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The Minimal Matching Energy of (n, m)-Graphs with a Given Matching Number

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Abstract

The matching energy of a graph is defined as the sum of the absolute values of the zeros of its matching polynomial. Let $\mathcal{G}_{n,m}$ be the set of connected graphs of order n and with m edges. In this note we determined the extremal graphs from $\mathcal{G}_{n,m}$ with $n \leq m \leq 2n-4$ minimizing the matching energy. Also we determined the minimal matching energy of graphs from $\mathcal{G}_{n,m}$ where $m = n-1+t$ and $1 \leq t \leq \beta-1$ and with a given matching number β. Moreover, the above extremal graphs have been completely characterized.

1 Introduction

We only consider finite, undirected and simple graphs throughout this paper. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The cardinality of $E(G)$ is usually denoted by $m(G)$. The *degree* of $v_i \in V(G)$, denoted by $d_G(v_i)$ or d_i

for short, is the number of vertices in G adjacent to v_i. In particular, $\Delta(G)$ denotes the maximum degree of vertices in G, and $\Delta_2(G)$ is the second maximum degree of vertices in G. For each $v_i \in V(G)$, the set of neighbors of the vertex v is denoted by $N_G(v_i)$. For a subset W of $V(G)$, let $G - W$ be the subgraph of G obtained by deleting the vertices of W and the edges incident with them. Similarly, for a subset E' of $E(G)$, we denote by $G - E'$ the subgraph of G obtained by deleting the edges of E'. If $W = \{v\}$ and $E' = \{xy\}$, the subgraphs $G - W$ and $G - E'$ will be written as $G - v$ and $G - xy$ for short, respectively. For any two nonadjacent vertices x and y of graph G, we let $G + xy$ be the graph obtained from G by adding an edge xy. In the following we always denote by S_n the star graph of order n, and by K_n the complete graph of order n. Other undefined notations and terminology on the graph theory can be found in [1].

For any graph G with edge set $E(G)$, if any two edges of $e_1, e_2, \ldots, e_k \in E(G)$ have no common vertices, we say that $\{e_1, e_2, \ldots, e_k\}$ is a k-matching of graph G. Moreover, we denote by $m(G, k)$ the number of k-matchings in G. In particular, $m(G, 1) = m(G)$ and $m(G, k) = 0$ when $k > \frac{n}{2}$ for any graph G of order n. For the sake of convenience, we set $m(G, 0) = 1$ for any graph G. Recall that the Hosoya index $z(G)$ [16] of a graph G is the sum of total number of all matchings, including the empty edge subset, in G. Thus, for a graph G of order n, we have

$$z(G) = \sum_{k=0}^{[\frac{n}{2}]} m(G, k) \ .$$

For some details of the results on the Hosoya index, please refer to [22–24, 26, 28, 29].

The matching polynomial of a graph G of order n is defined as

$$\alpha(G, \lambda) = \sum_{k \geq 0} (-1)^k m(G, k) \lambda^{n-2k} \ . \tag{1}$$

Moreover, the theory of matching polynomial of a graph G is well elaborated in [3, 9, 10, 14]. From the expression of matching polynomial (1) of graph G, a quasi-order on the set of graphs of order n can be deduced as follows:

$$G \succeq H \Longleftrightarrow m(G, k) \geq m(H, k) \ for \ \ k = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \ .$$

If $G \succeq H$ and there exists at least one integer k such that $m(G, k) > m(H, k)$, then we write $G \succ H$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a graph G, i.e., the eigenvalues of its $(0,1)$-adjacency matrix [4]. The energy of the garph G is defined as

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i| \ .$$

Nowadays the theory of energy of graphs have been well developed. Some details on the energy of graphs can be found in the book [21].

Recently Gutamn and Wagner [15] have first introduced the definition of matching energy of a graph G as follows:

$$ME(G) = \int_0^{+\infty} \frac{1}{x^2} \ln \left[\sum_{k \geq 0} m(G,k) x^{2k} \right] dx \ . \tag{2}$$

Also in [15] they pointed out that, for any forest G, its matching energy coincides its energy, and the following formula:

$$ME(G) = \sum_{k=1}^{n} \mu_i \ , \tag{3}$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the zeros of matching polynomial of graph G. Very recently the matching energy has attracted the attention of some researchers. Ji, Li and Shi [19] determined the extremal matching energies of all bicyclic graphs of order n. Li and Yan [20] characterized the maximal matching energy of some graphs with given parameters, including chromatic number and connectivity. The maximal matching energy of tricyclic graphs of order n have been determined in [2].

