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Clar Sets and Maximum Forcing Numbers of Hexagonal Systems *

Xiangqian Zhou, Heping Zhang†

School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, P. R. China

zhouxiangqian0502@126.com, zhanghp@lzu.edu.cn

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Abstract

Let H be a hexagonal system with a perfect matching. Xu et al. discovered that the maximum forcing number of H equals its Clar number. In this article we obtain a result: for any resonant set K of a peri-condensed hexagonal system H consisting of disjoint hexagons not meeting the boundary of H, if the subgraph obtained from H by deleting K and the boundary of H has a perfect matching or is empty, then the Clar number of H is at least $|K| + 2$. This fact improves the previous corresponding result due to Zheng and Chen. Based on the result, we prove that for each perfect matching M of H with the maximum forcing number, there exists a Clar set consisting of disjoint M-alternating hexagons of H.

1 Introduction

A *hexagonal system*, also called benzenoid system, is a 2-connected finite plane graph whose every interior face is bounded by a regular hexagon of side length one [17]. It can also be formed by a cycle with its interior in the infinite hexagonal lattice on the plane (graphene) [4]. A *perfect matching* of a hexagonal system H is a set of disjoint edges covering all vertices of H. This concept coincides with that of a Kekulé structure in organic chemistry. Since a hexagonal system with at least one perfect matching may be viewed as the carbon-skeleton of a benzenoid hydrocarbon molecule, various topological

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†Corresponding author.

properties of hexagonal systems have been extensively studied. The interested reader may refer to [2–4] with references therein.

A basic concept of Clar's aromatic sextet theory is that of Clar number, which can measure the stability of polycyclic benzenoid hydrocarbons [2]. According to Clar's theory, within a series of isometric benzenoid hydrocarbons, the one with larger Clar number is more stable [2].

Let M be a perfect matching of a graph G. A cycle C (resp. a path P) of G is said to be *M-alternating* if the edges of C (resp. P) appear alternately in and off M. For a subgraph F of G, let $G-F$ denote the graph obtained from G by deleting all the vertices of F together with their incident edges.

Let H be a hexagonal system with a perfect matching. A set K of disjoint hexagons of H is called a *resonant set* (or *cover*) if there is a perfect matching M of H such that all hexagons in K are M-alternating. It is obvious that K is a resonant set of H if and only if $H-K$ either has a perfect matching or is an empty graph. In particular, $\emptyset$ is regarded as a perfect matching of an empty graph (no vertices). A resonant set is *maximum* if its cardinality is maximum. A maximum resonant set of H is also called a *Clar set* or *Clar formula*, and its size is called the *Clar number* of H, denoted by $Cl(H)$. For the relevant researches on the Clar number and Clar formula, please see [5, 7, 9, 12, 18, 19, 22].

In 1985, Zheng and Chen [26] gave an important property for a maximum resonant set of a hexagonal system as follows.

Theorem 1.1. *[26] Let H be a hexagonal system and K a maximum resonant set of H. Then $H-K$ has a unique perfect matching.*

The proof of Theorem 1.1 is based on the following Lemma 1.2.

Lemma 1.2. *[26] Let H be a peri-condensed hexagonal system, K a resonant set of internal hexagons and $\partial(H)$ the boundary of the exterior face of H. If $H-K-\partial(H)$ has a perfect matching, then K is not a maximum resonant set.*

A hexagonal system H is said to be *fully benzenoid* if a maximum resonant set of H includes all vertices. Gutman and Salem showed [6] that a fully benzenoid has a unique maximum resonant set.

The *innate degree of freedom* of a Kekulé structure was defined by Randić and Klein [13] as the minimum number of double bonds which simultaneously belong to the given kekulé structure and to no other one, nowadays it is named "forcing number" by Harary et al. [10].

Let M be a perfect matching of a graph G. A *forcing set* S of M is a subset of M such that S is contained in no other perfect matchings of G. The *forcing number* of M, denoted by $f(G,M)$, is the smallest cardinality over all forcing sets of M. The *maximum* (resp. *minimum*) *forcing number* of G is the maximum (resp. minimum) value of forcing numbers of all perfect matchings of G, denoted by $F(G)$ (resp. $f(G)$). For the relevant researches on the matching forcing problem, we refer to [1, 11, 23–25].

