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Laplacian Spectrum, Laplacian–Energy–Like Invariant, and Kirchhoff Index of Graphs Constructed by Adding Edges on Pendent Vertices

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Abstract

Let G be a simple, undirected, connected and finite graph. For $i = 1, \ldots, r$, let K_{i,s_i} be a star in G with $s_i > 1$ and let H_i be an arbitrary graph of order s_i. Let $G(H_1, \ldots, H_r)$ be the graph obtained from G and the graphs $H_1, \ldots, H_r$ by identifying the vertices of H_i with the pendent vertices of K_{1,s_i}. It is proved that (i) if $\mu \neq 0$ and $\mu \neq 1$ is a Laplacian eigenvalue of G, then μ is a Laplacian eigenvalue of $G(H_1, \ldots, H_r)$ and (ii) for $1 \leq i \leq r$, if μ is a Laplacian eigenvalue of H_i, $\mu \neq 0$ or $\mu = 0$ with an eigenvector orthogonal to the all ones vector, then $1 + \mu$ is a Laplacian eigenvalue of $G(H_1, \ldots, H_r)$. Let $LEL(G)$ and $Kf(G)$ be the Laplacian–energy–like invariant and the Kirchhoff index of G, respectively. The above results are used to find the differences $LEL(G(H_1, \ldots, H_r)) - LEL(G)$ and $Kf(G(H_1, \ldots, H_r)) - Kf(G)$. These differences do not depend on the edges between non-pendent vertices of G.

1 Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. The Laplacian matrix of G is the $n \times n$ matrix $L(G) = D(G) - A(G)$ where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix of vertex degrees. It is well known that $L(G)$ is a positive semidefinite matrix and that $(0, \mathbf{e}_n)$ is an eigenpar of $L(G)$ where $\mathbf{e}_n$ is the corresponding all ones vector of n components. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues of

G. Fiedler [5] proved that G is a connected graph if and only if the second smallest eigenvalue Laplacian of G is positive. This eigenvalue is called algebraic connectivity of G and is denoted by $a(G)$. The largest Laplacian eigenvalue of G is called the Laplacian spectral radius of G.

We recall the following result.

Theorem 1. [15], *Corollary 4.2. Let G be a connected graph on n vertices. Suppose that $v_1, v_2, \ldots, v_s$ are s pendent vertices of G adjacent to a common vertex v. Let $\widetilde{G}$ be a graph obtained from G by adding any t, $0 \leq t \leq \frac{s(s-1)}{2}$, edges among $v_1, v_2, \ldots, v_s$. If $a(G) \neq 1$ then $a(\widetilde{G}) = a(G)$.*

A vertex of degree 1 is a called a pendent vertex. Theorem 1 tell us that if $a(G) \neq 1$ then $a(G)$ is also the algebraic connectivity of the graph obtained by adding edges between some of the pendent vertices adjacent to a common vertex in G. The next theorem tell us that the corresponding result also holds for the Laplacian spectral radius.

Theorem 2. [6], *Theorem 2.3. Let G be a connected graph on n vertices. Suppose that $v_1, v_2, \ldots, v_s$ are s pendent vertices of G adjacent to a common vertex v. Let $\widetilde{G}$ be a graph obtained from G by adding any t, $0 \leq t \leq \frac{s(s-1)}{2}$, edges among $v_1, v_2, \ldots, v_s$. Then the Laplacian spectral radius of G is also the Laplacian spectral radius of $\widetilde{G}$.*

A vertex in a graph is called a quasi-pendent vertex if it is adjacent to a pendent vertex. Let $p(G)$ and $q(G)$ be the number of pendent vertices and quasi-pendent vertices of a graph G, respectively. We recall the following result, due to I. Faria [4], concerning a lower bound for the multiplicity of 1 as a Laplacian eigenvalue of a graph.

Theorem 3. *For any graph G,*

$$m_G(1) \geq p(G) - q(G)$$

where $m_G(1)$ denotes the multiplicity of 1 as a Laplacian eigenvalue of G.

As usual, $K_{1,s}$ denotes a star on $s+1$ vertices.

