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Note on Three Results on Randić Energy and Incidence Energy*

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Abstract

In this paper, we show that the proofs of Theorem 4 and Theorem 5 in "Randić spectral radius and Randić energy" by Ş. B. Bozkurt et al. [*MATCH Commun. Math. Comput. Chem.* **64** (2010), 321–334] are not correct. We modify their result and give another upper bound for Randić energy. We also show that the proof of Theorem 4.1 in "On incidence energy of graphs" by K. C. Das et al. [*Linear Algebra Appl.* **446** (2014), 329–344] is not correct.

1 Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. If two vertices v_i and v_j of G are adjacent, then we use the notation $v_i \sim v_j$. Let d_i denote the degree of a vertex v_i, where $i = 1, 2, \cdots, n$. In 1975, Milan Randić proposed the Randić index [13], which is defined as:

$$R(G) = \sum_{i\sim j} \frac{1}{\sqrt{d_i d_j}},$$

where the summation runs over all pairs of adjacent vertices v_i and v_j.

The *Randić matrix* of G is the $n \times n$ matrix $\mathbf{R} = \| R_{ij} \|$ defined as

$$R_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}} & \text{if } v_i \sim v_j \ , \\ 0 & \text{otherwise.} \end{cases}$$

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This matrix was studied [1, 2, 14, 15] in connection with the Randić index. The Randić matrix also plays an outstanding role in the Laplacian spectral theory, for details see [4,5]. The Randić eigenvalues $\rho_1, \rho_2, \ldots, \rho_n$ of a graph G are the eigenvalues of its Randić matrix $\mathbf{R}$. Since $\mathbf{R}$ is a real symmetric matrix, all its eigenvalues are real numbers and thus we can order them so that $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$. The greatest eigenvalue ρ_1 is called the *Randić spectral radius* of the graph G. The concept of Randić energy of a graph G, denoted by $RE(G)$, was introduced in [5] as:

$$RE = RE(G) = \sum_{i=1}^{n} |\rho_i|.$$

The incidence energy, invented by Jooyandeh et al. [11], based on the incidence matrix. The incidence matrix of G is the $n \times m$ matrix I whose (i, j)-entry is 1 if a vertex v_i is incident to an edge e_j and 0 otherwise. The incidence energy IE of a graph G is defined as the sum of the singular values of the incidence matrix I.

Let A be the $(0, 1)$-adjacency matrix of G and D be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L = D - A$. This matrix has nonnegative eigenvalues $n \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. The signless Laplacian matrix of G is $Q = D + A$. This matrix has nonnegative eigenvalues $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$. It is known that [6]

$$II^t = D + A = Q. \tag{1}$$

Recall that the singular values of a matrix M are the nonnegative square roots of MM^t or M^tM and that these matrices have the same nonzero eigenvalues. From these facts and (1), it follows that [9, 10]

$$IE = IE(G) = \sum_{i=1}^{n} \sqrt{q_i}.$$

Bozkurt et al. [4] got several lower bounds for the Randić spectral radius ρ_1, and then they gave a upper bound for Randić energy of a simple connected graph. But they made some errors in the proofs of Theorem 4 and Theorem 5 in [4]. And on the incidence energy of a graph, Das et al. [7] gave lower and upper bounds for it. However, the proof of Theorem 4.1 in [7] has problem.

In this note we show the errors in the proofs of Theorem 4 and Theorem 5 in [4] and modify their result to give another upper bound for Randić energy of a simple connected graph. Also, we show that the proof of Theorem 4.1 in [7] is not correct.

2 Auxiliary definitions

We follow the notation and terminology in [4], [7], as well as [3].

Definition 1. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and Randić matrix* $\mathbf{R}$*. Then the Randić degree of v_i, denoted by R_i, is defined as*

$$R_i = \sum_{j=1}^{n} R_{ij}.$$

G is called k-Randić regular if $R_i = k$, for all $i = 1, 2, \cdots, n$.