For planar bipartite graphs, Pachter and Kim revealed a minimax fact that connects the forcing number of a perfect matching and its alternating cycles as follows.

Theorem 1.3. *[16] Let M be a perfect matching of a plane bipartite graph G. Then $f(G,M) = c(M)$, where $c(M)$ is the maximum number of disjoint M-alternating cycles of G.*

For a hexagonal system H with a perfect matching M, let $h(M)$ denote the maximum number of disjoint M-alternating hexagons of H. Theorem 1.3 implies $f(H,M) = c(M) \geq h(M)$. Second equality does not hold alway. Let us see an example in Fig. 1. The bold edges of Coronene form a perfect matching M' whose forcing number equals 2, but the graph has only one M'-alternating hexagon. However, Xu et al. [20] obtained the following result by finding a perfect matching M of H so that $F(H) = f(H,M) = h(M)$.

Theorem 1.4. *[20] Let H be a hexagonal system with perfect matchings. Then $Cl(H) = F(H)$.*

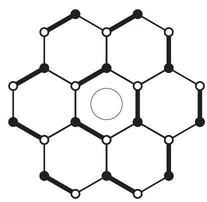

Figure 1: Coronene.

In this article, we show that for every perfect matching M of a hexagonal system H with the maximum forcing number, i.e. $F(H) = f(H,M)$, there exist $F(H)$ disjoint M-alternating hexagons in H. That is, $f(H,M) = h(M)$. To prove this, we mainly improve

Lemma 1.2 to obtain Lemma 2.1 in Section 2. Based on this crucial lemma, in Section 3 we describe clearly structure properties for a maximum set of disjoint M-alternating cycles of H for any perfect matchings M with the maximum forcing number, then we give a proof of this main result.

2 A crucial lemma

In this section, all hexagonal systems considered are placed in the plane so that an edge-direction is vertical. A *peak* (resp. *valley*) of a hexagonal system is a vertex whose neighbors are below (resp. above) it. For convenience, the vertices of a hexagonal system are colored with white and black such that any pair of adjacent vertices receive different colors and the *peaks* are black.

Let H be a hexagonal system. The boundary of H means the boundary of the infinite face, denoted by $\partial(H)$. An edge on the boundary of H is a *boundary edge*. A hexagon of H is called an *external hexagon* if it contains a boundary edge and an *internal hexagon* otherwise. H is said to be *cata-condensed* if all vertices lie on its boundary and *peri-condensed* otherwise.

We state a crucial lemma as follows.

Lemma 2.1. *Let H be a peri-condensed hexagonal system and K a resonant set consisting of internal hexagons of H. Suppose $H - K - \partial(H)$ has a perfect matching or is an empty graph. Then $Cl(H) \geq |K| + 2$.*

In order to prove the lemma, we need some further terminology and a known result.

Let M be a perfect matching of a hexagonal system H. An edge of H is called an *M-double edge* if it belongs to M and an *M-single edge* otherwise. An M-alternating cycle C of H is said to be *proper* if each edge of C in M goes from white end-vertex to black end-vertex along the clockwise direction of C.

The symmetric difference of two finite sets A and B is defined as $A \oplus B := (A \cup B) - (A \cap B)$. Given a perfect matching M of a hexagonal system H. If C is an M-alternating cycle (or hexagon) of H, then the symmetric difference $M \oplus C$ is another perfect matching of H and C is an $(M \oplus C)$-alternating cycle of H. Here C may be viewed as its edge-set. Let P be a set of some hexagons of H and let F be a subgraph of H. The set of the common hexagons of P and F is denoted by $P \cap F$.

A hexagonal system H is called a *linear chain* if the centers of all hexagons lie on a straight line. Zhang et al. obtained the following result in [21, Theorem 4].

Theorem 2.2. *[21] A hexagonal system H has $Cl(H) = 1$ if and only if H is a linear chain.*

For a cycle C of a hexagonal system H, let $I[C]$ denote the subgraph of H formed by C together with its interior.

Proof of Lemma 2.1. By the assumption, we can choose a cycle C of H satisfying that

(i) the graph $I[C]$ is a peri-condensed hexagonal system, and

(ii) C is disjoint with each member of K and $H - K - C$ has a perfect matching,

and $I[C]$ contains as few hexagons as possible subject to (i) and (ii). Set $H' := I[C]$ and $K' := K \cap H'$.