Definition 1. *Let G be a connected graph, possesing a vertex to which s pendent vertices are attached, $s > 1$. Let H be an arbitrary graph of order s. Then $G(H)$ denotes the graph obtained from G and H by identifying the vertices of H with the*

s pendent vertices above mentioned. That is, $G(H)$ is the graph with vertex set $V(G(H)) = V(G)$ and edge set $E(G(H)) = E(G) \cup E(H)$.

Definition 2. *Let G is a connected graph, possesing vertices, $i = 1, \ldots, r$, to which $s_i > 1$ pendent vertices are attached. Let H_i be an arbitrary graph of order s_i. Then $G(H_1, \ldots, H_r)$ denotes the graph obtained from G and the graphs $H_1, \ldots, H_r$ by identifying the vertices of H_i with the s_i pendent vertices above mentioned. That is, $G(H_1, \ldots, H_r)$ is the graph with vertex set $V(G(H_1, \ldots, H_r)) = V(G)$ and edge set $E(G(H_1, \ldots, H_r)) = E(G) \cup \cup_{i=1}^{r} E(H_i)$.*

The graph H and the graphs H_i in Definitions 1 and 2 are not necessarily connected graphs. Thus, if H or H_i is a no-connected graph then 0 is not a simple Laplacian eigenvalue of H or H_i, respectively.

We prove that (i) if $\mu \neq 0$ and $\mu \neq 1$ is a Laplacian eigenvalue of G then μ is a Laplacian eigenvalue of $G(H_1, \ldots, H_r)$ and (ii) for $1 \leq i \leq r$, if μ is a Laplacian eigenvalue of H_i, $\mu \neq 0$ or $\mu = 0$ with an eigenvector orthogonal to the corresponding all ones vector, then $1 + \mu$ is a Laplacian eigenvalue of $G(H_1, \ldots, H_r)$.

We observe that (i) generalizes Theorems 1 and 2. From Theorem 3, since $s_i > 1$, 1 is a Laplacian eigenvalue of G. On the other hand, (ii) says that 1 is also a Laplacian eigenvalue of $G(H_1, \ldots, H_r)$ whenever $\mu = 0$ is a Laplacian eigenvalue of H_i with an eigenvector orthogonal to the corresponding all ones vector. Moreover, if $\mu \neq 0$ is a Laplacian eigenvalue of H_i, (ii) says that $1 + \mu$ is a Laplacian eigenvalue of $G(H_1, \ldots, H_r)$.

For the connected graph G, the Laplacian-energy-like invariant of G is

$$LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i(G)}$$

and the Kirchhoff index of G is

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i(G)}$$

where

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$$

are the Laplacian eigenvalues of G.

For $1 \leq i \leq r$, let

$$\mu_1(H_i) \geq \mu_2(H_i) \geq \cdots \geq \mu_{s_i-1}(H_i) \geq \mu_{s_i}(H_i) = 0$$

be the Laplacian eigenvalues of the graphs H_i in Definition 2. In addition, we prove that

$$LEL(G(H_1,\ldots,H_r)) - LEL(G) = \sum_{i=1}^{r}\sum_{j=1}^{s_i}\sqrt{1+\mu_j\left(H_i\right)} - \sum_{i=1}^{r} s_i$$

and

$$Kf(G(H_1,\ldots,H_r)) - Kf(G) = n\left(\sum_{i=1}^{r}\sum_{j=1}^{s_i}\frac{1}{1+\mu_j(H_i)} - \sum_{i=1}^{r} s_i\right).$$

We see that the differences $LEL(G(H_1,\ldots,H_r)) - LEL(G)$ and $Kf(G(H_1,\ldots,H_r)) - Kf(G)$ do not depend on the edges between non-pendent vertices of G. That is, if $\widetilde{G} = G+e$ is obtained from G by adding an edge e between two non-pendent vertices of G then

$$LEL(G(H_1,\ldots,H_r)) - LEL(G) = LEL(\widetilde{G}(H_1,\ldots,H_r)) - LEL(\widetilde{G})$$

and

$$Kf(G(H_1,\ldots,H_r)) - Kf(G) = Kf(\widetilde{G}(H_1,\ldots,H_r)) - Kf(\widetilde{G})$$

2 Laplacian spectrum after adding edges to pendent vertices

A^T denotes the transpose of A, $|A|$ is the determinant of a square matrix A, I_s is the identity matrix of order s and $\mathbf{e}_s$ is the all ones column vector of size s.

