

The Chiral Graph Theory

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(Received July 2, 2014)

Abstract. The chiral graphs are modified graphs containing information on chirality elements defined by IUPAC: chirality vertices, axes, planes, and additionally topological chirality of the molecule. The chiral graphs are defined using two equivalent approaches: hypervertex and hyperedge. Chiral matrices and chiral topological indices are assigned to chiral graphs based on chiral incidence matrix, in which entries corresponding to chirality elements are represented by 5-tuple hypercomplex numbers. Standard operations on chiral incidence matrix, formulated either in hypervertex or hyperedge representation, allow to obtain chiral adjacency, valency, distance, Laplacian, Zagreb, and Randić matrices, containing information on chiral elements present in the chiral graph. The definition of algebra for the hypercomplex numbers allows for distinguishing among the types of chirality elements present in chiral topological indices unless the molecule is *meso* form and the index becomes real. The study forms a background for use of chiral topological indices in QSAR studies of chiral molecules.

1. Introduction

Chirality of a rigid object in three-dimensional (orientable, metric [1]) spaces is a symmetry-derived property. IUPAC defines chirality of the molecules in two ways [2]. One refers to symmetry of rigid objects in three-dimensional space: the molecule is chiral, if and only if, it belongs to the point symmetry groups containing only orientation preserving self-isometries [2]. Otherwise, it is achiral, *i.e.*, it has either a symmetry center, a symmetry plane, or an inverse rotation axis [3-5]. The second way states that the chiral molecule must be

nonsuperposable to its mirror image [2] which is more general than the first one and is valid for non-rigid objects in spaces more sophisticated than Euclidean R^3 space.

However, IUPAC lists also a set of structural elements of the molecule, which, if present, can make it chiral regardless whether it is rigid or not: an asymmetric center, an axial chirality, and a chirality plane [3-5]. These chirality elements are holding a set of groups, ligands, or an adjacent part of the structure in a spatial arrangement not superposable on its mirror image [2]. At the moment, IUPAC does not consider topological chirality of a whole molecule such as trefoil knot [6]. Note, that chirality of (i) a center can be treated as the property of a single vertex, (ii) an axis as the property of at least two vertices, or at least one edge, (iii) a plane as a property of at least three vertices or at least two edges, and (iv) the whole molecule only as the property of the entire molecule. The definition of chirality elements, is in fact, an operational definition stating "to make a molecule chiral insert a chirality element to it (avoiding formation of a *meso* form)".

Despite pioneering de Guye work on chirality measures in 1890 [7], for years, for chemists, a molecule was either chiral or achiral. A century later, chirality measures were categorized into two groups: based on the distance between (i) a molecule and an achiral reference, and (ii) a molecule and its mirror image [8]. At that time, the third category of graph theoretical measures of chirality was not considered because at that time the graph theory studies were rare. Extensive reviews of chirality measures and indices were done by Avnir *et al.* [9], Petitjean [10], Casanova i Casas [11], Natarajan and Basak [12], and Zhang *et al.* [13].

Nowadays, the category (i) continuous chirality measures (CCM), can be computed *via* Internet [14], and used to analyze peptides to nanotubes [15,16]. They are generalized to measure chirality of vectors, matrices and operators [17]. The category (ii) chirality measures, satisfying IUPAC nonsuperposability condition [2], have been developed by using the probability theory [18,19] and a concept of a product of 3D Cartesian and k -dimensional Property Space, $R^3 \times P^k$ [20-24]. In fact, chirality is not only a property of points in a space but also the features of these points: a regular tetrahedron having vertices in different colors is chiral. The Sinister-Rectus Chirality Measures, ^{SR}CMs calculated using property space [23], combine geometrical chirality with atom properties such as mass, charge, or a kind (a label). ^{SR}CMs were introduced by us and used for analysis of chirality of rigid heterofullerenes [21,22], flexible α -amino acids in crystals and in the gas-phase [20], and VCD spectra [24].

A major disadvantage of the similarity chirality measures is in the factorial increase of the number of enantiomer superpositions with the increase of the size of the molecule. Even if in the property space the combinatorial complexity is drastically reduced, the number still remains significant [20-22]. In search for less time-demanding chirality characteristics, we modified the chemical graph theory to better reproduce the molecular chirality.

The molecular graphs are thought to be only loosely related to chemical realm. Indeed, even though an ordinary graph is a representation of the Hückel hamiltonian matrix of π -electronic systems [25-29], the planar unsaturated, unsubstituted, molecules constitute only a small minority of known chemical substances. However, for non-rigid molecules, a strong link between chemical graph and molecular electronic structure can be found in the rigorous partition of electron density possible *via* the gradient vector field offered by the Atoms-In-Molecules (AIM) [30,31]. In the AIM theory, atoms and pairs of gradient paths in a molecule can be fully assigned to vertices and edges of a molecular graph [30,31].

It is not widely appreciated that some information on chirality is inherent to graph theory. For example, a chiral vertex can be defined as incident to three or four non-isomorphic subgraphs. This property, *GT-chirality*, allowed Robinson, Harary, and Balaban to enumerate chiral alkanes and monosubstituted alkanes [33-35]. In 1997, Mezey used the term *GT-chirality* for chiral lattice animal: a set of sidewise connected squares in the Euclidean plane [36]. Yet, the subgraph non-isomorphism is an elementary graph property requiring no graph embedding in an orientable space, and the lattice animals are a quite special subclass of cyclic graphs, for which name *LA-chirality* seems to be more appropriate. The concept of intrinsic chirality, *I-chirality*, introduced by Flapan and Weaver [37] for graphs of topological molecules such as Walba's Möbius Ladders [38] or trefoil knots [6], is also inherent to graph theory [39]. For example, the complete graphs K_{n+3} , $n \geq 1$, are *I-chiral* [37,39]. Although, the *GT*- and *I*-chirality concepts allow for some successful enumerations, a general proper distinction of all aspects of molecular chirality by graph theory methods seems to be difficult to accomplish. Thus, so far, the graph-theory studies on chiral molecules has been limited to distinguishing chirality centers by chirality-adjusted topological indices: numbers assigned to graphs.

Since breakthrough discovery of the Wiener index in 1947 [40], the importance of topological indices in QSAR analyses in medicinal chemistry and bioinformatics has systematically increased [41-52]. This is connected with the relative ease of obtaining numerical characteristics of molecules difficult or expensive for a one-time synthesis. Even though QSAR search methodology still yields a large number of random hints, if the forecast is

correct the potential profit can be enormous. The topological indices, generated from graph-theoretical matrices [53], are widely used in QSAR. In 2000, the number of QSAR descriptors was estimated to 1800 [54], but now *Dragon 6* QSAR software, including a lot of topological indices, provides 4885 molecular descriptors [55]. Unexpectedly, none of these descriptors accounts for chirality. Also, in the recent Katritzky group comprehensive review on QSPR, the chirality property was not mentioned even once [46]. In contrast, the number of chiral drugs increases, worldwide sales of single enantiomer drugs exceed \$150 billion, and the pharmaceutical industry introduces more and more enantiopure analogues of known racemic drugs [56]. Also, a huge effort is made to improve methods of asymmetric synthesis. *e.g.* [57-59], Hence, the topological indices distinguishing enantiomers are urgently needed.

Information about vertex chirality was introduced in different ways into the topological indices. In 1993, Pyka introduced the Optical Topological Index, I_{opt} [60]. It was the first-ever index differentiating the chiral molecules which was based on weighted vertex-distance matrix multiplied by a vector in which entries were -1 when the entry corresponded to D-center and 1 otherwise. The index was sum of all elements resulted from this multiplication. Another approach was based on the augmentation of square path matrices to rectangular matrices containing physical-property columns, where the sign was changed for chiral vertices entries of one these columns [13]. The index is obtained as a function of the largest eigenvalue of the rectangular matrix multiplied by its transpose. The following index of 2D-chirality uses similarity distance between codes of 2D-chiral polihexes [61], while a different 2D-chirality index introduces an absolute scale of the codes of 2D-chiral polihexes [62,63]. Several chirality-sensitive indices were obtained using weighted graphs and distance matrices modified by the chiral factors of +1 and -1 introduced for *S*- and *R*- configuration, respectively [64]. Further attempt was made by replacing of diagonal elements in the connectivity matrix (multiplied by charge matrix) by (a_i+c) for *R*- and (a_i-c) for *S*-configuration [65].