Claim 1. For any resonant set K_0 of H', $K_0 \cup (K \backslash K')$ is a resonant set of H.

Proof. Since $H - K - C$ has a perfect matching, H has a perfect matching M_0 such that each member in $K \cup \{C\}$ is M_0-alternating. So the restriction of M_0 on $H - H'$ is a perfect matching of $H - H'$, denoted by M_c. Let M_0' be a perfect matching of H' such that each member in K_0 is M_0'-alternating. Let $M' := M_0' \cup M_c$. Then M' is a perfect matching of H such that each member in $K_0 \cup (K \backslash K')$ is an M'-alternating hexagon. ■

From Claim 1 it suffices to prove that $Cl(H') \geq |K'| + 2$. If $K' = \emptyset$, by Theorem 2.2 we have that $Cl(H') \geq |K'| + 2$. From now on suppose that $K' \neq \emptyset$. Without loss of generality, let M be a perfect matching of H' such that the boundary C of H' and each member in K' are proper M-alternating cycles. We have the following claim.

Claim 2. H' has no external hexagons that are proper M-alternating.

Proof. Suppose to the contrary that an external hexagon h of H' is proper M-alternating. Then $M \oplus h$ is a perfect matching of H', and each component of $C \oplus h$ is a proper $(M \oplus h)$-alternating cycle. Since any two proper M-alternating hexagons of H' are disjoint, h is disjoint with each member of K'. Since $K' \neq \emptyset$, $C \oplus h$ has a component as a cycle C' which satisfies the above conditions (i) and (ii). But $I[C']$ has fewer hexagons than $I[C]$, contradicting the choice for C. Hence Claim 2 holds. ■

Along the boundary C of H', we will find two substructures of H' in its left-top corner and left-bottom corner as Figs. 3 and 4, respectively, as follows.

A *b-chain* of hexagonal system H' is a maximal horizontal linear chain consisting of the consecutive external hexagons when traversing (counter)clockwise the boundary $\partial(H')$. A b-chain is called *high* (resp. *low*) if all hexagons adjacent to it are below (resp. above) it. For example, in Fig. 2 D_0, D_1, D_2, $G_1, G_2, \ldots, G_9, G'_1, D_5, D_6$ and D_7 are b-chains. In particular, D_0, D_1, D_2 and G_1 are high b-chains, while G'_1, D_5 and D_6 are low b-chains. But $G_2, G_3, \ldots, G_9$ and D_7 are neither high nor low b-chains.

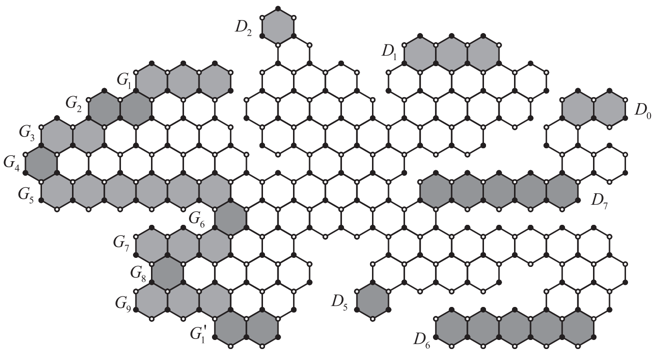

Figure 2: Various b-chains of a hexagonal system.

Choose a high b-chain and a low b-chain of H'. They are distinct. Otherwise H' itself is a linear chain, contradicting that H' is peri-condensed. From the high b-chain to the low b-chain along the boundary $\partial(H')$ counterclockwise, we pass through a sequence of consecutive b-chains. In this process, let G_1 be the last high b-chain and let G'_1 be the first low b-chain after G_1. Clearly, there is no other high b-chain and low b-chain between G_1 and G'_1. That is, those b-chains between G_1 and G'_1 descend monotonously.

From high b-chain G_1 we have a sequence of consecutive b-chains $G_1, G_2, \ldots, G_m$ with the following properties: (1) for each $1 \leq i < m$, G_{i+1} is next to G_i, and the left end hexagon of G_{i+1} lies on the lower left side of G_i, (2) either G_m is just the low b-chain G'_1 or G_{m+1} is the b-chain next to G_m such that G_{m+1} has no hexagon lies on the lower left side of G_m. Let G be a hexagonal chain of H' consisting of b-chains $G_1, G_2, \ldots, G_m$. Then G is a ladder-shape hexagonal chain.