In this section, H is a graph of order s with Laplacian eigenvalues

$$\mu_1\left(H\right) \geq \mu_2\left(H\right) \geq \cdots \geq \mu_{s-1}\left(H\right) \geq \mu_s\left(H\right) = 0$$

where $\mu_s\left(H\right) = 0$ is the Laplacian eigenvalue with eigenvector $\mathbf{e}_s$.

Lemma 1. *Let $K_{1,s}(H)$ as in Definition 1. Then the characteristic polynomial of $K_{1,s}(H)$ is*

$$|\lambda I - L(K_{1,s}(H))| = \lambda\left(\lambda - (s+1)\right)P_H\left(\lambda\right) \tag{1}$$

where

$$P_H\left(\lambda\right) = \prod_{i=1}^{s-1}\left(\lambda - \left(1+\mu_i(H)\right)\right) \tag{2}$$

Proof. We label the vertices of $K_{1,s}(H)$ with $1, 2, \ldots, s, s+1$ where $1, \ldots, s$ are used for the pendents vertices of $K_{1,s}$ and $s+1$ is used for its root. With this labeling $L(K_{1,s}(H))$ becomes

$$L\left(K_{1,s}(H)\right) = \begin{bmatrix} L\left(H\right) + I_s & -\mathbf{e}_s \\ -\mathbf{e}_s^T & s \end{bmatrix}.$$

Let $\mu = \mu_i(H)$, $1 \leq i \leq s-1$. There exists $\mathbf{x} \neq \mathbf{0}$ such that $L\left(H\right)\mathbf{x} = \mu\mathbf{x}$ with $\mathbf{e}_s^T\mathbf{x} = 0$. Hence

$$\begin{bmatrix} L\left(H\right) + I_s & -\mathbf{e}_s \\ -\mathbf{e}_s^T & s \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix} = \begin{bmatrix} \mu\mathbf{x} + \mathbf{x} \\ 0 \end{bmatrix} = (\mu + 1) \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}.$$

Then $\mu_1\left(H\right)+1, \mu_2\left(H\right)+1, \ldots, \mu_{s-1}\left(H\right)+1$ are eigenvalues of $L\left(K_{1,s}(H)\right)$. Moreover,

$$\begin{bmatrix} L\left(H\right) + I_s & -\mathbf{e}_s \\ -\mathbf{e}_s^T & s \end{bmatrix} \begin{bmatrix} -\mathbf{e}_s \\ s \end{bmatrix} = \begin{bmatrix} -\mathbf{e}_s - s\mathbf{e}_s \\ s + s^2 \end{bmatrix} = (s + 1) \begin{bmatrix} -\mathbf{e}_s \\ s \end{bmatrix}.$$

Then $s+1$ is also an eigenvalue of $L\left(K_{1,s}(H)\right)$. Therefore the characteristic polynomial of $L\left(K_{1,s}(H)\right)$ is given by (1) with $P_H(\lambda)$ as in (2). $\square$

Let I and E be the identity matrix and the matrix whose entries are zeros except for the entry in the last row and last column which is 1, respectively. The sizes of these matrices will be clear from the context where they occur. Moreover, $\underline{A}$ denotes the submatrix obtained from A by deleting its last row and its last column.

We recall the following fact. For square matrices A and B, we have

$$\begin{vmatrix} A & E \\ E^T & B \end{vmatrix} = |A|\,|B| - |\underline{A}|\,|\underline{B}|\,. \tag{3}$$

In addition, we recall that the complement of the complete graph K_n is the null graph, that is, $V(\overline{K_n}) = V(K_n)$ and $E(\overline{K_n}) = \emptyset$.

Theorem 4. *Let $G(H)$ as in Definition 1. Then*

$$|\lambda I - L(G(H))| = \lambda P_H\left(\lambda\right) R(\lambda) \tag{4}$$

and

$$|\lambda I - L(G)| = \lambda(\lambda - 1)^{s-1} R(\lambda). \tag{5}$$

where $P_H(\lambda)$ is the polynomial in (2) and $R(\lambda)$ is a polynomial of degree $n - s$ such that $R(0) \neq 0$.