From Formula (2) and the monotony of the function logarithm, the following two relations between the quasi-order defined as above and the matching energy, Hosoya index, respectively, can be deduced [15]:

$$G \succeq H \Longrightarrow ME(G) \geq ME(H) \, , \tag{4}$$

$$G \succeq H \Longrightarrow z(G) \geq z(H) \ . \tag{5}$$

Let $\mathcal{G}_{n,m}$ be the set of connected graphs of order n and with m edges. Denote by $\mathcal{G}_{n,m}(\beta)$ the set of connected graph from $\mathcal{G}_{n,m}$ and with matching number β where $2 \leq \beta \leq \lfloor \frac{m}{2} \rfloor$. In this note we characterized the extremal graphs from $\mathcal{G}_{n,m}$ where $n+1 \leq m \leq 2n-3$ minimizing the matching energy. Moreover, we determined the extremal graph from $\mathcal{G}_{n,n-1+t}(\beta)$ (where $1 \leq t \leq \beta - 1$) minimizing the matching energy.

2 Some lemmas

Before stating our main results, we will list or prove some lemmas as preliminaries, which will play an important role in the next proofs.

Lemma 2.1. *([9, 14]) For any graph G with $v_q \in V(G)$ and $e = v_i v_j \in E(G)$, we have*

(i) $m(G, k) = m(G - e, k) + m(G - \{v_i, v_j\}, k - 1)$;

(ii) $m(G, k) = m(G - v_q, k) + \sum_{v_r \in N_G(v_q)} m(G - v_q - v_r, k - 1)$.

Lemma 2.2. *([15]) Let G be a graph with $e \in E(G)$. Then we have*

$$ME(G - e) < ME(G).$$

Recall that the first Zagreb index of a graph G is defined ([12, 13]) as $M_1(G) = \sum_{v_i \in V(G)} d_i^2$. Some results of first Zagreb index can be seen in [5–7, 25]. For convenience, we let $\binom{a}{b} = 0$ for two positive integers a and b with $a < b$.

Lemma 2.3. *([15]) For any connected graph G with m edges, we have*

$$m(G, 2) = \binom{m}{2} + m - \frac{1}{2} M_1(G) \ .$$

Proof. Note that $\binom{1}{2} = 0$ for any pendent vertex v_p in the graph G. From the result in [15], we have

$$\begin{aligned} m(G, 2) &= \binom{m}{2} - \sum_{v_i \in V(G)} \binom{d_i}{2} \\ &= \binom{m}{2} - \frac{1}{2} \sum_{v_i \in V(G)} d_i^2 + \frac{1}{2} \sum_{v_i \in V(G)} d_i \\ &= \binom{m}{2} + m - \frac{1}{2} M_1(G) \ . \end{aligned}$$

■

For any integer m satisfying $n + 1 \leq m \leq 2n - 4$, we denote by $G_{n,m}$ a graph of order n and with m edges in which maximum degree is $n - 1$ and the second maximum degree is $m - n + 2$. The structure of graph $G_{n,m}$ can be seen in Fig. 1. Moreover, a graph $G'_{n,n+2}$ is shown in Fig. 2.

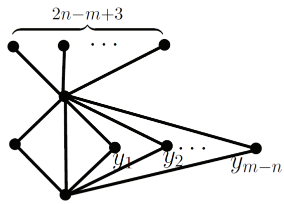

Figure 1: The graph $G_{n,m}$

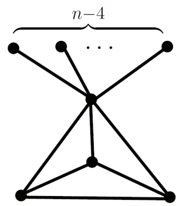

Figure 2: The graph $G'_{n,n+2}$ in $\mathcal{G}_{n,n+2}$

Lemma 2.4. ([7,27]) *For any graph $G \in \mathcal{G}_{n,m}$ with $n+1 \leq m \leq 2n-4$, we have*

$$M_1(G) \leq n(n-1) + (m-n+1)(m-n+6)$$

with equality holding if and only if $G \cong G_{n,m}$ for $m = n+1$ or $n+3 \leq m \leq 2n-4$; and $G \cong G_{n,m}$ or $G'_{n,n+2}$ for $m = n+2$.

