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# General Sum–Connectivity Index with $\alpha \geq 1$ for Bicyclic Graphs[1]

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**Abstract**

The general sum–connectivity index of a graph $G$ is a molecular descriptor defined as $\chi_\alpha(G) = \sum_{uv \in E(G)} (d(u) + d(v))^\alpha$ where, $d(u)$ denotes the degree of vertex $u$ in $G$ and $\alpha$ is a real number. The aim of this paper is to obtain the graph with the maximum general sum–connectivity index among the connected bicyclic graphs of order $n$ for $\alpha \geq 1$.

## 1 Introduction

Following standard notations in graph theory [2], let $G = (V(G), E(G))$ be a simple, undirected and connected graph with $V(G)$ the set of its vertices and $E(G)$ the set of its edges. For a vertex $u \in V(G)$ let $d_G(u)$ denote the degree and $N_G(u)$ the set of its neighbors. Where there is no danger of confusion, we shall give the simplified notation $d(u)$ for the degree of $u$. We will use the notations $P_r$ and $C_r$ respectively for a path and a cycle with $r$ edges. The distance between two vertices $u$ and $v$ of a connected graph, denoted by $d(u, v)$, is the length of a shortest path between them.

One important molecular descriptor is the Randić index defined in [8] with its generealization proposed in [1]:

$$R_\alpha(G) = \sum_{uv \in E(G)} (d(u)d(v))^\alpha \ .$$

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The classical Randić index is given by $\alpha = -1/2$ and it is one of the most used molecular descriptors in the QSAR and QSPR models. Like these descriptors, the sum–connectivity index [12] and the general sum–connectivity index introduced by Zu and Trijnastić in [13] and given by

$$\chi_\alpha(G) = \sum_{uv \in E(G)} (d(u) + d(v))^\alpha ,$$

were also proposed. Here $\chi_{-1/2}$ gives the classical sum–connectivity index, which is also studied and applied in QSAR, QSPR modeling.

Several extremal properties of the general sum–connectivity index have already been established for general graphs [13], multigraphs [9], trees [7, 9, 12] and unicyclic graphs [6, 10]. In this paper we want to extend the extremal study of the general sum–connectivity index to bicyclic graphs (connected graphs with $n$ vertices and $n+1$ edges). More precisely, we will find the graph with the largest value of $\chi_\alpha(G)$ among the bicyclic graphs of order $n$ for $\alpha \geq 1$.

## 2 Some initial transformations

For $n \leq 6$ we can easily see which are the connected bicyclic graphs of order $n$ with maximum general sum–connectivity index for $\alpha \geq 1$. If $n = 4$ we have a unique bicyclic graph and for $n = 5$, $n = 6$ the graphs with the largest value of the general sum–connectivity index are given in Fig. 1.

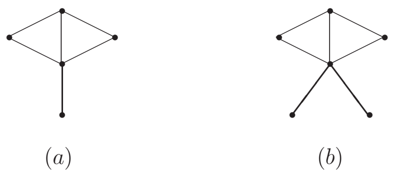

Figure 1: Bicyclic graphs with maximum $\chi_\alpha$: (a) $n = 5$; (b) $n = 6$.

Thus, we will consider in this paper that $|V(G)| = n \geq 7$.

Let $u$ and $v$ be two adjacent vertices with $d(v) \geq 2$ such that $N_G(u) \cap N_G(v) = \emptyset$ and the neighbors of the vertex $u$ except $v$, denoted $u_1, \ldots, u_r$ are pendent vertices. We begin with a particular case of the $t_1$-transform [9] through which all the pendent edges of vertex $u$ become incident edges of vertex $v$, as below:

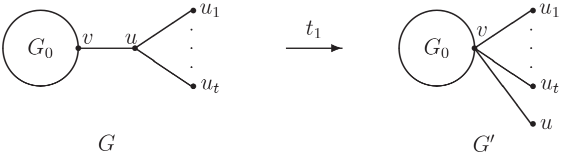


Figure 2: $t_1$-transform for pendent edges

Thus the transformation described above built the graph $t_1(G) = G - \{uu_1, \ldots, uu_t\} + \{vu_1, \ldots, vu_t\}$ obtained by removing $uu_1, \ldots, uu_t$ and adding $vu_1, \ldots, vu_t$, $t \geq 1$. We need the following result:

**Lemma 1.** *[9] Let $G$ and $G' = t_1(G)$ be the graphs from Fig. 2. Then, for $\alpha \geq 1$, we have $\chi_\alpha(G') > \chi_\alpha(G)$.*

Since a bicyclic graph has $n+1$ edges it can be obtained from a tree to which we add two other edges and thus forming some cycles. Then every bicyclic graph can be viewed as a (possibly empty) set of subtrees, each of them attached to one of the graph's cycles. Applying the $t_1$-transform for a finite number of times we easily see that we can reduce any of the above subtrees to a bunch of pendent edges incident to the subtree's cycle vertex of attachment.

As above, we define our next general sum–connectivity enhancing $t_2^i$-transform, with the purpose of further reducing our bicyclic graph to an even simpler case.

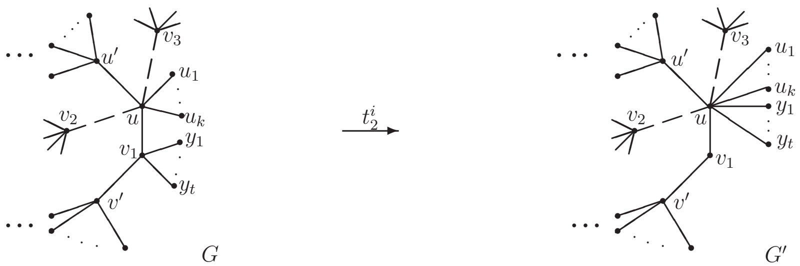


Figure 3: $t_2^i$-transform

Let $G$ be a graph as in Fig. 3 and we denote $d_G(u) - k = i \geq 2$. Suppose that the vertex $u$ has, besides its $k$ pendent neighbors, at least two (denoted by $u'$, $v_1$) and at most four non-pendent neighbors. Then, if $i = 3$ we denote by $v_2$ the third non-pendent neighbor of the vertex $u$ and for $i = 4$ we also have the vertex $v_3$. Thus we define the transformation $t_2^i(G) = G - \{v_1y_1, \ldots, v_1y_t\} + \{uy_1, \ldots, uy_t\}$. We prove that modifying in this manner the graph, the value of the general sum–connectivity index $\chi_\alpha$ strictly increases. But first we give a simple result that we will need several times throughout this paper.

**Lemma 2.** *The real function $f : [0, \infty) \to \mathbb{R}$ defined by $f_{\alpha,a}(x) = (x + a)^\alpha - x^\alpha$ is strictly increasing for all $\alpha > 1$, $a > 0$.*

**Lemma 3.** *Let $G$, $G' = t_2^i(G)$ as in Fig. 3 such that $uv_1 \in E(G)$, $t \geq 1$, $k \geq 0$, $d_G(v_1) - t = 2$ and $d(u') \geq d(v')$. Then $\chi_\alpha(G') > \chi_\alpha(G)$ for all $\alpha \geq 1$.*

*Proof.* We can write $i = 2 + \beta + \gamma$, where $\beta = 1$ indicates the existence of the vertex $v_2$ (otherwise $\beta = 0$) and likewise, $\gamma = 1$ indicates the existence of the vertex $v_3$ (otherwise $\gamma = 0$). With the established notations from the above figure, we have:
$\chi_\alpha(G') - \chi_\alpha(G) = [(d(u') + k + t + i)^\alpha - (d(u') + k + i)^\alpha] + [(d(v') + 2)^\alpha - (d(v') + t + 2)^\alpha] + k[(t + k + i + 1)^\alpha - (k + i + 1)^\alpha] + t[(t + k + i + 1)^\alpha - (t + 3)^\alpha] + \beta[(d(v_2) + t + k + i)^\alpha - (d(v_2) + k + i)^\alpha] + \gamma[(d(v_3) + t + k + i)^\alpha - (d(v_3) + k + i)^\alpha]$. Obviously the sum of the last four square parentheses in this expression is strictly positive. Now for the first two we need the above lemma and if $i \geq 2$, $d(u') \geq d(v')$ then $f_{\alpha,t}(d(u') + k + i) \geq f_{\alpha,t}(d(v') + 2)$, from which we conclude that the $t_2^i$-transform strictly increases $\chi_\alpha$. ∎