Definition 2. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and Randić matrix* $\mathbf{R}$*. Let its Randić degree sequence be $\{R_1, R_2, \ldots, R_n\}$. Then the second Randić degree of v_i, denoted by S_i, is defined as*

$$S_i = \sum_{j=1}^{n} R_{ij} R_j.$$

G is called pseudo k-Randić regular if $S_i/R_i = k$, for all $i = 1, 2, \cdots, n$.

For any graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, define a sequence ${L_i}^{(1)}$, ${L_i}^{(2)}, \cdots, {L_i}^{(p)}, \cdots$ for every vertex v_i, where $p \in \mathbb{N}$. Let ${L_i}^{(1)} = {R_i}^{\alpha}$, where $\alpha \in \mathbb{R}$, and for $p \geq 2$,

$${L_i}^{(p)} = \sum_{i \sim j} \frac{1}{\sqrt{d_i d_j}} {L_j}^{(p-1)}.$$

Definition 3. *[8] For a graph G, the first Zagreb index M_1 of G, denoted by $M_1(G)$, is the sum of squares of the vertex degrees of G.*

3 Error in the proof of Theorem 4 of [4]

In [4], the authors proved the following several lower bounds for ρ_1.

Theorem 1. *[4] Let G be a simple connected graph with n vertices and let $R(G)$ be its Randić index. Then*

$$\rho_1 \geq \frac{2R(G)}{n}. \tag{1}$$

The equality holds in (1) if and only if G is Randić regular.

Theorem 2. *[4] Let G be a simple connected graph with n vertices, and the vertex set be $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let R_i be the Randić degree of v_i. Then*

$$\rho_1 \geq \sqrt{\frac{1}{n} \sum_{i=1}^{n} {R_i}^2}. \tag{2}$$

The equality holds in (2) if and only if G is Randić regular.

Theorem 3. *[4] Let G be a simple connected graph, α be a real number, and p be an integer. Then*

$$\rho_1 \geq \sqrt{\frac{\sum_{i=1}^{n} \left(L_i{}^{(p+1)}\right)^2}{\sum_{i=1}^{n} \left(L_i{}^{(p)}\right)^2}}. \tag{3}$$

The equality holds in (3) if and only if

$$\frac{L_1{}^{(p+1)}}{L_1{}^{(p)}} = \frac{L_2{}^{(p+1)}}{L_2{}^{(p)}} = \cdots = \frac{L_n{}^{(p+1)}}{L_n{}^{(p)}}.$$

By setting $\alpha = 1$ and $p = 1$ in (3), Corollary 1 directly follows.

Corollary 1. *[4] Let G be a simple connected graph with n vertices, and the vertex set be $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let its Randić degree sequence be $\{R_1, R_2, \ldots, R_n\}$, the second Randić degree sequence be $\{S_1, S_2, \ldots, S_n\}$. Then*

$$\rho_1 \geq \sqrt{\frac{\sum_{i=1}^{n} S_i{}^2}{\sum_{i=1}^{n} R_i{}^2}}. \tag{4}$$

The equality holds in (4) if and only if G is pseudo k-Randić regular, for some k.

The authors proved the following theorem to compare those lower bounds for ρ_1.

Theorem 4. *(Theorem 4 of [4]) The lower bound for ρ_1 given in (3) improves the lower bounds given in (1), (2), and (4).*

We will show that their result on the comparison between the lower bounds given in (3) and (4) is not correct. They pointed out that for any fixed value $\alpha \in \mathbb{R}$ and $p \in \mathbb{N}$,

$$\rho_1 \geq \sqrt{\frac{\sum_{i=1}^{n} \left(L_i{}^{(p+1)}\right)^2}{\sum_{i=1}^{n} \left(L_i{}^{(p)}\right)^2}} \geq \sqrt{\frac{\sum_{i=1}^{n} S_i{}^2}{\sum_{i=1}^{n} R_i{}^2}}.$$