3 Main results

In [15] the extremal (n,n)-graph maximizing the matching energy has been determined, which is just S_n^+ obtained by adding a new edge in a star S_n. In the next theorem we determine the extremal graph from $\mathcal{G}_{n,m}$ with $n+1 \leq m \leq 2n-3$ maximizing the matching energy, which can be viewed as a more general one of the above result for (n,n)-graphs.

Theorem 3.1. *For any graph $G \in \mathcal{G}_{n,m}$ with $n+1 \le m \le 2n-4$, we have*

$$\begin{aligned} ME(G) \;&\geq\; 2\left[\sqrt{\frac{m+\sqrt{m^2-4(m-n+1)(n-3)}}{2}}\right. \\ &+\left.\sqrt{\frac{m-\sqrt{m^2-4(m-n+1)(n-3)}}{2}}\right] \end{aligned}$$

with equality holding if and only if $G \cong G_{n,m}$ for $n+3 \le m \le 2n-4$ or $m = n+1$; and $G \cong G_{n,m}$ or $G'_{n,n+2}$ for $m = n+2$.

Proof. For any graph $G \in \mathcal{G}_{n,m}$, we have $m(G,0) = 1 = m(G_{n,m},0)$, $m(G,1) = m = m(G_{n,m},1)$ and $m(G,k) \ge 0 = m(G_{n,m},k)$ for $k \ge 3$. Next we should prove that, for any graph $G \in \mathcal{G}_{n,m}$,

$$m(G,2) \ge m(G_{n,m},2) \tag{6}$$

for $n+3 \le m \le 2n-4$ or $m = n+1$, and

$$m(G,2) \ge m(G_{n,m},2) = m(G'_{n,n+2},2) \tag{7}$$

for $m = n+2$.

By Lemmas 2.3 and 2.4, Eq.s (6) and (7) hold immediately.

Now the only task for proving this theorem is to compute the value of $ME(G_{n,m})$ for $n+1 \le m \le 2n-3$ and $ME(G'_{n,n+2})$. Thanks to Lemmas 2.3 and 2.4, again, we have

$$\begin{aligned} m(G_{n,m},2) &= \binom{m}{2} + m - \frac{1}{2}\left[n(n-1) + (m-n+1)(m-n+6)\right] \\ &= (m-n+1)(n-3)\ . \end{aligned}$$

Then the matching polynomial of $G_{n,m}$ is

$$\alpha(G_{n,m},\lambda) \;=\; \lambda^n - m\lambda^{n-2} + (m-n+1)(n-3)\lambda^{n-4}.$$

Thus the non-zero roots of $\alpha(G_{n,m},\lambda)$ are $\sqrt{\frac{m+\sqrt{m^2-4(m-n+1)(n-3)}}{2}}$ with twice and $\sqrt{\frac{m-\sqrt{m^2-4(m-n+1)(n-3)}}{2}}$ with twice. Therefore our result in this theorem follows. ■

From Theorem 3.1, the following corollary can be easily deduced.

Corollary 3.2. *([19]) Let G be an $(n,n+1)$-graph. Then we have*

$$ME(G) \;\geq\; 2\left[\sqrt{\frac{n+1+\sqrt{n^2-6n+25}}{2}} + \sqrt{\frac{n+1-\sqrt{n^2-6n+25}}{2}}\right]$$

with equality holding if and only if $G \cong G_{n,n+1}$.

Based on the relation (5), by a very similar reasoning as that in the proof of Theorem 3.1, the following corollary is straightforward.

Corollary 3.3. *([8, 22]) For any graph $G \in \mathcal{G}_{n,m}$ with $n+2 \le m \le 2n-4$, we have*

$$z(G) \;\geq\; m(n-2) - (n-1)(n-3) + 1$$

with equality holding if and only if $G \cong G_{n,m}$ for $n+3 \le m \le 2n-4$; and $G \cong G_{n,m}$ or $G'_{n,n+2}$ for $m = n+2$.

After obtaining the result in Theorem 3.1, we naturally ask the following problem: i*f matching number of graphs from $\mathcal{G}_{n,m}$ are given, what are the extremal graphs maximizing the matching energy under this condition?* Equivalently, which graph has the maximal matching energy among all graphs from $\mathcal{G}_{n,m}(\beta)$?