Proof. Let v be the root of $K_{1,s}$. Let $d(v)$ be the degree of v as a vertex of G. Observe that $d(v)$ is also the degree of v as a vertex of $G(H)$. Then $d(v) = s + t$ where t is the number of the non-pendent vertices of G adjacent to v. We label the vertices of G and $G(H)$ as follows: the labels $1, 2, \ldots, s$ are used for the pendent vertices of $K_{1,s}$, the label $s + 1$ is used for v and the labels $s + 2, \ldots, n - (t - 1), \ldots, n$ are used for the rest of the vertices in which $n - (t - 1), \ldots, n$ are the labels for the non-pendent vertices adjacent to v. With this labeling

$$L(G) = \begin{bmatrix} A & F \\ F^T & B \end{bmatrix}$$

and

$$L(G(H)) = \begin{bmatrix} C & F \\ F^T & B \end{bmatrix} \tag{6}$$

where A and C are matrices of order $(s + 1) \times (s + 1)$ given by

$$A = \begin{bmatrix} I_s & -\mathbf{e}_s \\ -\mathbf{e}_s^T & s + t \end{bmatrix}$$

and

$$C = \begin{bmatrix} L(H) + I_s & -\mathbf{e}_s \\ -\mathbf{e}_s^T & s + t \end{bmatrix}. \tag{7}$$

The matrices B and F are the same for $L(G)$ and $L(G(H))$. The matrix F of order $(s + 1) \times (n - s - 1)$ has entries equal to 0 except for the entries

$$(s + 1, n - t + 1), (s + 1, n - t + 2), \ldots, (s + 1, n)$$

all of them equal to -1. From (6) the characteristic polynomial of $L(G(H))$ is

$$|\lambda I - L(G(H))| = \begin{vmatrix} \lambda I - C & -F \\ -F^T & \lambda I - B \end{vmatrix}.$$

Subtracting the last column from the $(t - 1)-$ precedent columns and then, in the resultant matrix, subtracting the last row from the $(t - 1)-$ precedent rows, we obtain

$$|\lambda I - L(G(H))| = \begin{vmatrix} \lambda I - C & E \\ E^T & \lambda I - D \end{vmatrix}$$

for some matrix D and a matrix E having entries equal to 0 except for the entry in the last row and last column which is 1. We use (3) to get

$$|\lambda I - L(G(H))| = |\lambda I - C| \, |\lambda I - D| - |\lambda I - \underline{C}| \, |\lambda I - \underline{D}|.$$

From (7), we have

$$|\lambda I - C| = \begin{vmatrix} \lambda I_s - (L(H) + I_s) & \mathbf{e}_s \\ \mathbf{e}_s^T & \lambda - s - t \end{vmatrix}.$$

By linearity on the last column, we have

$$\begin{aligned} |\lambda I - C| &= \begin{vmatrix} \lambda I_s - (L(H) + I_s) & \mathbf{e}_s + 0 \\ \mathbf{e}_s^T & \lambda - s + (-t) \end{vmatrix} \\ &= \begin{vmatrix} \lambda I_s - (L(H) + I_s) & \mathbf{e}_s \\ \mathbf{e}_s^T & \lambda - s \end{vmatrix} + \begin{vmatrix} \lambda I_s - (L(H) + I_s) & 0 \\ \mathbf{e}_s^T & (-t) \end{vmatrix}. \end{aligned}$$

Hence

$$|\lambda I - C| = |\lambda I_{s+1} - L(K_{1,s}(G))| + (-t)\,|\lambda I_s - (L(H) + I_s)|\,.$$

Applying Lemma 1, we obtain

$$\begin{aligned} |\lambda I - C| &= \lambda(\lambda - (s+1))\,P_H(\lambda) - t(\lambda - 1)\,P_H(\lambda) \\ &= P_H(\lambda)\,[\lambda(\lambda - s - 1) - t(\lambda - 1)] \end{aligned}$$

where $P_H(\lambda)$ is the polynomial in (2). Moreover

$$|\lambda I - \underline{C}| = |\lambda I_s - (L(H) + I_s)| = (\lambda - 1)\,P_H(\lambda)\,.$$