The Golbraikh, Bonchev, and Tropsha significant study introduced chirality (a_i+c) , (a_i-c) , (δ_i+c) , and (δ_i-c) corrections by real or complex numbers to *R*- and *S*-vertex valencies to calculate connectivity, Zagreb, and charge topological descriptors and their extensions [66]. Lukovits and Linert used a function antisymmetric between exchange of *S*- and *R*- to modify valencies of the chiral vertices in the connectivity index [67].

Another approach to chiral topological descriptors for molecule and molecular fragments was based on the trace of valency plus adjacency ($A+V$) matrix, or its power, modified by statistic

and chemistry derived factors using a trinary variable ω_c accounting for vertex chirality [68]. Yet another attempt (including also *E/Z* and axial equatorial isomerism) was the modification of the molecular multigraph adjacency matrix, or its power, multiplied by property vectors whose elements were chirality corrected by a trinary variable, producing a chirality index as a quadratic form built from adjacency matrix elements [69].

An interesting, simple method was based on the calculation of topological indices, especially Kier and Hall indices, multiplied by a chiral factor defined for chiral vertices [70]. A very promising method is based on calculation of a connectivity matrix descriptor δ_i , for each substituent attached to center of chirality, as a sum of series of descriptors of values decreasing as finite geometric series and the calculation of chirality indices as series of unary, binary, ternary, and quaternary forms of connectivity descriptors whose binary terms differ for *R*- and *S*-enantiomers [71]. Two recent approaches consider either permutation groups of graphs [72] or kernels of graphs [73,74] to distinguish chirality centers.

Finally, I was kindly informed by the Reviewer of this paper, that non-rigid stereoisomers can also be precisely described by the concept of finite ordered point sets in oriented *n*-dimensional space developed by Dreiding and Wirth [75] and within the frame of the category theory by Dreiding, Dress, and Haegi [76]. These approaches have been continued and developed by Gugisch [77, 78] within the formalism of the oriented matroids theory. Although, the matroid theory is closely related to the graph theory, as it is a generalization of linear independence in vector spaces it goes far beyond a single graph theory [79,80].

Thus, the methods used so far in chemical graph theory to distinguish chiral structures refer to transformations of 3D molecular structure conserving information on chirality centers into a graph. This has been known in organic chemistry since the introduction of Fischer projections in 1891 [81,82]. Now the projections system is precise [83], and is only not developed for topological isomers, for which the reduced regular diagrams used in topology can be adopted [84,85]. Thus, if in chemistry the projections of 3D structures into the plane are widely used, why not use them to modify chemical graph theory towards the chiral graph theory? These tools may be more suitable for the needs of the chemical graph theory. The aim of the present paper is to show two possible ways of such formulation of the chiral graph theory. We show that the approach presented here allows to obtain chirality discriminating indices for any chirality element and their combination, and for most of chemical graph theory matrices important for chemical applications.

2. Theory

Let us sketch the idea of the *chiral graph theory* in a form dedicated to chemistry. Here, a graph is defined for an arbitrary finite set V and a set $P_k(V)$ denoting the family of all k -element subsets of V . A graph G is an ordered pair of vertices V and a set of edges $E \in P_k(V)$, $G=(V,E)$. Assume that stereoisomerism of a molecule is fully known. Now, consider two approaches to code chirality center, chirality axis, chirality plane, and topological chirality [2-5,84] in a *chiral molecular graph* $*G$: the *hypervertex* and the *hyperedge* approach. In the former, information about chirality elements is encoded *via* vertices additionally possessing four different properties g_1, g_2, g_3 , and g_4 , whereas in the latter, it is encoded *via* additional four different types of hyperedges E_1, E_2, E_3 , and E_4 . The subscripts 1, 2, 3, and 4 assign the property of being a chirality center, axis, plane, and topological chirality to a hypervertex or a hyperedge. The *chiral graphs* in the *hypervertex*, $*G_{hv}$, and the *hyperedge*, $*G_{he}$, approach are defined in (1) and (2):

$$\begin{aligned}
 &{}^*G_{hv} = ({}^*V, E): \\
 &{}^*V = \{(v, g_1, g_2, g_3, g_4) \mid v \in V, E \subseteq {}^*V \times {}^*V \\
 &g_i(v) = \begin{cases} -1 & \text{for the R configuration} \\ 0 & \text{for the achiral element, } i=1,2,3,4 \\ +1 & \text{for the S configuration} \end{cases} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 &{}^*G_{he} = (V, {}^*E): \\
 &{}^*E = \left\{ (E, E_1, E_2, E_3, E_4) \mid e_i \subseteq \underbrace{V \times \dots \times V}_{k=1,2,3,m} \right\} \\
 &E_i = \begin{cases} -1 & \text{for the R configuration} \\ 0 & \text{for the achiral element, } i=1,2,\dots,m \\ +1 & \text{for the S configuration} \end{cases} \quad (2)
 \end{aligned}$$

E_i can be 1-ary, 2-ary, ..., m -ary relation between vertices v , while by assuming different fixed k different unequivalent versions of the hyperedge chiral graphs can be defined.

The two approaches are only slight modifications of the common graph or hypergraph. The hypervertex chiral graph is isomorphic to the common graph with chirality attributes assigned to vertices, and the hyperedge chiral graph is a hypergraph [86-91] with chirality attributes assigned to hyperedges. The two approaches can be combined into the hypervertex-hyperedge approach if necessary. Designations of chirality elements in chiral graphs can be those recommended by IUPAC [83].

The practical aim of the chiral graph theory is to calculate chiral topological indices for use in QSAR. They can be derived from chiral graph matrices (adjacency, valency, distance, *etc.*), calculated based on the chiral incidence matrix in common graph theory expressions (3)-(6) and the appropriate chiral graph theory matrices symbolized by the left superscript asterisk.

Let us recall some of expressions important to graph theory. The (square, symmetric, vertex) adjacency $A=A(n,n)$ and valency $V=V(n,n)$ matrices can be calculated from the rectangular incidence matrix $I=I(n,k)$ of a graph of n -vertices and k -edges, whose entries are unit if a vertex i is incident to an edge j and zero otherwise, by multiplying I by its transpose I^T :

$$I \cdot I^T = A + V \quad (3)$$

The entries of A are unit if vertices i and j are adjacent and zero otherwise, while the diagonal elements of V are equal to the numbers of incident edges, *i.e.*, equal to the vertex valencies.

The symmetric distance matrix D , whose d_{ij} entries denote the minimal number of edges connecting vertices i and j , is essential for the chemical graph theory, which was born with the discovery of the Wiener index of this matrix [40]. For acyclic graphs, D can be expressed as a sum (4) of series of the symmetric A_i adjacency matrices, where i denotes the vertex adjacency through i edges, for $i=\min(p_{j,k})$ where $p_{j,k}$ are paths between vertices j and k , and j and k runs over all edges, and l is the longest path in graph:

$$D = \sum_{i=1}^l i \cdot A_i \quad (4)$$

Importantly, the A_i adjacency matrix can be expressed using A^i matrix: the i -th power of the A matrix (5), whose r,s entry is equal to the number of paths with repetition of length i between r and s vertices. The unit entries of A^i indicate vertices connected by path without repetition and are in the same positions in A_i , while all the other entries in A^i correspond to zeros in A_i :

$$A_i = A_{k \neq 1 \rightarrow 0}^i \quad (5)$$

where $k \neq 0$ denotes that each entry different from unit in A^i is set equal to 0 in A_i .

