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Graphs with Smallest Resolvent Estrada Indices

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Abstract

The graphs and trees with smallest resolvent Estrada indices (EE_r) are characterized. The connected graph of order n with smallest EE_r-value is the n-vertex path. The second–smallest such graph is the $(n-1)$-vertex path with a pendent vertex attached at position 2. The tree with third–smallest EE_r is the $(n-1)$-vertex path with a pendent vertex attached at position 3, conjectured to be also the connected graph with third–smallest EE_r . Based on a computer–aided search, we established the structure of a few more trees with smallest EE_r .

The details of the theory of resolvent Estrada index are outlined in the paper [1], that appears in the same issue of this journal. Thus, for a graph G of order n, the resolvent Estrada index is defined as

$$EE_r = EE_r(G) = \sum_{i=1}^{n} \left(1 - \frac{\lambda_i}{n-1}\right)^{-1} \tag{1}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of G.

In [1] it is established that for e being an edge of G, $EE_r(G-e) < EE_r(G)$. From this relation immediately follows that the graph of order n with maximal EE_r is the complete graph

K_n and that the connected graph with minimal EE_r is some tree. The tree with maximal EE_r was shown [1] to be the star. On the other hand, the tree with minimal EE_r (thus, the connected graph with minimal EE_r) was not determined in [1]. The aim of the present note is to fill this gap.

Since for all graphs of order n (except for the complete graph K_n), $|\lambda_i/(n-1)| < 1$, the summand on the right–hand side of Eq. (1) can be expanded into a convergent power series as

$$\left(1 - \frac{\lambda_i}{n-1}\right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{\lambda_i}{n-1}\right)^k$$

and therefore the resolvent Estrada index can be expanded as

$$EE_r(G) = \sum_{k=0}^{\infty} \frac{M_k(G)}{(n-1)^k} \tag{2}$$

where $M_k(G)$ is the k-th spectral moment of G, defined as

$$M_k = M_k(G) = \sum_{i=1}^{n} (\lambda_i)^k \ .$$

Recall that for bipartite graphs (and for trees in particular), all odd spectral moments are equal to zero.

In the paper [2], Hanyuan Deng proved that for P_n and S_n being the n-vertex path and star, and T being any other tree of the same order, the inequalities

$$M_{2k}(P_n) \leq M_{2k}(T) \leq M_{2k}(S_n)$$

hold for all k. It is easy to verify that for $k \geq 2$, $M_{2k}(P_n) < M_{2k}(T)$. These results, combined with Eq. (2), directly imply:

Theorem 1. *Among all connected graphs of order n , $n \geq 1$, the path P_n has minimal resolvent Estrada index.*

In order to find the graph with second minimal EE_r-value, we use another result of Hanyuan Deng [3].

Let the vertices of the path P_h be numbered consecutively by $1, 2, \ldots, h$. Let R be a bipartite graph of order $n-h$, and let v be its non-isolated vertex. Construct the graph $P_h(j)R$ by identifying the vertex v of R with the vertex j of P_h . Then for all $k \geq 0$,

$$M_{2k}(P_h(1)R) \leq M_{2k}(P_h(2)R) \leq M_{2k}(P_h(3)R) \leq \cdots \leq M_{2k}(P_h(\lfloor h/2 \rfloor)R). \tag{3}$$

In [3], the relations (3) have been proven only for $h = 5$ (cf. Lemma 3 in [3]), but a fully analogous reasoning applies also to larger values of h.

It is evident that the smallest deviation of the (minimal) $M_{2k}(P_n)$-value will happen if the graph R in $P_h(j)R$ is as small as possible, i.e., if R contains only two vertices. In what follows, we denote these graph by $P_{n-1}(j)$. Thus, $P_{n-1}(j)$ is the tree obtained by attaching a pendent vertex at position j of the $(n-1)$-vertex path.

Bearing the above in mind, according to (3), the second-minimal M_{2k}-value will be that of $P_{n-1}(2)$. This implies:

Theorem 2. *Among all connected graphs of order* n , $n \geq 4$, *the tree* $P_{n-1}(2)$ *has the second-minimal resolvent Estrada index.*

In fact, for a complete proof of Theorem 2 we would need to show that $EE_r(P_{n-1}(2)) < EE_r(C_n)$, where C_n is the cycle of order n. The fact that C_n has many more self-returning walks than $P_{n-1}(2)$ is almost self-evident. Just recall that C_n has more edges than $P_{n-1}(2)$. In addition, if n is odd, then some of the odd spectral moments of C_n are greater than zero.

The tree with third-minimal EE_r-value also immediately follows from the inequalities (3):

Theorem 3. *Among all trees of order* n , $n \geq 6$, *the tree* $P_{n-1}(3)$ *has the third-minimal resolvent Estrada index.*

We conjecture that $P_{n-1}(3)$ has third-minimal EE_r-value among all connected graphs of order n. However, for a proof of this conjecture we would have to demonstrate that $EE_r(P_{n-1}(3)) < EE_r(C_n)$ and $EE_r(P_{n-1}(3)) < EE_r(U_n)$, where U_n is the unicyclic graph obtained by joining a new vertex to the vertices 1 and 2 of P_{n-1} . Note that by deleting an edge from C_n or from the cycle of U_n we obtain either P_n (the minimal graph) or $P_{n-1}(2)$ (the second-minimal graph).

By means of a computer–aided investigation of the resolvent Estrada indices of trees, we arrived at a simple regularity which we state as:

Conjecture 1. *Among all trees of order* n , $n \geq 2\ell$, *the tree* $P_{n-1}(\ell)$ *has the* ℓ*-th minimal resolvent Estrada index, for* $\ell = 2, 3, \ldots$.

For $6 \leq n \leq 14$, the trees with the fourth– up to seventh-minimal resolvent Estrada indices (when such do exist) are depicted in Fig. 1.

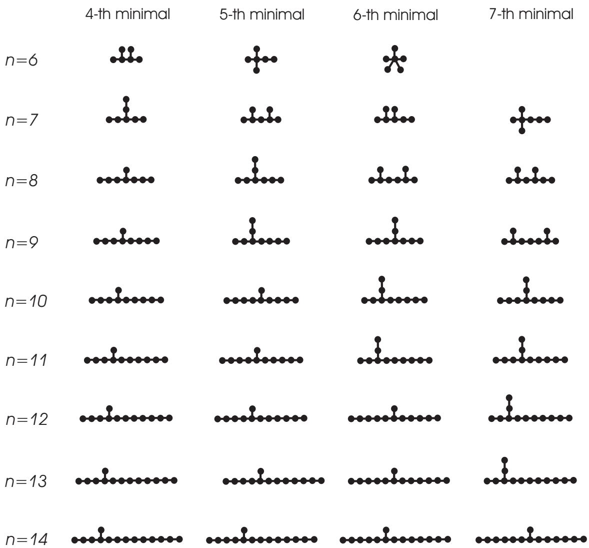

Fig. 1. Trees with small resolvent Estrada index.

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