Replacing these identities in the above expression of $|\lambda I - L(G(H))|$ and then factoring, we get

$$\begin{aligned} & |\lambda I - L(G(H))| \\ = \; & P_H(\lambda)\,[(\lambda(\lambda - s - 1) - t(\lambda - 1))\,|\lambda I - D| - (\lambda - 1)\,|\lambda I - \underline{D}|]. \end{aligned}$$

Let

$$T(\lambda) = (\lambda(\lambda - s - 1) - t(\lambda - 1))\,|\lambda I - D| - (\lambda - 1)\,|\lambda I - \underline{D}|\,.$$

Then

$$|\lambda I - L(G(H))| = P_H(\lambda)\,T(\lambda). \tag{8}$$

Observe that the polynomial $T(\lambda)$ does not depend on H. If $H = \overline{K_s}$, the null graph, then $G(H) = G$ and $P_{\overline{K_s}}(\lambda) = (\lambda - 1)^{s-1}$. Hence taking $H = \overline{K_s}$ in (8), we obtain

$$|\lambda I - L(G)| = (\lambda - 1)^{s-1}T(\lambda). \tag{9}$$

Since G is a connected graph, 0 is a simple eigenvalue of $L(G)$. From (9), it follows that $T(\lambda) = \lambda R(\lambda)$, where $R(\lambda)$ is a polynomial of degree $n - s$ such that $R(0) \neq 0$. Hence $|\lambda I - L(G(H))| = \lambda P_H(\lambda) R(\lambda)$ and $|\lambda I - L(G)| = \lambda(\lambda - 1)^{s-1} R(\lambda)$, and the proof is complete. □

Theorem 5. *Let $G(H)$ as in Definition 1.*

(i) If $\mu \neq 0$ and $\mu \neq 1$ is a Laplacian eigenvalue of G then μ is a Laplacian eigenvalue of $G(H)$, and

(ii) if μ is a Laplacian eigenvalue of H, $\mu \neq 0$ or $\mu = 0$ with an eigenvector orthogonal to the corresponding all ones vector, then $1 + \mu$ is a Laplacian eigenvalue of $G(H)$.

Proof. (i) Let $\mu \neq 0$ and $\mu \neq 1$ be a Laplacian eigenvalue of G. From (5), we get $R(\mu) = 0$. Then, replacing in (4), we have $|\mu I - L(G(H))| = 0$.

(ii) Let μ be a Laplacian eigenvalue of H, $\mu \neq 0$ or $\mu = 0$ with an eigenvector orthogonal to the corresponding all ones vector. Then $1 + \mu$ is a zero of the polynomial $P_H(\lambda)$ in (2), that is, $P_H(1 + \mu) = 0$. Replacing in (4), we obtain $|(1 + \mu)I - L(G(H))| = 0$. □

Theorem 6. *Let $G(H_1, \ldots, H_r)$ as in Definition 2.*

(i) If $\mu \neq 0$ and $\mu \neq 1$ is a Laplacian eigenvalue of G then μ is a Laplacian eigenvalue of $G(H_1, \ldots, H_r)$, and

(ii) if μ is a Laplacian eigenvalue of H_i, $1 \leq i \leq r$, $\mu \neq 0$ or $\mu = 0$ with an eigenvector orthogonal to the corresponding all ones vector, then $1 + \mu$ is a Laplacian eigenvalue of $G(H_1, \ldots, H_r)$.

Proof. We apply induction on r. The case $r = 1$ is given in Theorem 5. Let $r > 1$. Suppose that the theorem holds for $G(H_1, \ldots, H_{r-1})$ and that $G(H_1, \ldots, H_{r-1}, H_r)$ is the graph obtained from $G(H_1, \ldots, H_{r-1})$ and H_r by identifying the vertices of H_r with the pendent vertices of K_{1,s_r}. The below Theorem 5 will be applied to $G = G(H_1, \ldots, H_{r-1})$ with $H = H_r$.

(i) Let $\mu \neq 0$ and $\mu \neq 1$ be a Laplacian eigenvalue of G. By the induction hypothesis, μ is a Laplacian eigenvalue of $G(H_1, \ldots, H_{r-1})$. From Theorem 5, it follows that μ is a Laplacian eigenvalue of $G(H_1, \ldots, H_r)$.