There are also simple relations between A and V and the ZG_1 and ZG_2 Zagreb matrices [92-94], Laplacian matrix L [95] and Randić matrix R [96-98] (the Randić version of the connectivity matrix) multiplied by the first multiplicative Zagreb index:

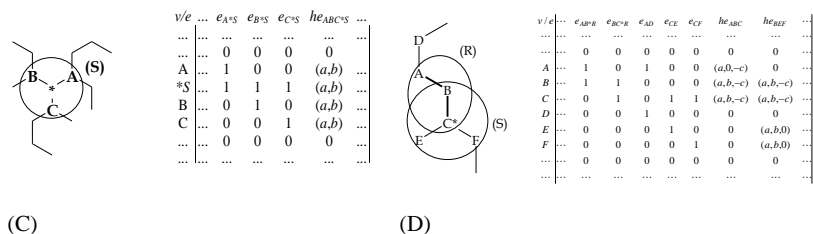


Fig. 1. Chiral graphs with corresponding fragments of incidence matrices in hypervertex (A) and (C) and hyperedge (B) and (D) representations, respectively. In (A) and (C), A, B, C denote C-atoms adjacent to the chiral atom *, in (B) and (D), A, B, C* denote atoms participating in chiral axis (in bold) while B, E, F are adjacent to chiral atom C*. (S) and (R) denote configuration; circles denote hyperedges, v , e , and he stand for vertex, edge, and hyperedge; $(a,b)=a+b \cdot i$; $(a,b,c)=a+b \cdot i_1+c \cdot i_2$; i_k is k -th imaginary element. In general, all the elements of $*I$ are quintuples instead of 1 and 0 in I .

Thus, $(1,0,0,0)$ denotes an ordinary vertex or a vertex in an ordinary edge; $(0,-1,0,0,0)$ denotes a chirality center in a hypervertex or a hyperedge of R - configuration; $(0,0,1,0,0)$ denotes chirality axis hypervertex or a vertex in chirality axis in a hyperedge of S - configuration, whereas $(0,-1,1,0,0)$ or $(0,-1,1,0,0)$ denotes that the vertex is simultaneously an R - chirality center involved in an S - chirality axis. The chirality of planes and topological molecules are represented analogously.

The presence of several chirality centers is common, the co-existence of chirality centers and a chiral axis is not strange, while the occurrence of all types of chirality elements in one molecule seems to be quite an exceptional case. For the former two cases, the quintuples in the incidence matrices can be reduced to either ordered pairs or triples as the 4-th, 5-th, and eventually 3-rd quintuple element is zero. Hereafter, we use the notation (a,b,\dots) instead of n -tuples with 0,-1, and 1 elements. Instead of quintuples with only unit values we plan to use the a,b,\dots as adjustable parameters which may be fitted in QSAR routines to reproduce specific molecular properties. To make use of equations (3)-(6) and to calculate $*A$, $*V$, and $*D$ matrices from $*I$, one has to multiply $*I$ by its transpose $*I^T$, i.e., to know how to multiply the quintuples. The following 5-tuple algebra of the hypercomplex numbers with four imaginary basis elements, i_1, i_2, i_3 , and i_4 , satisfies the following conditions:

The addition \oplus is defined as common addition $+$ of real numbers, and the multiplication \otimes is a generalization of complex number multiplication. In general the hypercomplex numbers [99] are described in terms of Clifford algebras isomorphic to matrix rings over R or C fields or H ring, where R , C , and H stand for real, complex, and quaternion numbers [100]. The hypercomplex numbers defined in (7) we call the M -hypercomplex numbers.

$$\begin{aligned}
 &1. (a, b, c, d, e) = a + b \cdot i_1 + c \cdot i_2 + d \cdot i_3 + e \cdot i_4 \quad a, b, c, d, e \in R \text{ and} \\
 &\forall k, l = 1, 2, 3, 4: i_k \cdot i_l = \begin{cases} -1 & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases} \\
 &2. (a_1, b_1, c_1, d_1, e_1) \oplus (a_2, b_2, c_2, d_2, e_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2, e_1 + e_2) \\
 &3. (a_1, b_1, c_1, d_1, e_1) \otimes (a_2, b_2, c_2, d_2, e_2) = (a_1 \cdot a_2 - b_1 \cdot b_2 - c_1 \cdot c_2 - d_1 \cdot d_2 - e_1 \cdot e_2, a_1 \cdot b_2 + a_2 \cdot b_1, \\
 &a_1 \cdot c_2 + a_2 \cdot c_1, a_1 \cdot d_2 + a_2 \cdot d_1, a_1 \cdot e_2 + a_2 \cdot e_1)
 \end{aligned} \tag{7}$$

The set of quintuples with the \oplus addition, $(0,0,0,\dots,0)$ as the addition neutral element, forms a commutative group. However, the set of quintuples with \otimes multiplication is not associative, it is commutative, distributive with respect to the \oplus addition, the $(1,0,0,\dots,0)$ is the neutral \otimes multiplication element which is unique, and there exists a unique reciprocal element to $(a_1, b_1, c_1, d_1, e_1)$ equal to $(a_1, b_1, c_1, d_1, e_1)^{-1}$. Thus, the algebra of the M -hypercomplex numbers forms a ring over the R^5 field. Use of the M -hypercomplex numbers allows for both the calculation of chiral matrices from $*I$ and chiral topological indices derivable from these matrices. Let us finally remark, the fact that the M -hypercomplex numbers can be conjugated at the 2-nd, 3-rd, 4-th, or 5-th element. Thus, a conjugate transpose matrix can be defined in 4! ways. Here, we use exclusively the total transpose of $*I$.

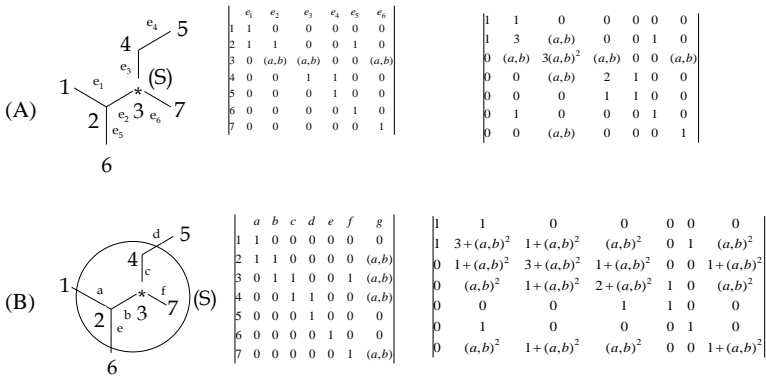


Fig. 2. Chiral graphs $*G_{hv}$ and $*G_{he}$ of 2,3S-dimethylpentane in the hypervertex (A) and hyperedge (B) representations and the corresponding $*I$ and $(*A+*V)$ matrices.

If not all of the chirality elements are present in a molecule, the algebra of M -hypercomplex numbers is reduced either to the algebra of complex numbers or hypercomplex triples or quadruples. When the chirality elements are absent the algebra is the algebra of real numbers, and chiral graph and incidence matrix are reduced to the common graph theory forms.

Let us calculate $(*A+*V)$ matrix of exemplary $*G_{\text{hv}}$ and $*G_{\text{he}}$ chiral graphs using Eq. (3) (Figs. 2 and 3). First, remark that zeros in $(*A+*V)$ are in the same positions as in the corresponding $(A+V)$ matrix of an achiral graph or hypergraph.

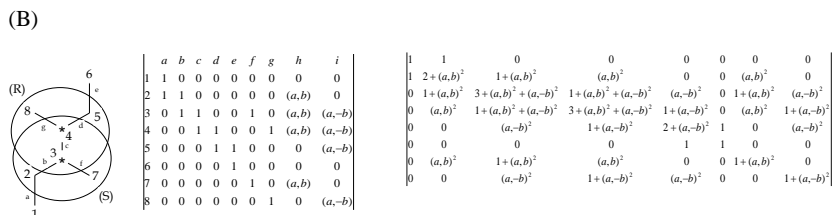
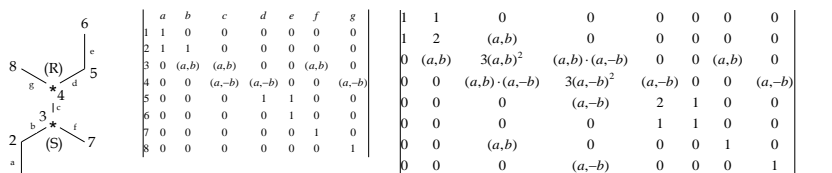
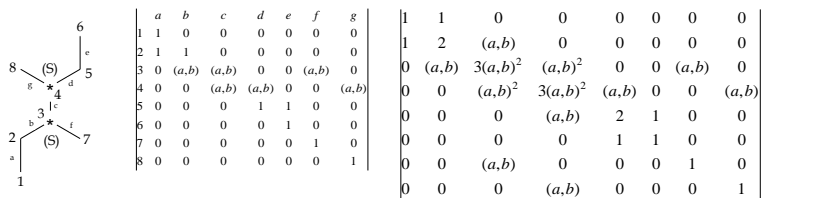


Fig. 3. Chiral graphs of 3S,4S-dimethylhexane (A) and the 3S,4R-dimethylhexane *meso* form in the hypervertex (B) and the hyperedge (C) representations and the corresponding $*I$ and $(*A+*V)$ matrices.