Unfortunately, the second inequality does not hold for all $\alpha \in \mathbb{R}$ and $p \in \mathbb{N}$. For example, let P_3 be the path with exactly three consecutive vertices v_1,v_2,v_3, we add a new vertex v_4 to P_3 and make v_4 adjacent to v_1 and v_2. Denote the new graph by G and its Randić matrix $\mathbf{R} = \| R_{ij} \|$. Clearly, we have $R_{12} = R_{21} = \frac{1}{\sqrt{6}}$, $R_{14} = R_{41} = \frac{1}{2}$, $R_{23} = R_{32} = \frac{1}{\sqrt{3}}$,

$R_{24} = R_{42} = \frac{1}{\sqrt{6}}$ and the rest $R_{ij} = 0$. From the definition of Randić degree, we have $R_1 = \frac{1}{\sqrt{6}} + \frac{1}{2}$, $R_2 = \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}$, $R_3 = \frac{1}{\sqrt{3}}$, $R_4 = \frac{1}{\sqrt{6}} + \frac{1}{2}$. Let us consider the case $p = 1$. Then

$$\sqrt{\frac{\sum\limits_{i=1}^{n}\left(L_i{}^{(p+1)}\right)^2}{\sum\limits_{i=1}^{n}\left(L_i{}^{(p)}\right)^2}} = \sqrt{\frac{\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} R_{ij}R_j{}^{\alpha}\right)^2}{\sum\limits_{i=1}^{n}(R_i{}^{\alpha})^2}},$$

$$\sqrt{\frac{\sum\limits_{i=1}^{n} S_i{}^2}{\sum\limits_{i=1}^{n} R_i{}^2}} = \sqrt{\frac{\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} R_{ij}R_j\right)^2}{\sum\limits_{i=1}^{n} R_i{}^2}}.$$

To compare $\sqrt{\frac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} R_{ij}R_j{}^{\alpha}\right)^2}{\sum_{i=1}^{n}(R_i{}^{\alpha})^2}}$ with $\sqrt{\frac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} R_{ij}R_j\right)^2}{\sum_{i=1}^{n} R_i{}^2}}$, we only need to consider whether $\frac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} R_{ij}R_j{}^{\alpha}\right)^2}{\sum_{i=1}^{n}(R_i{}^{\alpha})^2} - \frac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} R_{ij}R_j\right)^2}{\sum_{i=1}^{n} R_i{}^2}$ is not less than zero.

Let

$$h(\alpha) = \frac{\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} R_{ij}R_j{}^{\alpha}\right)^2}{\sum\limits_{i=1}^{n}(R_i{}^{\alpha})^2} - \frac{\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} R_{ij}R_j\right)^2}{\sum\limits_{i=1}^{n} R_i{}^2}.$$

Then, for this graph G we have

$$\begin{aligned} h(\alpha) \;=\; & [(R_{12}R_2{}^{\alpha} + R_{14}R_4{}^{\alpha})^2 + (R_{21}R_1{}^{\alpha} + R_{23}R_3{}^{\alpha} + R_{24}R_4{}^{\alpha})^2 + (R_{32}R_2{}^{\alpha})^2 \\ & +(R_{41}R_1 + R_{42}R_2)^2]/(R_1{}^{2\alpha} + R_2{}^{2\alpha} + R_3{}^{2\alpha} + R_4{}^{2\alpha}) \\ & -[(R_{12}R_2 + R_{14}R_4)^2 + (R_{21}R_1 + R_{23}R_3 + R_{24}R_4)^2 + (R_{32}R_2)^2 \\ & +(R_{41}R_1 + R_{42}R_2)^2]/(R_1{}^2 + R_2{}^2 + R_3{}^2 + R_4{}^2). \end{aligned}$$

With the aid of a computer, we figured out the value of $h(\alpha)$ when $\alpha \in [-10, 10]$. When $0 \leq \alpha \leq 1$ or $4.73018 \leq \alpha \leq 10$, we have $h(\alpha) \geq 0$, but when $1 < \alpha < 4.73018$ or $-10 < \alpha < 0$, $h(\alpha) < 0$. Hence the inequality $\sqrt{\frac{\sum_{i=1}^{n}\left(L_i{}^{(p+1)}\right)^2}{\sum_{i=1}^{n}\left(L_i{}^{(p)}\right)^2}} \geq \sqrt{\frac{\sum_{i=1}^{n} S_i{}^2}{\sum_{i=1}^{n} R_i{}^2}}$ does not hold for all $\alpha \in \mathbb{R}$ and $p \in \mathbb{N}$. That implies that the lower bound for ρ_1 given in (4) may not be better than the lower bounds given in (3). Thus, from the proof given in [4], we can only say that the lower bound for ρ_1 given in (4) improves the lower bounds given in (1) and (2).