(ii) Let μ be a Laplacian eigenvalue of H_i, $1 \leq i \leq r - 1$, $\mu \neq 0$ or $\mu = 0$ with an eigenvector orthogonal to the corresponding all ones vector. By the induction hypothesis, $1+\mu$ is a Laplacian eigenvalue of $G(H_1,\ldots,H_{r-1})$. Using Theorem 5, part (i) when $\mu\neq 0$ or part (ii) when $\mu=0$, we conclude that $1+\mu$ is a Laplacian eigenvalue of $G(H_1,\ldots,H_r)$. Let now μ be a Laplacian eigenvalue of H_r, $\mu\neq 0$ or $\mu=0$ with an eigenvector orthogonal to the corresponding all ones vector. Again, we apply Theorem 5 to obtain that $1+\mu$ is a Laplacian eigenvalue of $G(H_1,\ldots,H_r)$. $\square$

3 Laplacian–energy–like invariant and Kirchhoff index of graphs constructed by adding edges to pendent vertices

The signless Laplacian matrix of G is the $n\times n$ matrix $L^+(G)=D(G)+A(G)$. It is known that $L^+(G)$ is a positive semi-definite matrix and if G is a bipartite graph then $L^+(G)$ and $L(G)$ have the same characteristic polynomial [1]. Let

$$0=\mu_n(G)\leq\mu_{n-1}(G)\leq\cdots\leq\mu_1(G)$$

$$\mu_n^+(G)\leq\mu_{n-1}^+(G)\leq\cdots\leq\mu_1^+(G)$$

be the eigenvalues of $L(G)$ and $L^+(G)$, respectively.

The line graph $\mathcal{L}(G)$ is the graph whose vertex set is in one-to-one correspondence with the set of edges of G where two vertices of $\mathcal{L}(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common [12]. Let $V(G)=\{v_1,v_2,\ldots,v_n\}$ and let $E(G)=\{e_1,e_2,\ldots,e_m\}$ be the vertex set and edge set of the graph G, respectively. The incidence matrix of G is the $n\times m$ matrix $I(G)$ whose $(i,j)-$entry is 1 if v_i is incident to e_j and 0 otherwise. It is known [1] that

$$I(G)\,I(G)^T=D(G)+A(G)=L^+(G). \tag{10}$$

The notion of the energy of a graph was introduced by Gutman in 1978 as the sum of the absolute values of its adjacency eigenvalues, it is studied in chemistry and used to approximate the total π-electron energy of a molecule [7,9]. In [11], the Laplacian energy of G is introduced as follows

$$LE(G)=\sum_{j=1}^{n}\left|\mu_j(G)-\frac{2m}{n}\right|.$$

Similarly, the signless Laplacian energy of G is defined by

$$L^{+}E\left(G\right)=\sum_{j=1}^{n}\left|\mu_{j}^{+}(G)-\frac{2m}{n}\right|.$$

In [10] relations between the energy of the line graph of G, the Laplacian energy and signless Laplacian energy of G are established. In [14] the authors define the Laplacian–energy–like invariant $LEL(G)$ of G as

$$LEL\left(G\right)=\sum_{j=1}^{n}\sqrt{\mu_{j}(G)}\ .$$

In [16] a lower bound for $LEL(G)$ in terms of the maximum degree is given and also an upper bound and a lower bound for the Laplacian–energy–like invariant of the line graph of a regular graph G are obtained. More recently, in [2], lower and upper bounds for $LEL(G)$ are obtained, in terms of the order, number of edges, maximum vertex degree, and number of spanning trees, of the graph G.

The Kirchhoff index of a connected graph G of order n is

$$Kf(G)=n\sum_{j=1}^{n-1}\frac{1}{\mu_{j}(G)}\ .$$

In [3] the authors compare the $Kf(G)$ and $LEL(G)$, among other results, they arrive at a complete comparison of $Kf(G)$ and $LEL(G)$ for trees, unicyclic graphs, and bicyclic graphs.