Second, note that each complex number or a product of complex numbers in $(*A+*V)$ corresponds to 1 in $(A+V)$. Third, observe that although the ranks of $(*A+*V)_{hv}$ and $(*A+*V)_{he}$ are equal, the structure of the matrices in the two representations is different. Fourth, notice that the $(*A+*V)_{hv}$ matrix of 3*S*,4*S*-dimethylhexane is different from that of *meso* 3*S*,4*R*-dimethylhexane (Fig. 3).

Indeed, the number of complex conjugate numbers in the $(*A+*V)$ matrix in the two representations of the latter molecule is equal (Fig. 3). This is a simple consequence of both symmetry of the chiral graph and symmetry of the $(*A+*V)$ matrices. The latter property of the *meso* form of chiral graphs is very important because most of the common chiral indices for these graphs are real, whereas those of the chiral stereoisomers are (hyper)complex.

Based on chiral $*G_{hv}$ graphs and $(*A+*V)_{hv}$ matrices presented in Figs. 2 and 3, one can conceive the following definitions of the $*A_{hv}$ and $*V_{hv}$ matrices:

$$*A_{hv} = \begin{cases} 0 & \Leftrightarrow i \text{ is not adjacent to } j \\ 1 & \Leftrightarrow i \text{ is adjacent to } j \\ (a, b) & \Leftrightarrow i \text{ is adjacent to } j \text{ and either } i \text{ or } j \text{ is chiral} \\ (a, b_1)(a, b_2) & \Leftrightarrow i \text{ is adjacent to } j \text{ and both } i \text{ and } j \text{ are chiral} \end{cases} \quad (8)$$

where b , b_1 , and b_2 symbolize imaginary values corresponding to either *S*- or *R*-configuration.

$$*V_{hv} = \begin{cases} v_i & \Leftrightarrow i \text{ is achiral} \\ v_i(a, b)^2 & \Leftrightarrow i \text{ is chiral} \end{cases} \quad (9)$$

where v_i is valency and b symbolizes an imaginary value of either *S*- or *R*-configuration.

The definition of the analogous $*A_{he}$ and $*V_{he}$ chiral matrices in the hyperedge approximation

$$*A_{he} = \begin{cases} 0 & \Leftrightarrow i \text{ is not adjacent to } j \\ 1 & \Leftrightarrow i \text{ is adjacent to } j \\ (a, b)^2 & \Leftrightarrow i \text{ is adjacent to } j \text{ by an } S \text{ hyperedge} \\ (a, -b)^2 & \Leftrightarrow i \text{ is adjacent to } j \text{ by an } R \text{ hyperedge} \\ 1 + k(a, b)^2 + m(a, b)^2 & \Leftrightarrow i \text{ is adjacent to } j \text{ by an edge and } k \text{ and } m \text{ hyperedges of } S \text{ and } R \text{ configuration.} \end{cases} \quad (10)$$

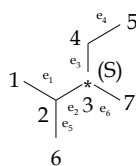
is the following:

where k and m are positive integers.

$$*V_{he} = \begin{cases} v_i & \Leftrightarrow i \text{ is achiral} \\ v_i + k(a, b)^2 + m(a, b)^2 & \Leftrightarrow i \text{ is chiral and belongs to an edge and } k \text{ and } m \text{ hyperedges of } S \text{ and } R \text{ configuration.} \end{cases} \quad (11)$$

Determination of common distance matrix is straightforward: it requires determining the minimal number of edges from vertex i to j . This can also be simple for chiral graphs, but to understand the rules, let us first determine $*D$ from (4) and (5), *i.e.*, determine $*A_i$ for $i=1,2,3,\dots, l$ —the longest path in a chiral graph. To achieve this aim one has to obtain all the i -th powers of $*A_1$, to set the entries in $*A^i$ corresponding to repetitive paths to zero (5), and sum the $*A_i$ matrices (4). The $*A_i$ and $*D$ matrices of $*G_{hv}$ and $*G_{he}$ of 2,3S-dimethylpentane are presented in Fig. 4.

Here, both cyclic graphs and the hyperedge representation are considered. Presence of a hyperedge containing at least three vertices implies the presence of a cyclic path between these very vertices. Therefore, it should be stressed that the minimal path condition used in definition of the $*A_i$ adjacency matrix (4) means that the non-zero entry appears only in the matrix of minimal subscript, even if for greater subscripts a non-repetitive path of given length also exists. This is often the case for chiral graphs in the hyperedge representation and also this is why $*A_4$ for 2,3S-dimethylpentane is the zero matrix.



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & (a,b) & 0 & 0 & 1 & 0 \\ 0 & (a,b) & 0 & (a,b) & 0 & 0 & (a,b) \\ 0 & 0 & (a,b) & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (a,b) & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & (a,b) & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (a,b)^2 & 0 & 0 & (a,b)^2 \\ (a,b) & 0 & 0 & 0 & (a,b) & (a,b) & 0 \\ 0 & (a,b)^2 & 0 & 0 & 0 & 0 & (a,b)^2 \\ 0 & 0 & (a,b) & 0 & 0 & 0 & 0 \\ 1 & 0 & (a,b) & 0 & 0 & 0 & 0 \\ 0 & (a,b)^2 & 0 & (a,b)^2 & 0 & 0 & 0 \end{pmatrix}$$

(A)

(B)

$$\begin{pmatrix} 0 & 0 & 0 & (a,b)^2 & 0 & 0 & (a,b)^2 \\ 0 & 0 & 0 & 0 & (a,b)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (a,b)^2 & 0 & 0 & 0 & 0 & (a,b)^2 & 0 \\ 0 & (a,b)^2 & 0 & 0 & 0 & 0 & (a,b)^2 \\ 0 & 0 & 0 & 0 & (a,b)^2 & 0 & (a,b)^2 \\ (a,b)^2 & 0 & 0 & 0 & (a,b)^2 & (a,b)^2 & 0 \end{pmatrix}$$

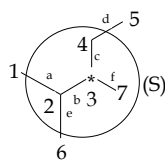
$$\begin{pmatrix} 0 & 0 & 0 & 0 & (a,b)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (a,b)^2 & 0 & 0 & 0 & 0 & (a,b)^2 & 0 \\ 0 & 0 & 0 & 0 & (a,b)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2(a,b) & 3(a,b)^2 & 4(a,b)^2 & 2 & 3(a,b)^2 \\ 1 & 0 & (a,b) & 2(a,b)^2 & 3(a,b)^2 & 1 & 2(a,b)^2 \\ 2(a,b) & (a,b) & 0 & (a,b) & 2(a,b) & 2(a,b) & (a,b) \\ 3(a,b)^2 & 2(a,b)^2 & (a,b) & 0 & 1 & 3(a,b)^2 & 2(a,b)^2 \\ 4(a,b)^2 & 3(a,b)^2 & 2(a,b) & 1 & 0 & 4(a,b)^2 & 3(a,b)^2 \\ 2 & 1 & 2(a,b) & 3(a,b)^2 & 4(a,b)^2 & 0 & 3(a,b)^2 \\ 3(a,b)^2 & 2(a,b)^2 & (a,b) & 2(a,b)^2 & 3(a,b)^2 & 3(a,b)^2 & 0 \end{pmatrix}$$

(C)

(D)

(E)