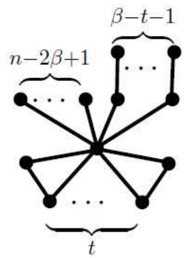

Figure 3: The graph $F_t(n, \beta)$

Before dealing with this problem, we first introduce some notations. Recall that *friendship graph* F_k is a graph of order $2k+1$ obtained from k triangles intersecting in a single vertex. An edge e in F_k is called *linking edge* if e is incident with the vertex of degree $2k+1$ in it. Denote by $F_t(n, \beta)$ (see Fig. 3) a graph obtained by attaching $n-2\beta+1$ pendent edges and $\beta-t-1$ paths of length 2 to the vertex of degree $2t+1$ in F_t. Clearly, we have $F_t(n, \beta) \in \mathcal{G}_{n,n-1+t}(\beta)$ with $1 \le t \le \beta-1$. A vertex v of a tree T is called a *branching point* if $d(v) \ge 3$. Let $T_n(n_1, n_2, \ldots, n_m)$ be the tree of order n obtained by inserting, respectively, $n_1 - 1, \ldots, n_m - 1$ vertices into the m edges of the star S_{m+1}, where $n_1 + \ldots + n_m = n-1$. For convenience, when considering the trees $Tn(n_1, n_2, \ldots n_k, \ldots, n_m)$ we use the symbols $n_k^{l_k}$ to indicate that the number of n_k is $l_k > 1$ in the following. For example, $T_{16}(2, 2, 3, 3, 5)$ will be written as $T_{16}(2^2, 3^2, 5)$.

Lemma 3.4. *([18]) Let T be a tree of order n and with matching number β. Then*

$$m(T,k) \ \geq \ m(T_n(2^{\beta-1},1^{n-2\beta+1}),k) \quad \text{for } k=0,1,\ldots,\beta$$

with all equalities holding if and only if $T \cong T_n(2^{\beta-1},1^{n-2\beta+1})$.

From the definition of quasi-order introduced in Section 1 and Formula (4), the following corollaries can be obtained immediately.

Corollary 3.5. *For any tree $T \in \mathcal{G}_{n,n-1}(\beta)$, we have*

$$ME(T) \geq ME(T_n(2^{\beta-1},1^{n-2\beta+1}))$$

with equality holding if and only if $T \cong T_n(2^{\beta-1},1^{n-2\beta+1})$.

Corollary 3.6. *Let G be a graph of order n and with matching number β. Then we have*

$$m(G,k) \ \geq \ m(\beta K_2 \cup (n-2\beta)K_1,k) \quad \text{for } k=0,1,\ldots,\beta$$

with all equalities holding if and only if $T \cong \beta K_2 \cup (n-2\beta)K_1$.

In the following we will prove a generalized result of Lemma 3.4.

Theorem 3.7. *For any graph $G \in \mathcal{G}_{n,n-1+t}(\beta)$ with $1 \leq t \leq \beta-1$, we have $ME(G) \geq ME(F_t(n,\beta))$ with equality holding if and only if $G \cong F_t(n,\beta)$.*

Proof. We prove this result by induction on t. Firstly we deal with the case when $t=1$. From the definition of the set $\mathcal{G}_{n,n-1+t}(\beta)$, we find that, for any graph $G \in \mathcal{G}_{n,n-1+t}(\beta)$ with $t=1$, there exists an edge $e=v_iv_j$ in a unique cycle of G such that $e \notin M$ where M is a maximum matching of G. Note that $G-e \in \mathcal{G}_{n,n-1}(\beta)$ and $G-\{v_i,v_j\}$ is with matching number $\beta-2$. In view of Lemma 2.1, for $k=0,1,2\ldots,\beta$, we have

$$\begin{aligned} m(G,k) &= m(G-e,k)+m(G-\{v_i,v_j\},k-1) \\ &\geq m(T_n(2^{\beta-1},1^{n-2\beta+1}),k)+m((\beta-2)K_2 \cup (n-2\beta+2)K_1,k-1) \\ &\qquad \text{by Lemma 3.4 and Corollary 3.6.} \end{aligned}$$

Similarly, by choosing $e=v_i'v_j'$ as an edge in the triangle incident with the vertex of maximum degree in $F_1(n,\beta)$, for $k=0,1,2\ldots,\beta$, we have

$$\begin{aligned} m(F_1(n,\beta),k) &= m(F_1(n,\beta)-e,k)+m(F_1(n,\beta)-\{v_i',v_j'\},k-1) \\ &= m(T_n(2^{\beta-1},1^{n-2\beta+1}),k)+m((\beta-2)K_2 \cup (n-2\beta+2)K_1,k-1). \end{aligned}$$

By the definition of quasi-order, we have $G \succeq F_1(n,\beta)$ for any graph $G \in \mathcal{G}_{n,n}(\beta)$ with equality holding if and only if $G \cong F_1(n,\beta)$. Thanks to Formula (4), again, our result holds for $t = 1$.