In [13], the authors introduce the concept of the incidence energy $IE\left(G\right)$ of G as the sum of the singular values σ_i of the incidence matrix $I(G)$. It is well known that the singular values of a matrix M are the nonnegative square root of MM^T. From this fact and (10), it follows that

$$IE\left(G\right)=\sum_{j=1}^{n}\sqrt{\mu_{j}^{+}(G)}.$$

Clearly, for bipartite graphs, $LEL(G)=IE(G)$.

Let $G(H_1,\ldots,H_r)$ as in Definition 2. We search for a relationship between $LEL(G)$ and $LEL(G(H_1,\ldots,H_r))$.

Theorem 7. *Let $G(H)$ as in Definition 1. Let*

$$\mu_1\left(H\right)\geq\mu_2\left(H\right)\geq\cdots\geq\mu_{s-1}\left(H\right)\geq\mu_s\left(H\right)=0\ .$$

be the Laplacian eigenvalues of H*. Then*

$$LEL(G(H)) - LEL(G) = \sum_{j=1}^{s} \sqrt{1 + \mu_j\,(H)} - s \ . \tag{11}$$

Proof. From Theorem 4, we have

$$|\lambda I - L(G(H))| = \lambda P_H\,(\lambda)\, R(\lambda) \tag{12}$$

and

$$|\lambda I - L(G)| = \lambda(\lambda - 1)^{s-1} R(\lambda) \ . \tag{13}$$

where $P_H(\lambda)$ is the polynomial in (2) and $R(\lambda)$ is a polynomial of degree $n - s$ such that $R(0) \neq 0$. Hence, from (12) and (13),

$$LEL(G(H)) = \sum_{j=1}^{s-1} \sqrt{1 + \mu_j(H)} + \sum_{\mu : R(\mu)=0} \sqrt{\mu} \ . \tag{14}$$

and

$$LEL(G) = (s - 1) + \sum_{\mu : R(\mu)=0} \sqrt{\mu} \ . \tag{15}$$

Sustracting (15) from (14), we obtain

$$LEL(G(H)) - LEL(G) = \sum_{j=1}^{s} \sqrt{1 + \mu_j(H)} - s \ .$$

The proof is complete. □

Let $G(H_1, \ldots, H_r)$ as in Definition 2. We know that $G(H_1)$ is the graph obtained from G and H_1 by identifying the vertices of H_1 with the pendent vertices of K_{1,s_1}. Moreover, for $i = 2, \ldots, r$, $G(H_1, \ldots, H_i)$ is the graph obtained from $G(H_1, \ldots, H_{i-1})$ and H_i by identifying the vertices of H_i with the pendent vertices of K_{1,s_i}.

Theorem 8. *Let* $G(H_1, \ldots, H_r)$ *as in Definition 2. For* $1 \leq i \leq r$*, let*

$$\mu_1\,(H_i) \geq \mu_2\,(H_i) \geq \cdots \geq \mu_{s_i - 1}\,(H_i) \geq \mu_{s_i}\,(H_i) = 0$$

be the Laplacian eigenvalues of the graphs H_i*. Then*

$$LEL(G(H_1, \ldots, H_r)) - LEL(G) = \sum_{i=1}^{r} \sum_{j=1}^{s_i} \sqrt{1 + \mu_j\,(H_i)} - \sum_{i=1}^{r} s_i \ . \tag{16}$$

Proof. By a repeated application of (11), we obtain

$$LEL\left(G\left(H_1\right)\right) - LEL(G) = \sum_{j=1}^{s_1} \sqrt{1+\mu_j\left(H_1\right)} - s_1$$

$$LEL\left(G\left(H_1, H_2\right)\right) - LEL(G\left(H_1\right)) = \sum_{j=1}^{s_2} \sqrt{1+\mu_j\left(H_2\right)} - s_2$$

$$LEL\left(G\left(H_1, H_2, H_3\right)\right) - LEL(G\left(H_1, H_2\right)) = \sum_{j=1}^{s_3} \sqrt{1+\mu_j\left(H_3\right)} - s_3$$

$$\vdots$$

$$LEL\left(G\left(H_1, \ldots, H_{r-1}\right)\right) - LEL(G\left(H_1, \ldots, H_{r-2}\right)) = \sum_{j=1}^{s_{r-1}} \sqrt{1+\mu_j\left(H_{r-1}\right)} - s_{r-1}$$

$$LEL\left(G\left(H_1, \ldots, H_r\right)\right) - LEL(G\left(H_1, \ldots, H_{r-1}\right)) = \sum_{j=1}^{s_r} \sqrt{1+\mu_j\left(H_r\right)} - s_r \ .$$

Adding these equalities, (16) is obtained. □

Now we derive a relationship between $Kf(G(H_1, \ldots, H_r))$ and $Kf(G)$.