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1+(a,b)^2 & (a,b)^2 & 0 & 1 & (a,b)^2 \\ 0 & 1+(a,b)^2 & 0 & 1+(a,b)^2 & 0 & 0 & 1+(a,b)^2 \\ 0 & (a,b)^2 & 1+(a,b)^2 & 0 & 1 & 0 & (a,b)^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & (a,b)^2 & 1+(a,b)^2 & (a,b)^2 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1+(a,b)^2 & (a,b)^2 & 0 & 1 & (a,b)^2 \\ 0 & 0 & 0 & 0 & (a,b)^2 & 0 & 0 \\ 1+(a,b)^2 & 0 & 0 & 0 & 1+(a,b)^2 & 1+(a,b)^2 & 0 \\ (a,b)^2 & 0 & 0 & 0 & 0 & 0 & (a,b)^2 \\ 0 & (a,b)^2 & 1+(a,b)^2 & 0 & 0 & 0 & (a,b)^2 \\ 1 & 0 & 1+(a,b)^2 & (a,b)^2 & 0 & 0 & (a,b)^2 \\ (a,b)^2 & 0 & 0 & 0 & (a,b)^2 & (a,b)^2 & 0 \end{pmatrix}$$

(F)

(G)

$$\begin{array}{c}
 \left(\begin{array}{cccc|cccc}
 0 & 0 & 0 & 0 & (a,b)^2 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 (a,b)^2 & 0 & 0 & 0 & 0 & (a,b)^2 & 0 & 0 \\
 0 & 0 & 0 & 0 & (a,b)^2 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array} \right) \\
 \text{(H)}
 \end{array}
 \qquad
 \begin{array}{c}
 \left(\begin{array}{cccc|cccc}
 0 & 1 & 2+2(a,b)^2 & 2(a,b)^2 & 3(a,b)^2 & 2 & 2(a,b)^2 \\
 1 & 0 & 1+(a,b)^2 & (a,b)^2 & 2(a,b)^2 & 1 & (a,b)^2 \\
 2+2(a,b)^2 & 1+(a,b)^2 & 0 & 1+(a,b)^2 & 2+2(a,b)^2 & 2+2(a,b)^2 & 1+(a,b)^2 \\
 2(a,b)^2 & (a,b)^2 & 1+(a,b)^2 & 0 & 1 & 2(a,b)^2 & (a,b)^2 \\
 3(a,b)^2 & 2(a,b)^2 & 2+2(a,b)^2 & 1 & 0 & 3(a,b)^2 & 2(a,b)^2 \\
 2 & 1 & 2+2(a,b)^2 & 2(a,b)^2 & 3(a,b)^2 & 0 & 2(a,b)^2 \\
 2(a,b)^2 & (a,b)^2 & 1+(a,b)^2 & (a,b)^2 & 2(a,b)^2 & 2(a,b)^2 & 0
 \end{array} \right) \\
 \text{(I)}
 \end{array}$$

Fig. 4. Chiral graph of 2,3S-dimethylpentane in the hypervertex (A-E) and hyperedge (F-I) and representation and the $*A_1$, $*A_2$, $*A_3$, and $*A_4$ adjacency and $*D$ distance matrices.

Definitions (8)-(11) can be translated into intuitive and mnemotechnic rules to fill the entries of $*A_1$ and $*V_1$. To this aim, let us introduce the term *the touch of the vertex, 1-touch*, meaning that the path starts from or ends in the vertex. The *2-touch* is when the path is *going through the vertex* and the vertex is internal in the path or starts and ends in the same vertex. If the path with repetition starts from or ends in the internal vertex, then we say that this is 3- or, more general, *m-touch*.

Let us denote by $\omega-$; $-\omega-$; χ^S- ; $-\chi^S-$; χ^R- ; and $-\chi^R-$ terminal and internal achiral vertex of a path and the analogous chiral vertices in the two configurations, respectively. The following paths of length 1, *I*-paths, are possible in a chiral graph $\omega-\omega$; $\chi^S-\chi^S$; $\chi^R-\chi^R$; and $\omega-\chi^S$; $\omega-\chi^R$; and $\chi^R-\chi^S$. These *I*-paths correspond to the following elements of $*A_1$ matrix: 1, $(a,b)^2$, $(a,b)^2$, (a,b) , (a,b) , and $(a,b)\cdot(a,b)$, respectively. They can be interpreted in terms of touching the appropriate vertices, which introduces a factor to the product that 1-1 corresponds to vertices in the path. In the case of $*V$ in the hypervertex and hyperedge representations the rules (9) and (11) are simple and need no mnemotechnical help.

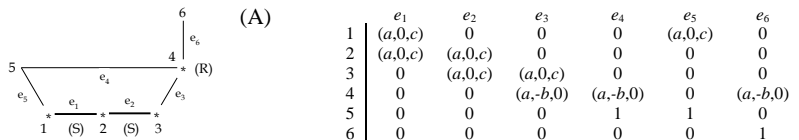
Using the same rules the following 4-paths $\omega-\chi^S-\omega-\chi^R$ and $\omega-\chi^S-\chi^S-\chi^R$ correspond to the following product of the appropriate entries of the $*A_4$ matrix: $1\cdot(a,b)^2\cdot 1\cdot(a,b) = (a,b)^2\cdot(a,b)$ and $1\cdot(a,b)^2\cdot(a,b)^2\cdot(a,b) = (a,b)^4\cdot(a,b)$. Obviously, the rule is valid irrespectively whether the path is with or without repetition and can be also used to analyze the $*A^i$ matrices in the hypervertex representation of a chiral graph.

For the hyperedge representation of the chiral graph the rule is different. The expression depends on the number of paths of the same length between given vertices and the kind and configuration of (hyper)edges in the path. In this representation the path corresponds to a sum whose addends depend on the kind of edge: $-$; $(he)^S$; and $(he)^R$ to which 1, $(a,b)^2$, and $(a,b)^2$

is assigned, respectively. Thus, the paths $v_1-v_2(\text{he})^S v_3(\text{he})^S v_4$ and $v_1(\text{he})^S v_2(\text{he})^R v_3(\text{he})^S v_4$ correspond to the following sums $1+(a,b)^2+(a,b)^2=1+2(a,b)^2$ and $(a,b)^2+(a,-b)^2+(a,b)^2=(a,b)^2$. However, if these two paths are different paths between the same vertices, the entry of $*A_4$ matrix will be equal to $1+3(a,b)^2$.

In the adjacency matrices of (*S*)-(4*R*)-methyl-cyclopenta-1,2-diene in the hypervertex representation entries with active two imaginary elements, such as $(a,-b,0)(a,0,c)=(a^2,-ab,ac)$, appear (Fig. 5). In the adjacency matrix for (*S*)-(4*R*)-methyl-cyclopenta-1,2-diene in the hypervertex representation entries appear with active two imaginary elements, such as $(a,-b,0)(a,0,c)=(a^2,-ab,ac)$. Fortunately, one can find the correct forms of the entries by simply counting of number of touches of the appropriate vertices in the center and axis in the non-repetitive path of given length. For example, $*a_{24}$ in $*A_2$ is equal to $(a,-b,0)(a,0,c)^3$ because the path from vertex 2 to vertex 4 touches three times vertices of the chiral axis and one vertex of the chiral center of *R*-configuration. Notice, that since (*S*)-(4*R*)-methyl-cyclopenta-1,2-diene is cyclic, the longest is the 3-path and $*A_4=[0]$.

In the hyperedge representation of (*S*)-(4*R*)-methyl-cyclopenta-1,2-diene with a chirality axes (Fig. 6), the longest path has length 2, because hyperedges create shortcuts between vertices. In full analogy to $*A_i$ of chiral graphs containing only chirality centers, here each path is represented by (sum of) squared terms, each of which represents an edge of a given type. For example, $*a_{24}$ in $*A_2$ is equal to $1+(a,-b,0)^2+(a,0,c)^2+(a,-b,0)^2 \cdot (a,0,c)^2$, because there is one path of length 2 from vertex 2 to 4 going through ordinary edges, one going through one ordinary edge and a hyperedge with one chiral center, one going through one ordinary edge and a hyperedge of the chiral axis, and one through one edge of the chiral center and another hyperedge of the chiral center.