Assume that our result holds for any graph $G \in \mathcal{G}_{n,n-1+k}(\beta)$ with k fewer than $t \leq \beta - 1$. For any graph $G \in \mathcal{G}_{n,n-1+t}(\beta)$ with M as its β-matching, we choose an edge $e = v_iv_j \in E(G)$ in a cycle of G but not in M. By Lemma 2.1, Corollary 3.6 and induction hypothesis, for $k = 0, 1, 2 \ldots, \beta$, we have

$$\begin{aligned} m(G,k) &= m(G-e,k) + m(G-\{v_i,v_j\},k-1) \\ &\geq m(F_{t-1}(n,\beta),k) + m((\beta-2)K_2 \cup (n-2\beta+2)K_1, k-1) \\ &\qquad\qquad \text{since } G-e \in \mathcal{G}_{n,n-2+t}(\beta) \\ &= m(F_t(n,\beta)-e',k) + m(F_t(n,\beta)-\{v_i',v_j'\},k-1) \\ &\quad \text{where } e' = v_i'v_j' \in E(F_t(n,\beta)) \text{ is a linking egde of } F_t \text{ in it} \\ &= m(F_t(n,\beta),k). \end{aligned}$$

Moreover, the above equality holds if and only if $G - v_iv_j \cong F_{t-1}(n,\beta)$ and $G - \{v_i,v_j\} \cong (\beta-2)K_2 \cup (n-2\beta+2)K_1$, that is, $G \cong F_t(n,\beta)$. Therefore our result holds for $k = t$, finishing the proof of this theorem. ■

In view of the definition of Hosoya index and an efficient tool [11] to it: $z(G) = z(G-v_iv_j) + z(G-\{v_i,v_j\})$, we can obtain $z(T_n(2^{\beta-1}, 1^{n-2\beta+1})) = 2^{\beta-2}(2n-3m+3)$ [17] (by induction on β) and

$$\begin{aligned} z(F_t(n,\beta)) &= z(F_t(n,\beta)-v_iv_j) + z(G-\{v_i,v_j\}) \\ &\quad \text{where } e = v_iv_j \text{ is a linking egde of } F_t \text{ in } F_t(n,\beta) \\ &= z(F_{t-1}(n,\beta)) + z((\beta-2)K_2 \cup (n-2\beta+2)K_1) \\ &= z(F_{t-1}(n,\beta)-v_kv_j) + z(G-\{v_k,v_j\}) + 2^{\beta-2} \\ &\quad \text{where } e = v_kv_j \text{ is a linking egde of } F_t \text{ in } F_{t-1}(n,\beta) \\ &= z(F_{t-2}(n,\beta)) + z((\beta-2)K_2 \cup (n-2\beta+2)K_1) + 2^{\beta-2} \\ &= z(F_{t-2}(n,\beta)) + 2 \times 2^{\beta-2} \\ &\quad \cdots\cdots \\ &= z(T_n(2^{\beta-1}, 1^{n-2\beta+1})) + t2^{\beta-2} \end{aligned}$$

$$= 2^{\beta-2}(2n-3m+3)+t2^{\beta-2}$$
$$= 2^{\beta-2}(2n-3m+t+3)$$

Based on Lemma 2.3 and quasi-order with Formula (4), respectively, the following tow corollaries can be deduced immediately.

Corollary 3.8. *Let* $1 \le t \le \beta - 1$ *be a integer and* $G \in \mathcal{G}_{n,n-1+t}(\beta)$*. Then we have*

$$M_1(G) \le (n-\beta+t)^2+3(\beta+t)+n-4$$

with equality holding if and only if $G \cong F_t(n,\beta)$*.*

Corollary 3.9. *Let* $1 \le t \le \beta - 1$ *be a integer and* $G \in \mathcal{G}_{n,n-1+t}(\beta)$*. Then we have*

$$z(G) \ge 2^{\beta-2}(2n-3m+t+3)$$

with equality holding if and only if $G \cong F_t(n,\beta)$*.*

By now we have completely determined the extremal graphs from $\mathcal{G}_{n,m}$ with $n \le m \le 2n-4$ and $\mathcal{G}_{n,n-1+t}(\beta)$ with $1 \le t \le \beta - 1$, respectively, minimizing the matching energy. Naturally we will ask: *what graphs from these two sets have the maximal matching energy, respectively?* Furthermore, *how about this problem when only limiting the order* n *and matching number* β *for the connected graphs?* These problems are unknown to us, maybe they will be our research task in the future.