Similarly, from low b-chain G'_1 we have a sequence of consecutive b-chains $G'_1, G'_2, \ldots, G'_s$

with the following properties: (1) for each $1 \le j < s$, G'_j is next to G'_{j+1}, and the left end hexagon of G'_{j+1} lies on the higher left side of G'_j, (2) either G'_s is just the high b-chain G_1 or G'_s is next to the b-chain G'_{s+1} such that G'_{s+1} has no hexagon lies on the higher left side of G'_m. Let G' be a hexagonal chain of H' consisting of b-chains $G'_1, G'_2, \ldots, G'_s$.

For example, given a high b-chain D_1 and a low b-chain D_5 in Fig. 2, we can get two required hexagonal chains $G = G_1 \cup G_2 \cup G_3 \cup G_4$ and $G' = G_9 \cup G'_1$.

Claim 3. Either G and G' are disjoint or the last b-chain G_m in G coincides with the first b-chain G'_s in G'.

To analyze the substructures G and G' of H', as [26] we label the hexagons of G and some edges as follows (see Fig. 3): let $S_{i,j}$, $1 \le i \le m$ and $1 \le j \le n(i)$, be the hexagons of b-chain G_i as Fig. 3, neither A nor A' is contained in H'. Denote by $e_{i,j}$ be the boundary edge of H' which is parallel to $e_{1,1}$ and belongs to $S_{i,j}$, $1 \le i \le m$ and $1 \le j \le n(i)$, and denote the other boundary edges in $S_{1,1}$ and $S_{m,n(m)}$ by a, a', e_0, e'_0 respectively, as shown in Fig. 3.

Since the boundary C of H' is a proper M-alternating cycle, all the edges e_0, e'_0, $e_{i,j}$, $1 \le i \le m$, $1 \le j \le n(i)$, are M-double edges. So we can draw a ladder-shape broke line segment $L_1 = P_0P_1 \cdots P_{q+1} (q \ge 1)$ satisfying the following conditions.

(A1) The endpoints P_0 and P_{q+1} of L_1 are the midpoints of the edges a and a', respectively. P_i $(1 \le i \le q)$ is the center of a hexagon S_i of H', P_iP_{i+1} $(0 \le i \le q)$ is orthogonal to one of the three edge directions, and P_{i+1} $(0 \le i \le q)$ lies on the lower left side or the left side of P_i according as i is even or odd (see Fig. 3). L_1 only passes through hexagons of H'. Clearly, the graph consisting of the hexagons intersected by L_1 is a hexagonal chain, denoted by H_1;

(A2) All the edges intersected by L_1 are M-single edges, all the M-double edges which are located in the region above L_1 are parallel to $e_{1,1}$ (see Fig. 3).

Note that there exists such a broke line segment such that it only passes through hexagons $S_{i,j}$, $1 \le i \le m$ and $1 \le j \le n(i)$. Among all those broke line segments, we can select one, also denoted by L_1, such that there are the maximum number of M-double edges above it.

Symmetrically we treat substructure G' of H' as follows. Let $T_{i,j}$, $1 \le i \le s$ and $1 \le j \le t(i)$, be the hexagons of b-chain G'_i, neither hexagon B nor hexagon B' is

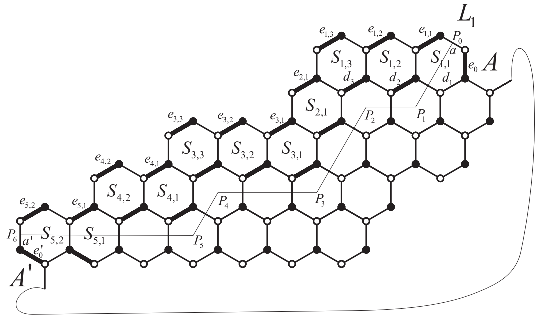

Figure 3: The hexagonal chain G on the left-top corner of H' (bold edges are M-double edges, $m = 6$, n(1)=3, n(2)=1, n(3)=3, n(4)=2, n(5)=2. And $A, A' \notin H'$.)

contained in H' as Fig. 4. Let $f_{k,\ell}$, $1 \leq k \leq s$ and $1 \leq \ell \leq t(k)$, be a series of boundary edges on this structure as indicated in Fig. 4. Since the boundary of H' is a proper M-alternating cycle, we can see that all the edges f_0, f'_0, $f_{k,\ell}$, $1 \leq k \leq s$ and $1 \leq \ell \leq t(k)$, are M-double edges (see Fig. 4).