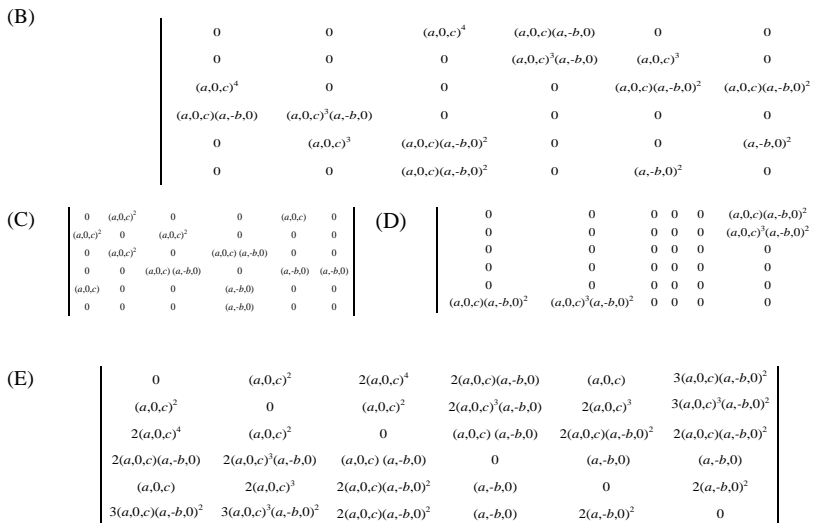


Fig. 5. Chiral graph of (S)-(4R)-methyl-cyclopenta-1,2-diene exhibiting chirality center and chiral axis 1-2-3 of (S) configuration and chirality centre 4 of (R) configuration in the hypervertex representation and the $*I$, $*A_1$, $*A_2$, and $*A_3$ adjacency matrices and $*D$ distance matrix, (A), (B), (C), (D), and (E), respectively.

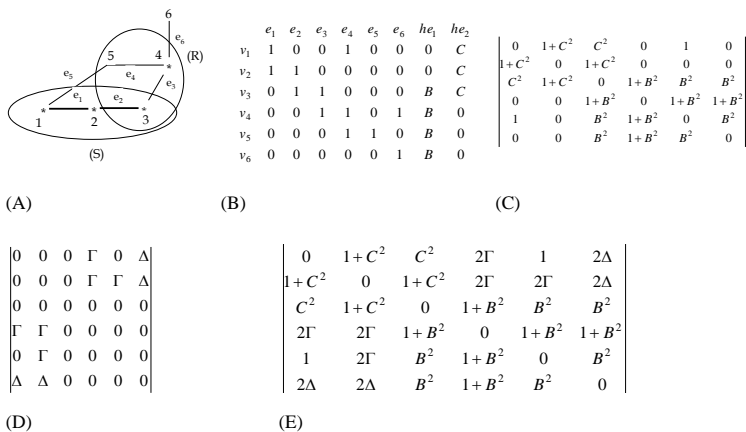


Fig. 6. Chiral graph of (S)-(4R)-methyl-cyclopenta-1,2-diene exhibiting chirality center and chiral axis in the hyperedge representation and the $*I$, $*A_1$, and $*A_2$, adjacency and $*D$ distance matrices, (A)÷(E), respectively. $B=(a,-b,0)$, $C=(a,0,c)$, $\Gamma=1+B^2+C^2+B^2C^2$, $\Delta=B^2+B^2C^2$.

The vast majority of chiral molecules of practical interest have only chirality centers and some exceptionally exhibit chirality axes. Therefore, here, we skip the discussion of more complex molecules, for which the $*I$, $*A_i$, and $*D$ matrices are constructed analogously. Let us demonstrate also some other chiral graph matrices (Figs. 7 and 8). As before, the two representations, the two configurations S and R , and the use of hypercomplex numbers yield different chiral Zagreb $*ZG_1$ and $*ZG_2$ matrices [93,94] (6) of 2,(3*S*)-dimethylpentane and (*S*)-(4*R*)-methyl-cyclopenta-1,2-diene, enabling the distinguishment between various configurations and chirality elements. Therefore, based on matrices in Fig. 7, one can easily write the matrices of all the stereoisomers. The Zagreb matrices play a role in QSPR studies of alkanes [53,101]. With the increase of molecular weight the number of chiral alkanes is much greater than of achiral ones [32]. The world production of olefins is enormous, therefore, even if the percentage of the C7+ fraction may seem small, in absolute numbers it is gigantic. Thus, chirality characteristics to alkanes, dormant feedstocks produced by gas, coal, and petroleum industry, may be of great potential.

The common Laplacian matrix of a graph (6) is a discrete form of the differential Laplacian quantum mechanics operator [102]. So far, the Laplacian matrix, L , and the derived indices are not particularly promising in chemistry [95], but since L is important for the spectral theory of graphs [103,104], and eigenvalues of L have emerged as important for information network [105]. Here we mention $*L$, because of possible future applications of the developed chiral graph theory outside chemistry. As $*L=*V* A$, it is easy to remark that the chiral Laplacian matrices of all optical isomers are distinctive.

The chiral Randić matrices $*R$ of the chiral graphs of (3*S*)-2,3-dimethylpentane and (*S*)-(4*R*)-methylcyclopenta-1,2-diene in the hypervortex and hyperedge representations (Fig. 9) show that for (3*S*)-2,3-dimethyl-pentane in the hypervortex representation $*R$ can be presented as real and identical to the R matrix of the ordinary graph. In this case, all information about chirality is gathered in the chiral Zagreb multiplicative index (6). Nevertheless, all the other Randić matrices presented in Fig. 9 are complex and correctly discriminate the enantiomers. The real form of the Randić matrix in the hypervortex approximation seems to be a rule rather than an exception. If so, in this case the chiral Randić index of chiral graphs is not discriminative. However, because R -derived connectivity indices are in extensive use in chemical graph theory and QSAR analyses [44-50], it is worthwhile to preserve signs of imaginary elements from the Zagreb multiplicative index to combine potency of R -derived indices with discriminative property of the hypercomplex numbers.

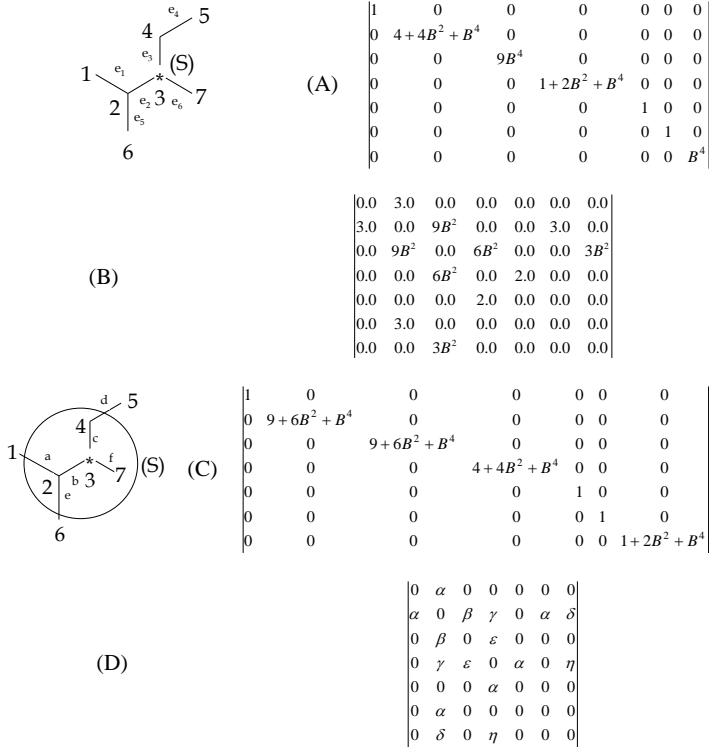


Fig. 7. Chiral graphs of 2,3*S*-dimethylpentane in the hypervertex and hyperedge representations and the corresponding $*ZG_1$ and $*ZG_2$ Zagreb matrices, (A), (C) and (B), (D), respectively. $B=(a,b)$, $\alpha=3+B^2$, $\beta=9+15B^2+7B^4+B^6$, $\gamma=6B^2+5B^4+B^6$, $\delta=3B^2+4B^4+B^6$, $\varepsilon=6+11B^2+6B^4+B^6$, $\zeta=3+7B^2+5B^4+B^6$, $\eta=2B^2+3B^4+B^6$.