# 3 Three particular types of bicyclic graphs

With the notations from [3] and ignoring the possible pendent subtrees that may appear, we have the following three types of bicyclic graphs:

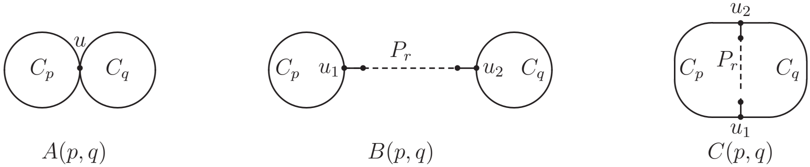


Figure 4: Types of bicyclic graphs

Thus, for the connected bicyclic graphs of order $n$ we denote by $A(p,q)$ the set of the graphs that have two cycles $C_p$ and $C_q$ with a single vertex $u$ in common. For the graphs in which these cycles are distinct and connected by a path (of length at least one) we use the notation $B(p,q)$. If $C_p$ and $C_q$ have in common a path $P_r$ ($r \geq 1$) we have a $C(p,q)$-graph.

In what follows we treat these three cases separately, with the purpose of determining the graph with maximum $\chi_\alpha$ in each category.

First we examine $A(p,q)$. To this category of graphs we apply the $t_2^2$-transform to all cycle edges which are not incident to the vertex $u$ and which have bunches of pendent edges at both ends. Thus, all remaining bunches of pendent edges to $C_p \cup C_q - \{u\}$ will be situated at distances of at least two one from another.

We now show that moving the remaining bunches of pendent edges in the vertex $u$, the index $\chi_\alpha$ continues to strictly increase. Thus we define a new transformation given by $t_3(G) = G - \{vy_1, \ldots, vy_t\} + \{uy_1, \ldots, uy_t\}$, where $v \in C_p \cup C_q - \{u\}$ is a vertex that has attached to it the set of the pendent edges $\{vy_1, \ldots, vy_t\}$, $t \geq 1$. Thus we have:

**Lemma 4.** *Denoting by $G' = t_3(G)$ we have $\chi_\alpha(G') > \chi_\alpha(G)$ for all $\alpha \geq 1$.*

*Proof.* Let $G \in A(p,q)$ be a graph as in Fig. 4 and let $\{uu_1, \ldots, uu_k\}$ be the (possibly empty) set of the pendent edges in the vertex $\{u\} = C_p \cap C_q$. The vertex $v$ with its pendent edges can be adjacent to vertex $u$ or $d(u,v) \geq 2$.

Case I: $uv \in E(G)$.

Suppose, for simplicity that $v \in C_p$ and let $N_{C_p}(v) = \{u, w\}$. Then, since all remaining bunches of pendent edges to $C_p \cup C_q - \{u\}$ are situated at distances of at least two one from another, we have $d_G(w) = 2 < d_G(u)$. Finally, since $d_G(u) - k = 4$, we can apply the $t_2^4$-transform to move in the vertex $u$ the edges pendent to $v$. We repeat this transformation whenever possible for the adjacent vertices of $u$ situated on the cycles.

Case II: $d(u,v) > 1$.

First observe that, since we already applied the $t_2^2$-transform whenever possible, both of $v$'s cycle neighbors have degree exactly 2 in $G$. Moreover, from the previous case we have that all the neighbors of $u$ situated on the cycles $C_p$ and $C_q$ have degree 2 in $G$ also.

Thus, we have:

$\chi_\alpha(G') - \chi_\alpha(G) = 2[4^\alpha - (t+4)^\alpha] + 4[(k+t+6)^\alpha - (k+6)^\alpha] + k[(t+k+5)^\alpha - (k+5)^\alpha] + t[(k+t+5)^\alpha - (t+3)^\alpha]$. Applying lemma 2 for the sum of the first two square parentheses, we have $f_{\alpha,t}(4) < f_{\alpha,t}(k+6)$ for every $k \geq 0, t \geq 1$ and the conclusion easily follows. ■

Applying the $t_3$-transform for all the bunches of pendent edges to $C_p \cup C_q - \{u\}$ we obtain the graph $G_1$ from Fig. 5. We observe now that - through all the transformations used so far - by bringing as many edges as possible in the well chosen vertex $u$, the general sum–connectivity index strictly increases. Based on this observation, it appears naturally to extract edges from the two cycles and attach them to the vertex $u$. We construct thus the transformation:

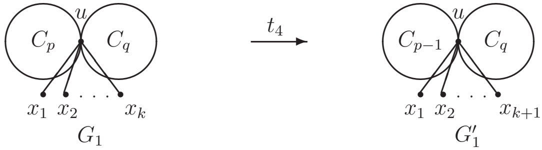


Figure 5: Decreasing of cycles of $A(p,q)$-graphs

**Lemma 5.** *Denoting by $A_n(p,q,k)$ the graph $G_1$ from Fig. 5 we have $\chi_\alpha(A_n(p,q,k)) < \chi_\alpha(A_n(p-1,q,k+1))$, for $p > 3$.*

*Proof.* A simple computation gives us $\chi_\alpha(A_n(p-1,q,k+1)) - \chi_\alpha(A_n(p,q,k)) = 4[(k+7)^\alpha - (k+6)^\alpha] + k[(k+6)^\alpha - (k+5)^\alpha] + (k+6)^\alpha - 4^\alpha > 0$. ■

**Theorem 1.** *If $\alpha \geq 1$ then $A_n(3,3,n-5)$ is the unique graph with the largest general sum–connectivity index among the graphs of order $n$ in $A(p,q)$.*

*Proof.* This result follows from the previous lemmas. If G is not isomorphic to $A(3,3,n-5)$, then by one of the transformations described above we can find another bicyclic graph of order $n$ having a greater general sum–connectivity index. Hence $A(3,3,n-5)$ maximizes the general sum–connectivity index in the $A(p,q)$ family of graphs (see Fig. 10(a)). ■

We will analyse now the family of graphs denoted by $B(p,q)$ (see Fig. 4). We first successively apply the $t_1$-transform until we obtain a graph with bunches of pendent edges attached to the cycles $C_p$, $C_q$ and to the path $P_r$. With the notations from Fig. 6(a) we apply the $t_2^2$-transform on the paths $C_p - \{u_1\}$, $C_q - \{u_2\}$ and $P_r - \{u_1, u_2\}$. Thus, on these paths the remaining bunches of pendent edges will be situated at a distance greater or equal to 2. Apart from these there will eventually remain some bunches of pendent edges in the vertices $v_1, v_2, v_3, v_4, w_1, w_2$ (see Fig. 6, where $w_1, w_2$ may coincide or disappear altogether if $r = 1$). Next, we gather all those remaining edges in the vertex $u_1$ or all in the vertex $u_2$ to strictly increase the index $\chi_\alpha$. For this purpose, we will apply a new transformation which will be handled in a certain manner. Let $y \in V(G)$ and $\{yy_1, \ldots, yy_t\}$ be the set of the pendent edges in vertex $y$, $t \geq 1$ and we define the new transformation as $t_5(G) = G - \{yy_1, \ldots, yy_t\} + \{u_iy_1, \ldots, u_iy_t\}$, $i \in \{1, 2\}$.

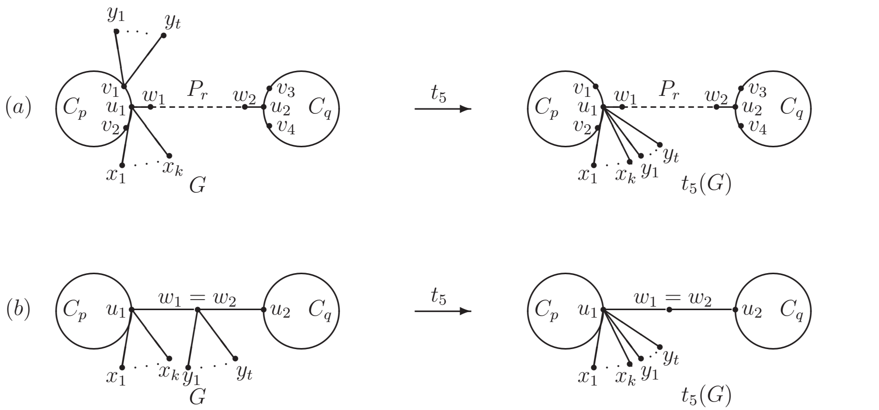


Figure 6: Different cases for shifting the pendent edges for a vertex $y \in N_G(u_1) \cup N_G(u_2)$