Theorem 9. *Let $G(H)$ as in Definition 1. Let*

$$\mu_1\left(H\right) \geq \mu_2\left(H\right) \geq \cdots \geq \mu_{s-1}\left(H\right) \geq \mu_s\left(H\right) = 0.$$

be the Laplacian eigenvalues of H. Then

$$Kf(G(H)) - Kf(G) = n \sum_{j=1}^{s} \frac{1}{1+\mu_j(H)} - ns \ . \tag{17}$$

Proof. We start by observing that $R(\lambda)$ is a polynomial of degree $n-s$, such that $R(0) \neq 0$. From (12) and (13),

$$Kf(G(H)) = n \sum_{i=1}^{s-1} \frac{1}{1+\mu_i(H)} + n \sum_{\mu:R(\mu)=0} \frac{1}{\mu} \ . \tag{18}$$

and

$$Kf(G) = n(s-1) + n \sum_{\mu:R(\mu)=0} \frac{1}{\mu} \ . \tag{19}$$

Subtracting (19) from (18), (17) is obtained. □

Theorem 10. *Let $G(H_1, \ldots, H_r)$ as in Definition 2. For $1 \leq i \leq r$, let*

$$\mu_1(H_i) \geq \mu_2(H_i) \geq \cdots \geq \mu_{s_i-1}(H_i) \geq \mu_{s_i}(H_i) = 0$$

be the Laplacian eigenvalues of the graphs H_i. Then

$$Kf(G(H_1, \ldots, H_r)) - Kf(G) = n\left(\sum_{i=1}^{r}\sum_{j=1}^{s_i}\frac{1}{1+\mu_j(H_i)} - \sum_{i=1}^{r} s_i\right). \tag{20}$$

Proof. By a repeated application of (17), we obtain

$$Kf(G(H_1)) - Kf(G) = n\sum_{j=1}^{s_1}\frac{1}{1+\mu_j(H_1)} - ns_1$$

$$Kf(G(H_1, H_2)) - Kf(G(H_1)) = n\sum_{j=1}^{s_2}\frac{1}{1+\mu_j(H_2)} - ns_2$$

$$Kf(G(H_1, H_2, H_3)) - Kf(G(H_1, H_2)) = n\sum_{j=1}^{s_3}\frac{1}{1+\mu_j(H_3)} - ns_3$$

$$\vdots$$

$$Kf(G(H_1, \ldots, H_{r-1})) - Kf(G(H_1, \ldots, H_{r-2})) = n\sum_{j=1}^{s_{r-1}}\frac{1}{1+\mu_j(H_{r-1})} - ns_{r-1}$$

$$Kf(G(H_1, \ldots, H_r)) - Kf(G(H_1, \ldots, H_{r-1})) = n\sum_{j=1}^{s_r}\frac{1}{1+\mu_j(H_r)} - ns_r\,.$$

Adding these equalities, (20) is obtained. □

Therefore, the differences $LEL(G(H_1,\ldots, H_r))$–$LEL(G)$ and $Kf(G(H_1,\ldots, H_r))$–$Kf(G)$ do not depend on the edges between non-pendent vertices of G. That is, if $\widetilde{G} = G + e$ is obtained from G by adding an edge e between two non-pendent vertices of G then

$$LEL(G(H_1, \ldots, H_r)) - LEL(G) = LEL(\widetilde{G}(H_1, \ldots, H_r)) - LEL(\widetilde{G})$$

and

$$Kf(G(H_1, \ldots, H_r)) - Kf(G) = Kf(\widetilde{G}(H_1, \ldots, H_r)) - Kf(\widetilde{G})\,.$$

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