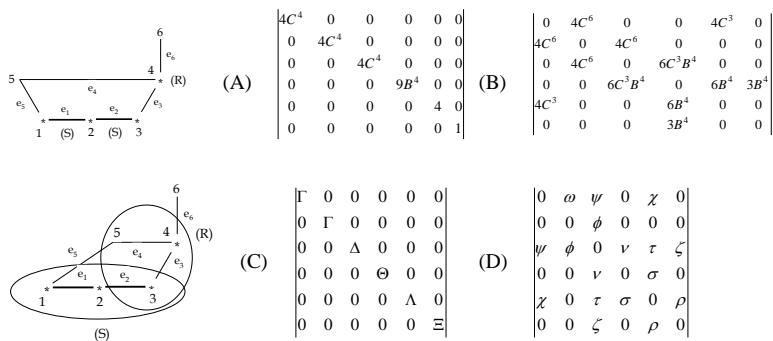


Fig. 8. Chiral graphs of (S)-(4R)-methyl-cyclopenta-1,2-diene in the hypervertex and hyper-edge representations and the corresponding $*ZG_1$ and $*ZG_2$ Zagreb matrices, (A), (C) and (B), (D), respectively. $B=(a,b,0), C=(a,0,c)$, $\Gamma=4+4C^2+C^4$, $\Delta=4+4B^2+4C^2+2B^2C^2+B^4+C^4$, $\Theta=9+6B^2+B^4$, $\Lambda=4+4B^2+B^4$, $\Xi=9+6B^2+B^4$; $\omega=4+8C^2+5C^4+C^6$, $\psi=4C^2+2B^2C^2+4C^4+B^2C^4+C^6$, $\chi=4+2B^2+2C^2+B^2C^2$, $\phi=4+2B^2+8C^2+3B^2C^2+5C^4+B^2C^4+C^6$, $\nu=6+11B^2+6B^4+B^6+3C^2+4B^2C^2+B^4C^2$, $\tau=4B^2+4B^4+B^6+8C^2+2B^2C^2+B^4C^2$, $\tau=2B^2+3B^4+B^6+2B^2C^2+B^4C^2$, $\sigma=6+11B^2+6B^4+B^6$, $\rho=3+7B^2+5B^4+B^6$.

As it was stressed earlier, there is a huge number of matrices used in chemical graph theory [53], but for a vast majority of them their chiral versions can be obtained just following the rules outlined in this paper for chiral incidence, adjacency, valency, distance, Laplacian, and Randić matrices. Hereafter, we show that the chiral topological indices obtained from chiral graph matrices also discriminates the enantiomers.

Let us remark first that several topological indices are obtained as half of the sum of the off-diagonal elements of topological matrices. Let \mathcal{D} (Croatian D with a stroke) and $p(\mathcal{D}(G))$ denote an arbitrary symmetric matrix connected with a graph G and the topological index of this matrix obtained just as a half of the sums of the off-diagonal elements, which we shall call the principal topological index of the matrix \mathcal{D} :

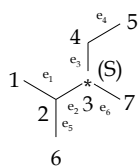
$$p(\mathcal{D}(G)) = \frac{1}{2} \sum_{\substack{i=1, j=1 \\ i \neq j}}^n d_{ij} \tag{12}$$

Then, the Wiener index is obtained when \mathcal{D} is the distance matrix, D , and $d_{ij}=d_{ij}$, the first Randić connectivity index is obtained when \mathcal{D} is Randić matrix and $d_{ij}=(\delta_i\delta_j)^{-1/2}$, where δ_i , is valency of the i -th vertex, the second Zagreb index is obtained when $\mathcal{D}=ZG_2$, and the detour index is obtained when \mathcal{D} is detour matrix in which A_{ij} elements are the longest distances between vertices i and j (without edge/vertex repetition). In the Balaban index J [106], the

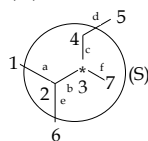
factor $1/2$ in the principal topological index is replaced by $\beta=\varepsilon(\mu+1)^{-1/2}$ in which ε is the number of edges and μ is the number of cycles, and the of diagonal elements \bar{d}_{ij} are equal to $(b_i b_j)^{-1/2}$ where $b_i=\sum d_{ik}$ and d_{ik} are the off-diagonal elements of the distance matrix.

Inspection into exemplary values of the Wiener and Zagreb indices for chiral (3S)-2,3-dimethylpentane and (S)-(4R)-methyl-cyclopenta-1,2-diene graphs in the hypervertex and hyperedge representations (Fig. 10) shows that the chirality elements present in molecules are reflected in the chiral topological indices as a simple consequence of the use of the (hyper) complex numbers. The corresponding Randić indices have complicated form and in some cases, e.g., in the hypervertex representation, must be multiplied by a factor build from the (hyper)complex numbers in order to preserve the discriminative properties.

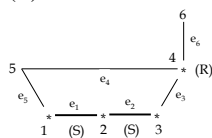
There are also topological indices or polynomials derived from secular equations associated to A , D , R , ZG , and other graph-topological matrices in the chemical graph theory [29,53]. For example, from the graph of characteristic polynomials for A and D , the Hosoya Z_A and Z_D indices, counting the number of choices of k disconnected edges in the graph, and the number of edges in the shortest path between vertices, respectively, may be calculated [107]. However, the analysis of analogous properties of chiral matrices is beyond this study.



(A)



(B)



(C)

$$\begin{vmatrix} 0 & 3^{-0.5} & 0 & 0 & 0 & 0 & 0 \\ 3^{-0.5} & 0 & 3^{-0.5} 3^{-0.5} & 0 & 0 & 3^{-0.5} & 0 \\ 0 & 3^{-0.5} 3^{-0.5} & 0 & 2^{-0.5} 3^{-0.5} & 0 & 0 & 3^{-0.5} \\ 0 & 0 & 2^{-0.5} 3^{-0.5} & 0 & 2^{-0.5} & 0 & 0 \\ 0 & 0 & 0 & 2^{-0.5} & 0 & 0 & 0 \\ 0 & 3^{-0.5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3^{-0.5} & 0 & 0 & 0 & 0 \end{vmatrix}$$

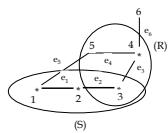
$$(\pi_1)^{1/2} = 1 \cdot 3 \cdot 3 \cdot (a,b)^2 \cdot 2 \cdot 1 \cdot 1 \cdot 1$$

$$\begin{vmatrix} 0 & A^{-0.5} & 0 & 0 & 0 & 0 & 0 \\ A^{-0.5} & 0 & BA^{-0.5} A^{-0.5} & (B-1)A^{-0.5} \Gamma^{-0.5} & 0 & A^{-0.5} & (B-1)A^{-0.5} B^{-0.5} \\ 0 & BA^{-0.5} A^{-0.5} & 0 & BA^{-0.5} \Gamma^{-0.5} & 0 & 0 & A^{-0.5} \\ 0 & (B-1)A^{-0.5} \Gamma^{-0.5} & BA^{-0.5} \Gamma^{-0.5} & 0 & \Gamma^{-0.5} & 0 & (B-1)B^{-0.5} \Gamma^{-0.5} \\ 0 & 0 & 0 & \Gamma^{-0.5} & 0 & 0 & 0 \\ 0 & A^{-0.5} & 0 & 0 & 0 & 0 & 0 \\ 0 & (B-1)A^{-0.5} B^{-0.5} & A^{-0.5} & (B-1)B^{-0.5} \Gamma^{-0.5} & 0 & 0 & 0 \end{vmatrix}$$

$$(\pi_1)^{1/2} = 1 \cdot (3+(a,b)^2)(3+(a,b)^2)(2+(a,b)^2) \cdot 1 \cdot 1 \cdot (1+(a,b)^2)$$

$$\begin{vmatrix} 0 & 2^{-1} & 0 & 0 & 2^{-1} & 0 \\ 2^{-1} & 0 & 2^{-1.5} \pi^{-1} & 0 & 0 & 0 \\ 0 & 2^{-1.5} \pi^{-1} & 0 & 2^{-1.5} 3^{-0.5} \varphi^{-1} & 0 & 0 \\ 0 & 0 & 2^{-1.5} 3^{-0.5} \varphi^{-1} & 0 & 2^{-0.5} 3^{-0.5} \varphi^{-1} & 3^{-0.5} \varphi^{-1} \\ 2^{-1} & 0 & 0 & 2^{-0.5} 3^{-0.5} \varphi^{-1} & 0 & 0 \\ 0 & 0 & 0 & 3^{-0.5} \varphi^{-1} & 0 & 0 \end{vmatrix}$$