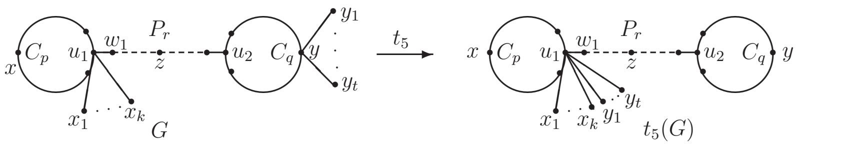


Figure 7: Shifting the pendent edges for vertex $y \in V(G)$, $d(y, u_i) > 1$, $i \in \{1, 2\}$

**Lemma 6.** *Let $G$ be a $B(p, q)$-graph as in Fig. 6 or 7. There exists a sequence of $t_5$-transforms, that strictly increase the value of the general sum–connectivity index after which all the pendent edges will be incident to the vertex $u_1$ or all will be incident to the vertex $u_2$.*

*Proof.* We construct the sequence in the following order:

Step 1. Let us consider $y \in N_G(u_i)$, $i \in \{1, 2\}$. In this case we move all the pendent edges from $y$ to its adjacent vertex $u_i$.

Since we first already applied the $t_2^2$-transform whenever possible, for every $x$ in $N_G(y) - \{u_1, u_2, y_1, \ldots, y_t\}$ we have $d(x) = 2$. We will first treat the case when $y$ is a cycle vertex.

Case 1.1. Let $y$ be in $N_G(u_i) \cap (C_p \cup C_q)$. With the notations from Fig. 6(a), $y$ is one of the vertices $v_1, v_2, v_3, v_4$. For these vertices all conditions in lemma 3 are fulfilled, so we can apply the $t_2^3$-transform to bring all the pendent edges from $v_1$ and $v_2$ in $u_1$ and from $v_3$ and $v_4$ in $u_2$.

Case 1.2. Let $y$ be in $N_G(u_i) \cap P_r$. With the notations in the figure, $y$ is one of the vertices $w_1, w_2$.

Let $k$ be the number of pendent edges attached to $u_1$, i. e., $k = d(u_1) - 3$.

$(a)$ Suppose $r = 1$.

If $k > 0$ we will move to $u_1$ all the pendent edges attached to $u_2$, otherwise we keep these pendent edges in the vertex $u_2$.

From case I we have $d(v_i) = 2$, for every $1 \leq i \leq 4$. So:
$\chi_\alpha(G') - \chi_\alpha(G) = 2[(k+t+5)^\alpha - (k+5)^\alpha] + 2[5^\alpha - (t+5)^\alpha] + t[(k+t+4)^\alpha - (t+4)^\alpha] + k[(k+t+4)^\alpha - (k+4)^\alpha]$. Here the last two parentheses are clearly positive and for the first two we apply lemma 2.

$(b)$ If $r = 2$ it follows that $y = w_1 = w_2$ (Fig. 6(b)).

We observe that we cannot use $t_2^3$ in this case. Thus we compare directly the values of $\chi_\alpha$ for $G$ and $G' = t_5(G)$. If $d(u_1) \geq d(u_2)$ we attach the pendent edges from $y$ to $u_1$. We denote by $c$ the number of pendent vertices adjacent to $u_2$ and using the notations from Fig. 6(b) we have:

$\chi_\alpha(G') - \chi_\alpha(G) = 2[(k+t+5)^\alpha - (k+5)^\alpha] + [(c+5)^\alpha - (t+c+5)^\alpha] + t[(k+t+4)^\alpha - (t+3)^\alpha] + k[(k+t+4)^\alpha - (k+4)^\alpha]$. Since $d(u_1) \geq d(u_2)$ $(k+3 \geq c+3)$, then from lemma 2 we have $f_{\alpha,t}(c+5) \leq f_{\alpha,t}(k+5)$, hence $\chi_\alpha$ strictly increases.

For $d(u_1) < d(u_2)$ we move the pendent edges from $y$ to vertex $u_2$ (the computations are the same as above).

$(c)$ For $r \geq 3$, first note that since we already applied the $t_2^2$-transform whenever possible, the neighbor of $y$ on $P_r - \{u_1, u_2\}$ has degree exactly 2. Thus we can apply the $t_2^3$-transform to bring the pendent edges from $y = w_1$ to $u_1$ and from $y = w_2$ to $u_2$.