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References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, MacMillan, New York, 1976.

[2] L. Chen, Y. Shi, The maximal matching energy of tricyclic graphs, *MATCH Commun. Math. Comput. Chem.* **73** (2015) 105–119.

[3] D. M. Cvetković, M. Doob, I. Gutman, A. Torgašev, *Recent Results in the Theory of Graph Spectra*, North–Holland, Amsterdam, 1988, Chapter 4.

[4] D. M. Cvetković, P. Rowlinson, S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge Univ. Press, Cambridge, 2010.

[5] K. C. Das, Sharp bounds for the sum of the squares of the degrees of a graph, *Kragujevac J. Math.* **25** (2003) 31–49.

[6] K. C. Das, Maximizing the sum of the squares of degrees of a graph, *Discr. Math.* **257** (2004) 57–66.

[7] H. Deng, A unified approach to the extremal Zagreb indices for trees, unicyclic graphs and bicyclic graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 597–616.

[8] H. Deng, The smallest Hosoya index in $(n, n+1)$-graphs, *J. Math. Chem.* **43** (2008) 119–133.

[9] E. J. Farrell, An introduction to matching polynomials, *J. Combin. Theory B* **27** (1979) 75–86.

[10] C. D. Godsil, I. Gutman, On the theory of the matching polynomial, *J. Graph Theory* **5** (1981) 137–144.

[11] I. Gutman, O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin, 1986.

[12] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. III. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.

[13] I. Gutman, B. Ruščić, N. Trinajstić, C. F. Wilcox, Graph theory and molecular orbitals. XII. Acyclic polyenes, *J. Chem. Phys.* **62** (1975) 3399–3405.

[14] I. Gutman, The matching polynomial, *MATCH Commun. Math. Comput. Chem.* **6** (1979) 75–91.

[15] I. Gutman, S. Wagner, The matching energy of a graph, *Discr. Appl. Math.* **160** (2012) 2177–2187.

[16] H. Hosoya, Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, *Bull. Chem. Soc. Japan* **44** (1971) 2332–2339.

[17] Y. Hou, On acyclic systems with minimal Hosoya index, *Discr. Appl. Math.* **119** (2002) 251–257.

[18] Y. Hou, On trees with the least energy and a given size of matching, *J. Systems Sci. Math. Sci.* **23** (2003) 491–494. (in Chinese).

[19] S. Ji, X. Li, Y. Shi, Extremal matching energy of bicyclic graphs, *MATCH Commun. Math. Comput. Chem.* **70** (2013) 697–706.

[20] S. Li, W. Yan, The matching energy of graphs with given parameters, *Discr. Appl. Math.* **162** (2014) 415–420.

[21] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[22] X. Pan, Z. Sun, The (n, m)-graphs of minimum Hosoya index, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 811–820.

[23] S. Wagner, I. Gutman, Maxima and minima of the Hosoya index and the Merrifield–Simmons index: A survey of results and techniques, *Acta Appl. Math.* **112** (2010) 323–346.

[24] K. Xu, On the Hosoya index and the Merrifield–Simmons index of graphs with a given clique number, *Appl. Math. Lett.* **23** (2010) 395–398.

[25] K. Xu, The Zagreb indices of graphs with a given clique number, *Appl. Math. Lett.* **24** (2011) 1026–1030.

[26] K. Xu, The smallest Hosoya index of unicyclic graphs with given diameter, *Math. Commun.* **17** (2012) 221–239.

[27] K. Xu, K. C. Das, S. Balachandran, Maximizing the Zagreb indices of (n, m)-graphs, *MATCH Commun. Math. Comput. Chem.* **72** (2014) 641–654.

[28] K. Xu, I. Gutman, The greatest Hosoya index of bicyclic graphs with given maximum degree, *MATCH Commun. Math. Comput. Chem.* **66** (2011) 795–824.

[29] K. Xu, J. Li, L. Zhong, The Hosoya indices and Merrifield–Simmoms indices of graphs with connectivity at most k, *Appl. Math. Lett.* **25** (2012) 476–480.