4 Error in Theorem 5 of [4]

With the lower bounds for ρ_1, the authors proved the following theorem in [4].

Theorem 5. *(Theorem 5 of [4]) Let G be a simple connected graph. Then, for any fixed values of $\alpha \in \mathbb{R}$ and $p \in \mathbb{N}$,*

$$RE(G) \leq \sqrt{\frac{\sum_{i=1}^{n}\left(L_i^{(p+1)}\right)^2}{\sum_{i=1}^{n}\left(L_i^{(p)}\right)^2}} + \sqrt{(n-1)\left[2\sum_{i\sim j}\frac{1}{d_id_j} - \frac{\sum_{i=1}^{n}\left(L_i^{(p+1)}\right)^2}{\sum_{i=1}^{n}\left(L_i^{(p)}\right)^2}\right]}. \tag{1}$$

The equality holds if and only if G is a complete graph or a connected graph satisfying

$$\frac{L_1^{(p+1)}}{L_1^{(p)}} = \frac{L_2^{(p+1)}}{L_2^{(p)}} = \cdots = \frac{L_n^{(p+1)}}{L_n^{(p)}} = k \geq \sqrt{\frac{2\sum_{i\sim j}\frac{1}{d_id_j}}{n}}$$

with three distinct Randić eigenvalues

$$k,\ \sqrt{\frac{2\sum_{i\sim j}\frac{1}{d_id_j}-k^2}{n-1}}\ \textit{and}\ -\sqrt{\frac{2\sum_{i\sim j}\frac{1}{d_id_j}-k^2}{n-1}}.$$

We will show that this result is not correct. In the proof of Theorem 5, the authors first derived that

$$\sum_{i=2}^{n}|\rho_i| \leq \sqrt{(n-1)\sum_{i=2}^{n}\rho_i^2} = \sqrt{(n-1)\left(2\sum_{i\sim j}\frac{1}{d_id_j} - \rho_1^2\right)}. \tag{2}$$

Note that $\rho_1 \geq 0$, therefore,

$$RE(G) \leq \rho_1 + \sqrt{(n-1)\left(2\sum_{i\sim j}\frac{1}{d_id_j} - \rho_1^2\right)}.$$

By Cauchy-Schwarz inequality they got

$$R_i^2 = \left(\sum_{j=1}^{n}R_{ij}\right)^2 \leq n\sum_{j=1}^{n}R_{ij}^2.$$

So,

$$\sum_{i=1}^{n}R_i^2 \leq n\sum_{i=1}^{n}\sum_{j=1}^{n}R_{ij}^2 = 2n\sum_{i\sim j}\frac{1}{d_id_j}.$$

And

$$S_i = \sum_{j=1}^{n}R_{ij}R_j \geq \sum_{j=1}^{n}R_{ij}^2,$$

then they got

$$\sum_{i=1}^{n} {S_i}^2 \geq \sum_{i=1}^{n}\left(\sum_{j=1}^{n} {R_{ij}}^2\right)^2 \geq \left(2\sum_{i\sim j}\frac{1}{d_i d_j}\right)^2. \tag{3}$$