Step 2. Suppose the distance $d(y, u_i) > 1$, $i \in \{1, 2\}$, where $y$ is a vertex situated on $C_p$, $C_q$ or $P_r$.

Observe that, after applying all the transformations from step 1, every non-pendent neighbor of $y$ has degree exactly 2.

Case 2.1. $y \neq u_2$ (Fig. 7).

Supposing that $k > 0$, we will move all the pendent edges from $y$ to $u_1$. In the case of a null value of $k$ we move all in the vertex $u_2$, with similar computations (even if the vertex $u_2$ also has no pendent edges attached to it, i.e, $d(u_2) = 3$).

Let us denote $d(w_1) = c$ and it is easy to observe that if $r = 1$, then $w_1 = u_2$ and $c = d(u_2)$, else we have $c = 2$. Thus:

$\chi_\alpha(G') - \chi_\alpha(G) = 2[(k+t+5)^\alpha - (k+5)^\alpha] + 2[4^\alpha - (t+4)^\alpha] + t[(k+t+4)^\alpha - (t+3)^\alpha] + k[(k+t+4)^\alpha - (k+4)^\alpha] + (k+t+c+3)^\alpha - (k+c+3)^\alpha$. Using lemma 2 the conclusion easily follows.

Case 2.2. $y = u_2$.

We apply whenever possible the $t_5$-transform from the previous step, thus, if $k = 0$, all the pendent edges are already attached to the vertex $u_2$, then we are done. Otherwise, we have some pendent edges attached to the vertex $u_1$, as to the vertex $u_2$. In this case, we collect all the pendent edges to the vertex $u_1$ by moving the pendent edges from $u_2$. We get:

$\chi_\alpha(G') - \chi_\alpha(G) = 3[(k+t+5)^\alpha - (k+5)^\alpha] + 3[5^\alpha - (t+5)^\alpha] + t[(k+t+4)^\alpha - (t+4)^\alpha] + k[(k+t+4)^\alpha - (k+4)^\alpha]$. Using lemma 2 this case is also resolved. ■

Now we modify the obtained graph by deleting edges from the two cycles and from the path joining them and reattaching them to the vertex $u_1$, by the transformations $t_6$ and $t'_6$, as below:

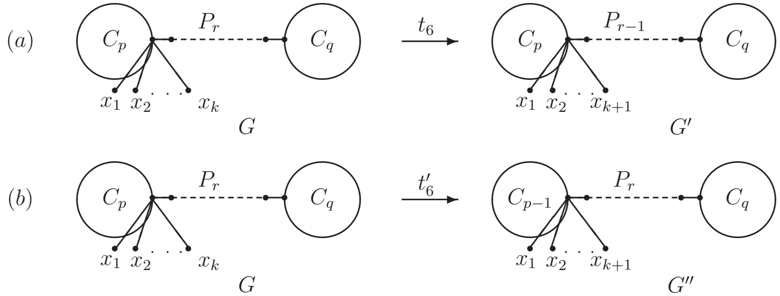


Figure 8: Transformations for $B(p,q)$-graphs that strictly increase $\chi_\alpha$

**Lemma 7.** *Let us denote by $B_n(p,q,r,k)$ the graph $G$ in Fig. 8. Then the transformations $t_6$ and $t'_6$ strictly increase the general sum–connectivity index for $\alpha \geq 1$:*

*(a)* $\chi_\alpha(B_n(p,q,r,k)) < \chi_\alpha(B_n(p,q,r-1,k+1))$ *for* $r > 2$*;*

*(b)* $\chi_\alpha(B_n(p,q,r,k)) < \chi_\alpha(B_n(p-1,q,r,k+1))$ *for* $p > 3$.

*Proof.* We can see that

$\chi_\alpha(B_n(p,q,r-1,k+1)) - \chi_\alpha(B_n(p,q,r,k)) = (k+5)^\alpha - 4^\alpha + k[(k+5)^\alpha - (k+4)^\alpha] + 3[(k+6)^\alpha - (k+5)^\alpha] > 0$ for all $\alpha \geq 1, k \geq 0$.