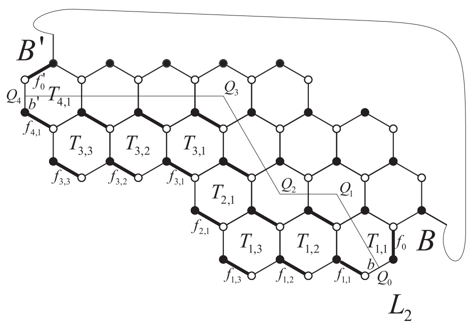

Figure 4: The hexagonal chain G' on the left-bottom corner of H' (bold edges are M-double edges, $s = 4$, t(1)=3, t(2)=1, t(3)=3, t(4)=1. And $B, B' \notin H'$.)

Like L_1, we also draw a ladder-shape broke line segment $L_2 = Q_0Q_1 \cdots Q_{r+1} (r \geq 1)$ as indicated in Fig. 4 so that the part below L_2 has as many M-double edges parallel to $f_{1,1}$ as possible. Let Q_i $(1 \leq i \leq r)$ be the center of a hexagon T_i of H'. Let H_2 be the hexagonal chain consisting of the hexagons intersected by L_2.

Clearly, both L_1 and L_2 have an odd number of turning points. We now have the following claim.

Claim 4. The boundary of H_1 (resp. H_2) is a proper M-alternating cycle and $m \geq 2$ (resp. $s \geq 2$).

Proof. We only consider H_1 (the other case is almost the same). Let d_i be the edge of $S_{1,i}$ opposite to $e_{1,i}$, $1 \leq i \leq n(1)$ (see Fig. 3). By Claim 2, $S_{1,1}$ is not an M-alternating hexagon. It implies that all edges $d_2, \ldots, d_{n(1)}$ are M-double edges. Hence, $S_{2,1}$ is a hexagon of H' and $m \geq 2$.

Let P_1 be the path induced by those vertices of H_1 which are just upon L_1. By the choice of L_1, we can see that P_1 is an M-alternating path with two end edges in M. Let P_2 be the path induced by those vertices of H_1 which are just below L_1. It suffices to show that P_2 is also an M-alternating path with two end edges in M.

Let $w_1(= e_0'), w_2, \ldots, w_{\ell_2}$ be a series of parallel edges on the bottom of H_1 and let $h_1(= e_0), h_2, \ldots, h_{\ell_1}$ be a series of vertical edges of H_1 on the right of P_0P_1 (see Fig. 5).

For $q = 1$, by the condition (A2) and $\{e_0, e_0'\} \subseteq M$, it follows that $h_1, h_2, \ldots, h_{\ell_1}$ (resp. $w_1, w_2, \ldots, w_{\ell_2}$) are forced by e_0 (resp. e_0') in turn and thus belong to M (see Fig. 5(a)). Therefore, P_2 is an M-alternating path with two end edges in M.

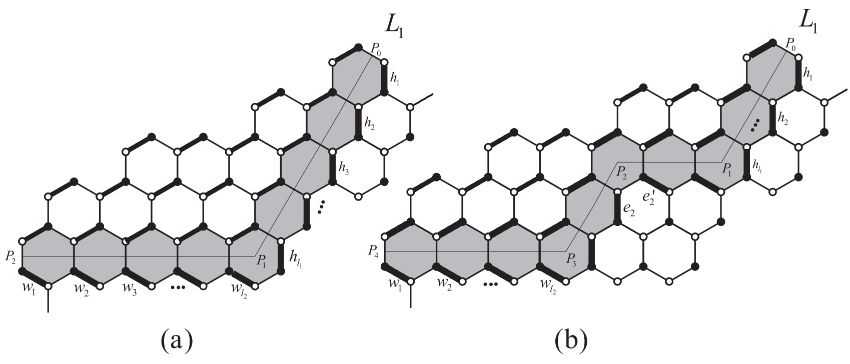

Figure 5: Illustration for Claim 4 in the proof of Lemma 2.1.

