2 \cup (n-2)\,K_1$ or $G \cong K_{n/2,\,n/2}$.

Proof: If $G \cong n\,K_1$, then the equality holds in (7). Otherwise, $m \geq 1$. Since $\mathbf{L}^+(G) = \mathbf{L}(G) + 2\,\mathbf{A}(G)$ is a positive semi definite matrix, by Lemma 2.1, we get

$$\mu_i \geq -2\,\lambda_{n-i+1} \quad (1 \leq i \leq n) \tag{8}$$

$$\text{and} \qquad q_i \geq 2\lambda_i \quad (1 \leq i \leq n). \tag{9}$$

Let ν^+ and ν^- be the number of positive and negative eigenvalues of graph G, respectively. Since the rank of the adjacency matrix is r, we have $r = \nu^+ + \nu^-$. From (1),

$$E(G) = \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{\nu^+} \lambda_i - \sum_{i=1}^{\nu^-} \lambda_{n-i+1}\,. \tag{10}$$

By Lemma 2.2, we have

$$\begin{aligned} LE(G) &= \max_{1\leq k\leq n} \left(2\sum_{i=1}^{k} \mu_i - \frac{4m\,k}{n} \right) \\ &\geq 2\sum_{i=1}^{\nu^-} \mu_i - \frac{4m\,\nu^-}{n} \\ &\geq -4\sum_{i=1}^{\nu^-} \lambda_{n-i+1} - \frac{4m\,\nu^-}{n} \qquad \text{by (8).} \end{aligned} \tag{11}$$

Similarly, from Lemma 2.3, we get

$$\begin{aligned} LE^+(G) &= \max_{1\leq k\leq n} \left(2\sum_{i=1}^{k} q_i - \frac{4m\,k}{n} \right) \\ &\geq 4\sum_{i=1}^{\nu^+} \lambda_i - \frac{4m\,\nu^+}{n} \qquad \text{by (9).} \end{aligned}$$

From the above two results, we get

$$LE(G) + LE^+(G) \geq 4\left(\sum_{i=1}^{\nu^+} \lambda_i - \sum_{i=1}^{\nu^-} \lambda_{n-i+1} \right) - \frac{4m}{n}\,(\nu^+ + \nu^-)\,.$$

Since $r = \nu^+ + \nu^-$, using (10), from the above result, we get the required result in (7). First part of the proof is done.

Now suppose that the equality holds in (7). Then all the inequalities in the above must be equalities. Thus we have $\mu_i = -2\,\lambda_{n-i+1}$ $(1 \leq i \leq n)$ and $q_i = 2\lambda_i$ $(1 \leq i \leq n)$. First we assume that G is disconnected. Then $\mu_{n-1} = 0$ and hence $\lambda_2 = 0$, that is, $q_2 = 0$. Hence $G \cong K_2 \cup (n-2)\,K_1$ as $m \geq 1$.

Next we assume that G is connected. Since $m \geq 1$, we have ν^+, $\nu^- \geq 1$. Therefore $q_1 = 2\lambda_1$ and hence G is regular graph, by Lemma 2.5. For d-regular graph, we have $\mu_i = d - \lambda_{n-i+1}$ $(1 \leq i \leq n)$. Moreover, $\mu_1 = -2\,\lambda_n$ and hence $\lambda_1 = d = -\lambda_n$, that is, G is bipartite graph, by Lemma 2.4. If $G \cong K_{n/2,\,n/2}$, then $LE^+(G) = LE(G) = n$, $E(G) = n$, $r = 2$ and hence the equality holds in (7). Otherwise, G has diameter at least 3 as G is bipartite regular. Since $d < k$ (k is the number of distinct eigenvalues in G), then G has at least four different eigenvalues in G and hence $\nu^+ \geq 2$. Therefore we have $q_2 = 2\lambda_2 = \mu_2 = d + \lambda_2$, that is, $d = \lambda_2 < \lambda_1 = d$, a contradiction. ■

Remark 3.2. *Two results* (6) *and* (7) *are incomparable. Sometimes our result in* (7) *is better than the result in* (6)*, but not always. For this we denote a graph H obtained from an isolated vertex joining by two edges to the centers of two stars S_3 and S_4, respectively. From the Table 1, one can easily check that our result in* (7) *is better than the result in* (6) *for graphs H and $K_{3,5}$, on the other hand the result in* (6) *is better than our result in* (7) *for graphs S_8 and S_8^+.*

Table 1.