Since $\chi_\alpha(B_n(p,q,r-1,k+1)) = \chi_\alpha(B_n(p-1,q,r,k+1)) = \chi_\alpha(B_n(p,q-1,r,k+1))$, then (b) is also true. ■

From the above proof we easily see that the $t'_6$-transform can be used to shrink the $C_p$ cycle as well as the $C_q$ cycle. Keeping in mind the requirements of this lemma, we can repeat the above transformations successively to obtain graphs with greater $\chi_\alpha$ until $r = 2$, $p = 3$ and $q = 3$, which gives us the graph $B_n(3,3,2,n-7)$.

**Theorem 2.** $B_n(3,3,1,n-6)$ *from Fig. 10(b) is the graph of order $n$ that maximizes the general sum–connectivity index for $\alpha \geq 1$ in the set $B(p,q)$.*

*Proof.* Using the above remark, all that remains is to compare the general sum–connectivity values for $B_n(3,3,2,n-7)$ and $B_n(3,3,1,n-6)$. Thus:

$\chi_\alpha(B_n(3,3,1,n-6)) - \chi_\alpha(B_n(3,3,2,n-7)) = 2(n-1)^\alpha + n^\alpha + (n-6)(n-2)^\alpha - 5^\alpha - 3(n-2)^\alpha - (n-7)(n-3)^\alpha = 2[(n-1)^\alpha - (n-2)^\alpha] + (n-7)[(n-2)^\alpha - (n-3)^\alpha] + n^\alpha - 5^\alpha$, which is surely positive for $\alpha \geq 1, n \geq 7$. ■

We continue now by finding the maximal graph in the category $C(p,q)$. We observe that the procedure of increasing the index $\chi_\alpha$ through transformations used for the $B(p,q)$ family of graphs, that bring as many edges as possible in a well selected vertex, can be also applied in this case. Thus we have:

**Lemma 8.** *Let $G$ be a graph such as in Fig. 9. There exists a sequence of $t_5$-transforms, that strictly increases the value of the general sum–connectivity index after which all the pendent edges will be incident to the vertex $u_1$ or all will be incident to the vertex $u_2$.*

*Proof.* The proof is identical to the case of the $B(p,q)$-graphs. We only need to use the cycles $C_p$ and $C_q$ excluding the common path $P_r$. Thus, if in the proof of lemma 6 we use the paths $C_p - P_r$ and $C_q - P_r$ instead of $C_p$ and $C_q$, then we get the conclusion. ■

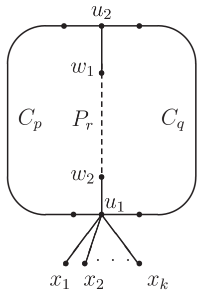


Figure 9: Graph in $C(p,q)$

From here we will further proceed as in the case of the $B(p,q)$ family by removing edges from the cycles and reattaching them to the vertex $u_1$. Denoting by $C_n(p,q,r,k)$ the graph in Fig. 9, we define the transformations $t_7$, $t_7'$, $t_7''$ by $t_7(C_n(p,q,r,k)) = C_n(p-1,q,r,k+1)$, $t_7'(C_n(p,q,r,k)) = C_n(p,q-1,r,k+1)$, $t_7''(C_n(p,q,r,k)) = C_n(p-1,q-1,r-1,k+1)$. Noting that by the transformations $t_7$ and $t_7'$ we remove edges from the paths $C_p - P_r$, $C_q - P_r$ (not from the entire cycle) and by $t_7''$-transform we remove only an edge from the path $P_r$ (so, implicitly, $p$ an $q$ decrease by one unit).

**Lemma 9.** *For $\alpha \geq 1$ we have:*

*(a)* $\chi_\alpha(C_n(p,q,r,k)) < \chi_\alpha(C_n(p-1,q,r,k+1))$ *for* $p-r > 2$;

*(b)* $\chi_\alpha(C_n(p,q,r,k)) < \chi_\alpha(C_n(p-1,q-1,r-1,k+1))$ *for* $r > 2$.