But, the second inequality in (8) does not hold. We can find a counterexample to the inequality $\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} {R_{ij}}^2\right)^2 \geq \left(2\sum\limits_{i\sim j}\frac{1}{d_i d_j}\right)^2$. Let G be a path with exactly three consecutive vertices v_1, v_2, v_3, denote its Randić matrix $\mathbf{R} =\| R_{ij} \|$. We have $R_{12} = R_{21} = \frac{1}{\sqrt{d_1 d_2}} = \frac{1}{\sqrt{2}}$, $R_{23} = R_{32} = \frac{1}{\sqrt{d_2 d_3}} = \frac{1}{\sqrt{2}}$, so $R_1 = R_3 = \frac{1}{\sqrt{2}}$, $R_2 = \sqrt{2}$, then we have

$$\sum_{i=1}^{3}\left(\sum_{j=1}^{3} {R_{ij}}^2\right)^2 = \left({R_{12}}^2\right)^2 + \left({R_{21}}^2 + {R_{23}}^2\right)^2 + \left({R_{32}}^2\right)^2 = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}+\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 = \tfrac{3}{2},$$

$$\left(2\sum_{i\sim j}\frac{1}{d_i d_j}\right)^2 = 4\left(\frac{1}{d_1 d_2} + \frac{1}{d_2 d_3}\right)^2 = 4\left(\frac{1}{2}+\frac{1}{2}\right)^2 = 4.$$

Thus, for this graph G we have $\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} {R_{ij}}^2\right)^2 < \left(2\sum\limits_{i\sim j}\frac{1}{d_i d_j}\right)^2$. Moreover,

$$\sum_{i=1}^{3} {S_i}^2 = (R_{12}R_2)^2 + (R_{21}R_1 + R_{23}R_3)^2 + (R_{32}R_2)^2 = 3 < 4.$$

So, (3) does not hold. Consequently, the authors derived that $\rho_1 \geq \sqrt{\frac{\sum_{i=1}^{n} {S_i}^2}{\sum_{i=1}^{n} {R_i}^2}} \geq \sqrt{\frac{2}{n}\sum\limits_{i\sim j}\frac{1}{d_i d_j}}$, is incorrect as well. And from Theorem 4, the authors proved that

$$\rho_1 \geq \sqrt{\frac{\sum\limits_{i=1}^{n}\left({L_i}^{(p+1)}\right)^2}{\sum\limits_{i=1}^{n}\left({L_i}^{(p)}\right)^2}} \geq \sqrt{\frac{\sum\limits_{i=1}^{n} {S_i}^2}{\sum\limits_{i=1}^{n} {R_i}^2}} \geq \sqrt{\frac{2}{n}\sum_{i\sim j}\frac{1}{d_i d_j}}. \tag{4}$$

Since we pointed out that both $\sqrt{\frac{\sum_{i=1}^{n}\left({L_i}^{(p+1)}\right)^2}{\sum_{i=1}^{n}\left({L_i}^{(p)}\right)^2}} \geq \sqrt{\frac{\sum_{i=1}^{n} {S_i}^2}{\sum_{i=1}^{n} {R_i}^2}}$ and $\sqrt{\frac{\sum_{i=1}^{n} {S_i}^2}{\sum_{i=1}^{n} {R_i}^2}} \geq \sqrt{\frac{2}{n}\sum\limits_{i\sim j}\frac{1}{d_i d_j}}$ do not hold, (4) is also incorrect.

Let f be the function given by

$$f(x) = x + \sqrt{(n-1)\left(2\sum_{i\sim j}\frac{1}{d_i d_j} - x^2\right)}.$$

Since f monotonically decreases for $x \geq \sqrt{\frac{2}{n}\sum\limits_{i\sim j}\frac{1}{d_i d_j}}$, the authors obtained that

$$RE(G) \leq f(\rho_1) \leq f\left(\sqrt{\frac{\sum\limits_{i=1}^{n}\left({L_i}^{(p+1)}\right)^2}{\sum\limits_{i=1}^{n}\left({L_i}^{(p)}\right)^2}}\right).$$

The above result is not correct because they did not get the correct range for ρ_1. And that leads to the incorrectness of Theorem 5.