$$(\pi_1)^{1/2} = (2+(a,0,c)^2)(2+(a,0,c)^2)(2+(a,b,0)^2+(a,0,c)^2)(3+(a,-b,0)^2) \cdot (2+(a,-b,0)^2)(1+(a,-b,0)^2)$$



$$\begin{pmatrix} 0 & (1+\pi^2)(X2)^2 & \pi^2(X2)\Lambda & 0 & (X2)(E2) & 0 \\ (1+\pi^2)(X2)^2 & 0 & (1+\pi^2)(X2)\Lambda & 0 & 0 & 0 \\ \pi^2(X2)\Lambda & (1+\pi^2)(X2)\Lambda & 0 & (1+\varphi^2)(E3)\Lambda & \varphi^2(E2)\Lambda & \varphi^2(E1)\Lambda \\ 0 & 0 & (1+\varphi^2)(E3)\Lambda & 0 & (1+\varphi^2)(E2)\Lambda & (1+\varphi^2)(E1)(E3) \\ (X2)(E2) & 0 & \varphi^2(E2)\Lambda & (1+\varphi^2)(E2)\Lambda & 0 & \varphi^2(E1)(E2) \\ 0 & 0 & \varphi^2(E1)\Lambda & (1+\varphi^2)(E1)(E3) & \varphi^2(E1)(E2) & 0 \end{pmatrix}$$

(D)

$$(\pi_1)^{1/2} = 2 \cdot (a,0,c)^2 \cdot 2 \cdot (a,0,c)^2 \cdot 2 \cdot (a,0,c)^2 \cdot 3 \cdot (a,-b,0)^3 \cdot 2 \cdot 1$$

Fig. 9. Chiral graphs of 2,3S-dimethylpentane and (S)-(4R)-methyl-cyclopenta-1,2-diene in the hypervertex and hyperedge representations and the corresponding Randić *R matrices and root square of π_1 : the first Zagreb multiplicative indices: (A), (B), (C), and (D), respectively. $A = (3 + (a, b)^2)$, $B = (1 + (a, b)^2)$, $\Gamma = (2 + (a, b)^2)$, $\pi = (a, 0, c)$, $\varphi = (a, -b, 0)$, $X1 = (1 + \pi^2)^{0.5}$, $X2 = (2 + \pi^2)^{0.5}$, $E1 = (1 + \varphi^2)^{0.5}$, $E2 = (2 + \varphi^2)^{0.5}$, $E3 = (3 + \varphi^2)^{0.5}$, $\Lambda = (2 + \pi^2 + \varphi^2)^{0.5}$.

	*W	*ZG ₂	*ZG ₁
	$5+8(a,b)+30(a,b)^2$	$1+2(a,b)^2+9(a,b)^3+6(a,b)^4+3(a,b)^6$	$8+6(a,b)^2+12(a,b)^4$
	$9+25(a,b)^2$	$26+47(a,b)^2+30(a,b)^4+6(a,b)^6$	$26+18(a,b)^2+4(a,b)^4$
	$2(a,0,c)+(a,0,c)^2+2(a,0,c)^3+2(a,0,c)^4+2(a,0,c)(a,-b,0)+7(a,0,c)(a,-b,0)^2+2(a,0,c)^3(a,-b,0)+3(a,0,c)^3(a,-b,0)^2+2(a,-b,0)+2(a,-b,0)^2$	$4(a,0,c)^3+8(a,0,c)^6+6(a,0,c)^3(a,-b,0)^3+9(a,-b,0)^4$	$5+12(a,0,c)^4+9(a,-b,0)^6$
	$12+9(a,0,c)^2+16(a,-b,0)^2+10(a,0,c)^2(a,-b,0)^2$	$27+25(a,0,c)^2+14(a,0,c)4+2(a,0,c)^6+41(a,-b,0)^2+27(a,-b,0)^3+6(a,-b,0)^5+13(a,0,c)^2(a,-b,0)^2+2(a,0,c)^4(a,-b,0)^2+3(a,0,c)^2(a,-b,0)^4$	$26+12(a,0,c)^2+3(a,0,c)^4+16(a,-b,0)^2+4(a,-b,0)^4+2(a,0,c)^2(a,-b,0)^2$

Fig. 10. Chiral graphs of 2,3S-dimethylpentane and (S)-(4R)-methyl-cyclopenta-1,2-diene in the hyper-vertex and hyperedge representations and the corresponding Wiener *W, Zagreb *ZG₂, and *ZG₁. The Randić *R indices have complicated form and were omitted.

An important consequence of using the M -hypercomplex numbers is the possibility to represent each chiral topological matrix and topological index as a matrix or an index of a common graph plus matrices or indices connected with each imaginary element corresponding to the appropriate chirality element, plus a mixed correction term which may be interpreted as reflecting an "interaction" of different chirality elements in a molecule.

Let us denote by *D and $p({}^*D)$ a chiral matrix and the derived principal chiral topological index expressed using M -(hyper)complex numbers. Then:

$$\begin{aligned} {}^*D &= \text{Re}({}^*D) + \text{HIm}({}^*D) \\ {}^*D &= D + \sum_{k=c,a,p,m} {}^*D^k + \sum_{k=c,a,p,m} {}^*D_{corr}^k \\ \sum_{k=c,a,p,m} {}^*D^k &= \sum_{k=c,a,p,m} i_k \cdot D^k \end{aligned} \quad (13)$$

and

$$\begin{aligned} p({}^*D) &= \text{Re}(p({}^*D)) + \text{HIm}(p({}^*D)) \\ p({}^*D) &= p(D) + \sum_{k=c,a,p,m} p({}^*D^k) + \sum_{k=c,a,p,m} p({}^*D_{corr}^k) \\ \sum_{k=c,a,p,m} p({}^*D^k) &= \sum_{k=c,a,p,m} i_k \cdot p(D^k) \end{aligned} \quad (14)$$

where Re and HIm denote real and hyperimaginary part of the number or matrix; presence of an asterisk in the left superscript denotes the chiral hyperimaginary matrix, which otherwise is real; c , a , p , and m denote the appropriate chirality elements; the right superscript denotes matrix with conserved only the entries corresponding to a given hyperimaginary element i .

Let us stress that the matrices and indices of different enantiomers differ by sign of the appropriate hyperimaginary elements. Moreover, the contribution of the k -th chirality element is included in the term multiplied by k -th hyperimaginary unit i_k . Last, but very importantly, the principal chiral topological indices of the *meso* forms are real because the same number of hyperimaginary i_k elements of the opposite signs occur in the chiral matrices and mutually cancel in the derived indices.

3. Conclusions

The chiral graph theory assigning the chiral matrices and indices to chiral graphs corresponding to a chiral molecule was formulated. To obtain the chiral graph a mapping of three-dimensional molecular structures into the chiral graphs was defined. The chiral graphs are modified graphs containing information on chirality elements (chiral vertices, axes, planes, and topological chirality of the whole molecule as in a trefoil knot molecule) and were

defined in agreement with IUPAC recommendations. The chiral graphs were defined using two different versions: hypervertex and hyperedge approach.

All results obtained within the frame of chiral graph theory are consequences of construction of chiral incidence matrix by changing the unit entries corresponding to chiral elements by appropriate 5-tuple hypercomplex numbers. Algebra of these hypercomplex numbers is tailored to distinguish chirality elements and derive all chiral matrices important to common graph theory from standard operations on chiral incidence matrix formulated either in hypervertex or hyperedge representation of a chiral graph. To illustrate this the chiral adjacency (through k -edges), valency, distance, Laplacian, Zagreb, and Randić matrices were derived for model chiral graphs exhibiting chirality center and chirality axis. The chirality-sensitive topological indices were constructed as well. In this way, the background for massive use of chiral topological indices in QSAR studies of chiral molecules was created.

Acknowledgments: Helpful comments and discussions with Prof. Piotr Mankiewicz from Institute of Mathematics of Polish Academy of Sciences in Warsaw, who passed away in February 2014, are greatly appreciated. A critical reading of the manuscript by Prof. Jerzy Ciosłowski from the Institute of Physics at the University of Szczecin is gratefully acknowledged. This work was supported by the National Science Centre in Poland Grant No. 2013/09/B/ST5/03664.

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