G	(7)	(6)
H	15.538	15
$K_{3,5}$	15.984	15.492
S_8	14.166	21
S_8^+	12.293	20

4 Upper bound on $E(G)$

Brouwer [3] has conjectured the following:

Conjecture 4.1. *Let G be a graph with n vertices. Then $S_k(G) = \sum_{i=1}^{k} \mu_i \leq m + \binom{k+1}{2}$ for $k = 1, 2, \ldots, n$.*

The conjecture is known to be true for (i) $k = 1$, $k = 2$, $k = n - 1$ and $k = n$; (ii) trees; (iii) unicyclic and bicyclic graphs; (iv) regular graphs; (v) split graphs (graphs whose vertex set can be partitioned into a clique and an independent set); (vi) cographs (graphs with no path on 4 vertices as an induced subgraph); (vii) graphs with at most 10 vertices.

In [2], Ashraf et al. gave the following conjecture:

Conjecture 4.2. *Let G be a graph with n vertices. Then $S_k^+(G) = \sum_{i=1}^{k} q_i \le m + \binom{k+1}{2}$ for $k = 1, 2, \ldots, n$.*

Moreover, they mentioned that Conjecture 4.2 is true for $k = 1,\ 2,\ n-1,\ n$. It is also true for trees and regular graphs. Yang et al. [20] show that Conjecture 4.2 is true for unicyclic and bicyclic graphs. Let $G(n, m)$ be a set of graphs of order n and m edges. Now we define

$$\Gamma_1 = \left\{ G : G \in G(n,\, m) \quad \text{and} \quad \sum_{i=1}^{k} \mu_i(G) \le m + \frac{k^2+k}{2}\ ,\ \ 1 \le k \le n \right\}$$

and

$$\Gamma_2 = \left\{ G : G \in G(n,\, m) \quad \text{and} \quad \sum_{i=1}^{k} q_i(G) \le m + \frac{k^2+k}{2}\ ,\ \ 1 \le k \le n \right\}.$$

From the above we conclude that tree, unicyclic, bicyclic and regular graphs are belongs to $\Gamma_1 \cap \Gamma_2$. We now obtain the following result.

Theorem 4.3. *Let G be a graph of order n with m (> 0) edges. If $G \in \Gamma_1 \cap \Gamma_2$, then*

$$E(G) \le m + \frac{r^2+r}{4} - 1\,, \tag{12}$$

where $r = rank(A)$.

Proof: Since $G \in \Gamma_1 \cap \Gamma_2$, we have $G \in \Gamma_1$ and $G \in \Gamma_2$. From (8) and (9), we get

$$\sum_{i=1}^{k} \mu_i \ge -2 \sum_{i=1}^{k} \lambda_{n-i+1} \quad \text{for} \quad 1 \le k \le n \tag{13}$$

and

$$\sum_{i=1}^{k} q_i \ge 2 \sum_{i=1}^{k} \lambda_i \quad \text{for} \quad 1 \le k \le n. \tag{14}$$

Let ν^- and ν^+ be the negative and positive eigenvalues of G, respectively. Since $G \in \Gamma_1$ and setting $k = \nu^-$ in (13), we get

$$E(G) = -2 \sum_{i=1}^{\nu^-} \lambda_{n-i+1} \le \sum_{i=1}^{\nu^-} \mu_i \le m + \frac{{\nu^-}^2 + \nu^-}{2}\,.$$

Since $G \in \Gamma_2$ and setting $k = \nu^+$ in (14), we get

$$E(G) = 2\sum_{i=1}^{\nu^+} \lambda_i \leq \sum_{i=1}^{\nu^+} q_i \leq m + \frac{{\nu^+}^2 + \nu^+}{2}\,.$$

From the above two results, we get

$$2E(G) \leq 2m + \frac{{\nu^-}^2 + {\nu^+}^2}{2} + \frac{\nu^- + \nu^+}{2}\,.$$

It is well known that $\nu^- + \nu^+ = r = rank(A)$. Using this result, from the above, we get

$$2E(G) \leq 2m + \frac{r^2 + r}{2} - \nu^-\nu^+\,.$$

Since $m > 0$, we have $\nu^-\nu^+ \geq 1$. From the above, we get the required result in (12). ■

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