In fact, we can modify the proof of Theorem 5 to get another result. First, we modify inequality (3). By the Cauchy-Schwarz inequality, we can get that

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n}{R_{ij}}^2\right)^2 \geq \frac{\left(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}{R_{ij}}^2\right)^2}{n}.$$

So,

$$\sum_{i=1}^{n}{S_i}^2 \geq \sum_{i=1}^{n}\left(\sum_{j=1}^{n}{R_{ij}}^2\right)^2 \geq \frac{\left(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}{R_{ij}}^2\right)^2}{n} = \frac{\left(2\sum\limits_{i\sim j}\frac{1}{d_id_j}\right)^2}{n}.$$

Hence,

$$\rho_1 \geq \sqrt{\frac{\sum\limits_{i=1}^{n}{S_i}^2}{\sum\limits_{i=1}^{n}{R_i}^2}} \geq \sqrt{\frac{2}{n^2}\sum_{i\sim j}\frac{1}{d_id_j}}. \tag{5}$$

Then, consider the monotonicity of function $f(x)$ where $\sqrt{\frac{2}{n^2}\sum\limits_{i\sim j}\frac{1}{d_id_j}} \leq x \leq \sqrt{2\sum\limits_{i\sim j}\frac{1}{d_id_j}}$. Since f monotonically increases for $\sqrt{\frac{2}{n^2}\sum\limits_{i\sim j}\frac{1}{d_id_j}} \leq x \leq \sqrt{\frac{2}{n}\sum\limits_{i\sim j}\frac{1}{d_id_j}}$ and monotonically decreases for $x \geq \sqrt{\frac{2}{n}\sum\limits_{i\sim j}\frac{1}{d_id_j}}$, from which follows that

$$RE(G) \leq f\left(\rho_1\right) \leq f\left(\sqrt{\frac{2}{n}\sum_{i\sim j}\frac{1}{d_id_j}}\right) = \sqrt{2n\sum_{i\sim j}\frac{1}{d_id_j}}. \tag{6}$$

Now let us suppose that the equality holds in (6). Then we have

$$\rho_1 = \sqrt{\frac{2}{n}\sum_{i\sim j}\frac{1}{d_id_j}}.$$

And from (2),

$$|\rho_i| = \sqrt{\frac{\left(2\sum\limits_{i\sim j}\frac{1}{d_id_j} - {\rho_1}^2\right)}{(n-1)}} = \sqrt{\frac{2}{n}\sum_{i\sim j}\frac{1}{d_id_j}} = \rho_1,$$

for $i = 2, 3, \cdots, n$. So G has at most two distinct Randić eigenvalues.

If G has only one Randić eigenvalue, since $\sum\limits_{i=1}^{n}\rho_i = 0$ and G is connected, $G \cong K_1$. If G has exactly two distinct Randić eigenvalues, then by Lemma 2 in [4], G is complete, i.e., $G \cong K_n$. So we have $\sum\limits_{i\sim j} \frac{1}{d_i d_j} = \frac{n(n-1)}{2} \cdot \frac{1}{(n-1)^2} = \frac{n}{2(n-1)}$, then $f\left(\sqrt{\frac{2}{n}\sum\limits_{i\sim j}\frac{1}{d_i d_j}}\right) = \frac{n}{\sqrt{n-1}}$. On the other hand, by directly calculating, $RE(K_n) = 2$ for $n \geq 2$. Hence, only when $G \cong K_2$, $f\left(\sqrt{\frac{2}{n}\sum\limits_{i\sim j}\frac{1}{d_i d_j}}\right) = RE(G)$. Conversely, one can easily see that the equality in (6) holds for the graph $G \cong K_1$ or K_2.

Summarizing all the arguments above, we have the following modified version of Theorem 5:

Theorem 6. *Let G be a simple connected graph with n vertices. Then*

$$RE(G) \leq \sqrt{2n \sum_{i\sim j} \frac{1}{d_i d_j}}.$$

The equality holds if and only if $G \cong K_1$ or $G \cong K_2$.

Actually, Theorem 6 is a special case of Theorem 4 given in [5], which was obtained by a different technique.

5 Error in the proof of Theorem 4.1 of [7]

In [7], the authors gave the following upper bound for the incidence energy IE of a connected graph in terms of n, m and first Zagreb index M_1.