*Proof.* These inequalities are proved in the same way as in Lemma 7. ■

With these preparations we have the following result:

**Theorem 3.** *The graph of order $n$ that maximizes the general sum connectivity index for $\alpha \geq 1$ in the family $C(p,q)$ is $C_n(3,3,1,n-4)$ (Fig. 10(c)).*

*Proof.* By the previous lemma, we strictly increase the value of $\chi_\alpha$ by repeated use of the transformations $t_7$, $t_7'$ and $t_7''$, that gives us the graph $C_n(4,4,2,n-5)$. Since lemma 9 cannot be applied for $r = 2$, we have to show that in this case the $t''_7$-transform also strictly increases $\chi_\alpha$. For this we see that $\chi_\alpha(C_n(3,3,1,n-4)) - \chi_\alpha(C_n(4,4,2,n-5)) = 2[(n+1)^\alpha - n^\alpha] + (n-5)[n^\alpha - (n-1)^\alpha] + (n+2)^\alpha - 5^\alpha$, which is obviously positive for $\alpha \geq 1$. ■

# 4 Maximum value of $\chi_\alpha$ for bicyclic graphs ($\alpha \geq 1$)

We have obtained so far, for each of the families $A(p,q)$, $B(p,q)$ and $C(p,q)$, the graph which maximizes the general sum–connectivity index $\chi_\alpha$ for $\alpha \geq 1$ (see Fig. 10).

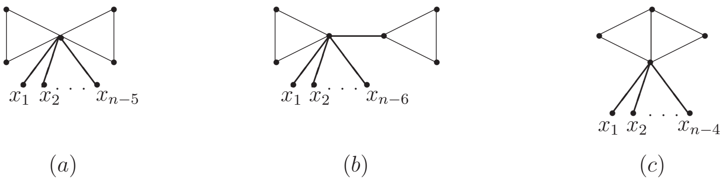


Figure 10: (a) $A_n(3,3,n-5)$; (b) $B_n(3,3,1,n-6)$; (b) $C_n(3,3,1,n-4)$.

We shall now find which is the graph with the greatest $\chi_\alpha$ index in the category of bicyclic graphs. Thus we have the following:

**Theorem 4.** $C_n(3,3,1,n-4)$ *is the unique graph with the largest general sum–connectivity index for* $\alpha \geq 1$ *among all the connected bicyclic graphs of order* $n \geq 4$.

*Proof.* Using the three theorems above all that remains is to compare the graphs from Fig. 10. Thus:

$\chi_\alpha(A(3,3,n-5)) = 2 \cdot 4^\alpha + 4(n+1)^\alpha + (n-5)n^\alpha$;

$\chi_\alpha(B(3,3,1,n-6)) = 2 \cdot 4^\alpha + 2 \cdot 5^\alpha + 2(n-1)^\alpha + n^\alpha + (n-6)(n-2)^\alpha$;

$\chi_\alpha(C(3,3,1,n-4)) = 2 \cdot 5^\alpha + 2(n+1)^\alpha + (n+2)^\alpha + (n-4)n^\alpha$.

Now we have that:

$\chi_\alpha(C(3,3,1,n-4)) - \chi_\alpha(B(3,3,1,n-6)) = 2(n+1)^\alpha + (n+2)^\alpha + (n-4)n^\alpha - 2 \cdot 4^\alpha - 2(n-1)^\alpha - n^\alpha - (n-6)(n-2)^\alpha = 2[(n+1)^\alpha - (n-1)^\alpha] + (n-6)[n^\alpha - (n-2)^\alpha] + (n+2)^\alpha + n^\alpha - 2 \cdot 4^\alpha$.

Since $\alpha \geq 1$ and $n \geq 7$, the expression above is strictly positive.

$\chi_\alpha(C(3,3,1,n-4)) - \chi_\alpha(A(3,3,n-5)) = 2[5^\alpha - 4^\alpha] + (n+2)^\alpha + n^\alpha - 2(n+1)^\alpha$. The square parenthesis is obviously positive and for the last three terms of the sum we shall consider the function $f : [0, \infty) \to \mathbb{R}$ defined by $f_\alpha(n) = n^\alpha$. Since $f$ is convex for $\alpha \geq 1$ then by Jensen's inequality we deduce the positivity of the last part of the sum. ■

**Remark 1.** We note that in the category of connected bicyclic graphs, the graph that maximizes the general sum–connectivity index for $\alpha \geq 1$ is the same that maximizes the Zagreb indices [3], the Merrifield–Simmons index [5] and minimizes the Hosoya index [4]. Moreover it is one of the two graphs that maximizes the Harary index [11].

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