Let $q \geq 3$. For even i, $2 \leq i \leq q-1$, let e_i'' be the slant edge of S_i below L_1. Let e_i and e_i' be the two edges of H' which are adjacent to e'' and below L_1 (see Fig. 6(a)). Clearly, e_i is parallel to e_0, and e_i' is parallel to e_0'. We assert that $e_i'' \notin M$. Otherwise, e_i'' is an M-double edge. Since C is a proper M-alternating cycle, e_i'' does not lie on the boundary

C of H'. Thus S_i' is a hexagon of H' (see Fig. 6(b)). Moreover, we can switch from L_1 to a new broke line segment L_1' which passes through S_i' and satisfies the conditions (A1–A2) (see Fig. 6(b)). But the part above L_1' has more M-double edges than above L_1, contradicting the choice for L_1. Thus the assertion is true. From condition (A2), we can see that $\{e_0, e_0', e_2, e_2', \ldots, e_{q-1}, e_{q-1}'\} \subseteq M$. It follows that P_2 is an M-alternating path with two end edges in M (see Fig. 5(b)). ■

For odd i, by Claim 4 S_i $(1 \le i \le q)$ and T_i $(1 \le i \le r)$ are all proper M-alternating hexagons, and the other hexagons of H_1 and H_2 are not M-alternating. For convenience, let $S_0 := S_{1,1}, S_{q+1} := S_{m,n(m)}, T_0 := T_{1,1}$ and $T_{r+1} := T_{s,t(s)}$. By Claim 2, we have that $S_0 \neq S_1$, $S_{q+1} \neq S_q$, $T_0 \neq T_1$ and $T_{r+1} \neq T_r$. Further, by Claim 4 we can see that each hexagon in K' either belongs to $H_1 \cup H_2$ or is disjoint with $H_1 \cup H_2$.

Let $K_1 := \{S_0, S_2, \ldots, S_{q+1}\}$ and $K_2 := \{T_0, T_2, \ldots, T_{r+1}\}$. To complete the proof of the lemma, there are two cases to be considered.

Case 1. H_1 and H_2 are disjoint (see Figs. 3 and 4).

It is straightforward to verify that $H_i - K_i$ has a perfect matching, $i = 1, 2$, so K_i is a resonant set of H_i and $|K_i| \ge |H_i \cap K'| + 1$.

Let $K'' := (K_1 \cup K_2) \cup (K' - K' \cap H_1 - K' \cap H_2)$. Similar to the proof of Claim 1, we have that K'' is a resonant set of H' and $|K''| \ge |K'| + 2$. Thus $Cl(H') \ge |K'| + 2$.

Case 2. H_1 intersects H_2 .

By Claim 3 the last b-chain G_m in G coincides with the first b-chain G_s' in G'. Hence $S_{q+1} = T_{r+1}$. By Claim 4 both boundaries of H_1 and H_2 are proper M-alternating cycles. It follows that only segment $P_q P_{q+1}$ of L_1 is identical to segment $Q_r Q_{r+1}$ of L_2. Hence $H_1 \cup H_2$ is a cata-condensed hexagonal system with exactly one branch hexagon S_q $(= T_r)$ as Fig. 7, and its boundary is also a proper M-alternating cycle. So H_1 and H_2 have exactly one common M-alternating hexagon. We also can see that $K_1 \cup K_2$ is a resonant set of $H_1 \cup H_2$, and $|K_1 \cup K_2| \ge |K' \cap (H_1 \cup H_2)| + 2$. Let $K'' := (K_1 \cup K_2) \cup (K' - K' \cap (H_1 \cup H_2))$. By Claim 1, we have that K'' is a resonant set of H' and $|K''| \ge |K'| + 2$. Thus $Cl(H') \ge |K'| + 2$.

Now the entire proof of the lemma is complete. ■

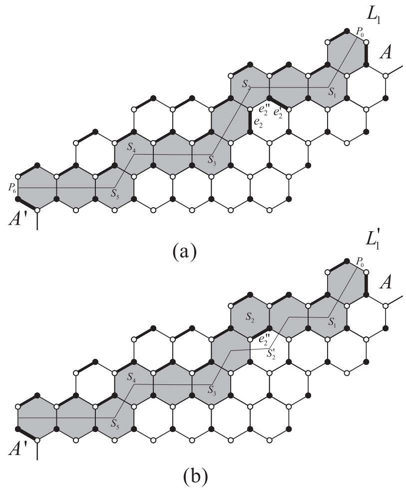

Figure 6: Illustration for Claim 4 in the proof of Lemma 2.1.