Theorem 7. *(Theorem 4.1 of [7]) Let G be a connected graph of order n with m edges. Then*

$$IE(G) \leq \sqrt{\frac{4m}{n}} + (n-1)^{\frac{3}{4}}(M_1(G) + 2m - \frac{16m^2}{n^2})^{\frac{1}{4}}. \tag{1}$$

Equality holds if and only if $G \cong K_n$.

We will show that the method used in the proof of the above theorem in [7] is not correct. In the proof of Theorem 7, the authors first obtained that

$$\sum_{i=1}^{n} q_i^2 = \sum_{i=1}^{n} d_G(v_i)[d_G(v_i) + 1] = M_1 + 2m.$$

Then, they considered two cases according to whether G is bipartite or not. When G is nonbipartite, since G is also connected, then $q_i > 0,\ i = 1, 2, \ldots, n$. By applying Jensen's inequality, they obtained

$$(\frac{1}{n-1}\sum_{i=2}^{n}\sqrt{q_i})^4 \leq \frac{1}{n-1}\sum_{i=2}^{n} q_i^2 = \frac{M_1 + 2m - q_1^2}{n-1},$$

that is,

$$\sum_{i=2}^{n} \sqrt{q_i} \leq (n-1)^{\frac{3}{4}}(M_1 + 2m - q_1^2)^{\frac{1}{4}}, \tag{2}$$

with equality holds if and only if $q_2 = q_3 = \cdots = q_n$. This implies

$$IE(G) = \sqrt{q_1} + \sum_{i=2}^{n} \sqrt{q_i} \leq \sqrt{q_1} + (n-1)^{\frac{3}{4}}(M_1(G) + 2m - q_1^2)^{\frac{1}{4}}. \tag{3}$$

When G is bipartite, since G is connected, then $q_n = 0$ and $q_i > 0$, for $i = 1, 2, \ldots, n-1$. Instead of (2) they had

$$\sum_{i=2}^{n} \sqrt{q_i} \leq (n-2)^{\frac{3}{4}}(M_1(G) + 2m - q_1^2)^{\frac{1}{4}} < (n-1)^{\frac{3}{4}}(M_1(G) + 2m - q_1^2)^{\frac{1}{4}}.$$

So, they again arrived at the inequality (3). Then, they defined an auxiliary function

$$f(x) = \sqrt{x} + (n-1)^{\frac{3}{4}}(M_1(G) + 2m - x^2)^{\frac{1}{4}},$$

whose first derivative is

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{(n-1)^{\frac{3}{4}}x}{2(M_1 + 2m - x^2)^{\frac{3}{4}}}.$$

Since

$$nq_1^2 \geq \sum_{i=1}^{n} q_i^2 = M_1(G) + 2m,$$

they verified that $f'(x) \leq 0$, i.e., $f(x)$ is a decreasing function, and then the maximum is achieved at $x = \frac{4m}{n}$, i.e., $f(x) \leq f(\frac{4m}{n})$.

Actually, this would imply that $f'(x) \leq 0$ in the interval $[\frac{4m}{n}, q_1]$, i.e., $f(x)$ is decreasing in this interval. But, this is not true in general. In the following, we will give a counterexample to show that $f'(x) \leq 0$ does not hold in $[\frac{4m}{n}, q_1]$. Let G be a star S_n, where $n \geq 13$. We have $m = n - 1$. Then

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{(n-1)^{\frac{3}{4}}x}{2(M_1 + 2m - x^2)^{\frac{3}{4}}} = \frac{(M_1(G) + 2m - x^2)^{\frac{3}{4}} - [(n-1)x^2]^{\frac{3}{4}}}{2\sqrt{x}(M_1(G) + 2m - x^2)^{\frac{3}{4}}}.$$

When $n \geq 13$, we find that $f'(\frac{4m}{n}) > 0$, and thus, $f(x)$ is no longer a decreasing function. In the proof of Theorem 7, the authors used the fact that $f(x)$ is a decreasing function to obtain the result of Theorem 7. So the proof of Theorem 7 is not correct.

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