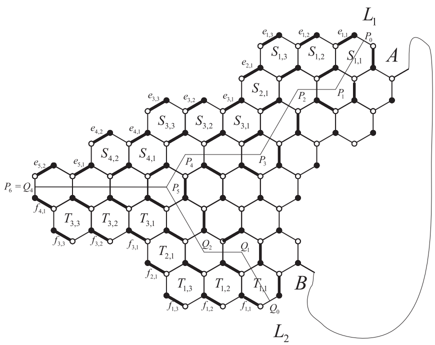

Figure 7: Illustration for Case 2 in the proof of Lemma 2.1.

3 Main results

We now state our main result as follows.

Theorem 3.1. *Let H be a hexagonal system with a perfect matching. For every perfect matching M of H such that $f(H, M) = F(H)$, there exist $F(H)$ disjoint M-alternating hexagons of H.*

By Theorem 1.3, there are $F(H)$ disjoint M-alternating cycles of H. It is well known that each M-alternating cycle of H has an M-alternating hexagon in its interior [24]. In order to prove the above theorem, we only need to prove the following lemma.

Let $\mathcal{C}$ be a set of disjoint cycles of a hexagonal system H. A member of $\mathcal{C}$ is called *minimal* if it contains no other members of $\mathcal{C}$ in its interior.

Lemma 3.2. *Let H be a hexagonal system. Let M be a perfect matching of H with the maximum forcing number and let $\mathcal{A}$ be a maximum set of disjoint M-alternating cycles of H. Then for any two members in $\mathcal{A}$ their interiors are disjoint, and for any $C \in \mathcal{A}$, $I[C]$ is a linear chain.*

Proof. Let $n := F(H) = f(H, M)$. By Theorem 1.3, $n = |\mathcal{A}|$. Suppose to the contrary that there exist two cycles in $\mathcal{A}$ so that their interiors have a containment relation. Then $\mathcal{A}$ has a non-minimal member C_0 and its interior contains only minimal members of $\mathcal{A}$.

Let $\mathcal{A}_0$ denote the set of minimal members of $\mathcal{A}$ whose interiors are contained in the interior of C_0. Then the restriction of M on $I[C_0]$ is also a perfect matching of $I[C_0]$, denoted by M_c. Note that each M-alternating cycle has an M-alternating hexagon in its interior [24]. Then each cycle in $\mathcal{A}_0$ can be replaced by an M-alternating hexagon, the set of these hexagons is a resonant set of $I[C_0]$, denoted by K. Clearly, K is disjoint with C_0, $|K| = \mathcal{A}_0$ and $I[C_0]-C_0-K$ has a perfect matching. By Lemma 2.1, $I[C_0]$ has a resonant set S such that $|S| \geq |K|+2$. Let M_0 be a perfect matching of $I[C_0]$ such that all hexagons in S are M_0-alternating. Let $M_1 := (M \backslash M_c) \cup M_0$ and $\mathcal{A}' := S \cup \big(\mathcal{A} - \{C_0\} - \mathcal{A}_0\big)$. Then M_1 is a perfect matching of H such that each member in $\mathcal{A}'$ is an M_1-alternating cycle. Note that $|\mathcal{A}'| \geq n+1$. By Theorem 1.3, we have that $f(H, M_1) \geq n+1$. This contradicts that the maximum forcing number of H is n. Therefore, for any two members in $\mathcal{A}$ their interiors are disjoint.

For any $C \in \mathcal{A}$, we assert that the Clar number of $I[C]$ is 1. Otherwise, $I[C]$ has a resonant set S' with $|S'| \geq 2$. Similar to the above discussion, we can obtain $n+1$ disjoint cycles which are M_2-alternating with respect to some perfect matching M_2 of H. By Theorem 1.3, we have that $F(H) \geq n+1$, a contradiction. Hence the assertion is true. By Theorem 2.2, for any $C \in \mathcal{A}$, $I[C]$ is a linear chain. ■

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