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Counterexamples to a Conjecture on Wiener Index of Common Neighborhood Graphs[1]

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Abstract

For a simple graph G, the common neighborhood graph $\mathrm{con}(G)$ is the graph with the same vertex set as G, with two vertices adjacent in $\mathrm{con}(G)$ if they have a common neighbor in G. We describe here constructions of counterexamples to a conjecture of Knor et al. [*MATCH Commun. Math. Comput. Chem.* **72** (2014), 000–000] that there exists an absolute constant C such that for every graph G it holds that $W(\mathrm{con}(G)) \leq C \cdot W(G)$, where $W(G)$ denotes the Wiener index of G.

1 Introduction

Let $G = (V, E)$ be a simple graph. For $u, v \in V$, the distance $d(u, v \,|\, G)$ between the vertices u and v in G is defined as the length of the shortest path between u and v in G,

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provided that u and v belong to the same connected component of G. If G is connected, the Wiener index $W(G)$ is defined as

$$W(G) = \sum_{\{u,v\} \in \binom{V}{2}} d(u, v \,|\, G),$$

where $\binom{V}{2}$ denotes the set of unordered pairs of elements of V, while if G is disconnected with the connected components $G_1, \ldots, G_c$, $c \geq 2$, the Wiener index $W(G)$ is defined as

$$W(G) = W(G_1) + \cdots + W(G_c).$$

The *common neighborhood graph* of G is the graph $\mathrm{con}(G) = (V, F)$ on the same vertex set as G such that two vertices u, v are adjacent in $\mathrm{con}(G)$ if and only if they have a common neighbor in G. Knor et al. [1] study the relations between Wiener indices of G and $\mathrm{con}(G)$, and conclude with

Conjecture 1 ([1])**.** *There is an absolute constant C such that for every graph G it holds that*

$$W(\mathrm{con}(G)) \leq C \cdot W(G).$$

Our goal here is to provide counterexamples to this conjecture. Inspiration for the construction of counterexamples came from observing that the smallest examples of graphs for which $W(G) < W(\mathrm{con}(G))$, which are depicted in Fig. 1, all have the same structure: *a bipartite graph with an odd cycle attached to one (or two) of its vertices.*

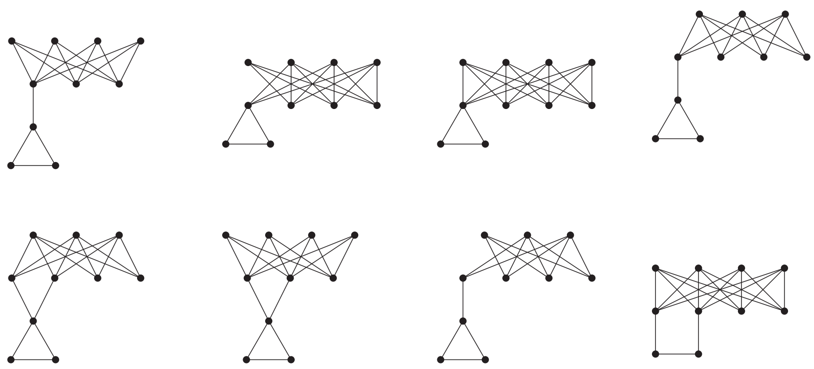

Figure 1: Smallest examples of graphs for which $W(G) < W(\mathrm{con}(G))$.

Note that in a connected graph G with a structure of this type, $\mathrm{con}(G)$ is necessarily connected due to the presence of the odd cycle. Moreover, the shortest path between vertices of different parts of the underlying bipartite graph goes along the odd cycle, implying that the distance between such vertices is bounded from below by the size of the odd cycle. Hence, if the odd cycle is long enough (not too long!) and the sizes of the parts of the bipartite graph are both linear in n, then one can obtain a counterexample to Conjecture 1 for an arbitrary constant C. We illustrate this with two infinite families of counterexamples given in following sections.

2 A family of counterexamples

The first family of counterexamples is based upon complete bipartite graphs. Let $Q_{n,k}$ be obtained from the disjoint union of the complete bipartite graph $K_{n,n}$ and the odd cycle C_{2k+1} by adding an edge between a vertex of $K_{n,n}$ and a vertex of C_{2k+1}. Let (X,Y) be a bipartition of $K_{n,n}$ and denote by ab, $a \in X$, $b \in C_{2k+1}$, the edge of $Q_{n,k}$ between $K_{n,n}$ and C_{2k+1}. Denote the remaining vertices along C_{2k+1} by $c_1, \ldots, c_{2k}$. An example of $Q_{n,k}$ is shown in Fig. 2.

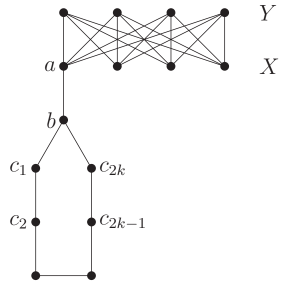

Figure 2: The graph $Q_{4,3}$.

Direct calculation yields Wiener indices of the building blocks of $Q_{n,k}$:

$$\begin{aligned} W(K_{n,n}) &= 2\binom{n}{2} \cdot 2 + n^2 \cdot 1 = 3n^2 - 2n, \\ W(C_{2k+1}) &= (2k+1)(1+2+\cdots+k) = \frac{k(k+1)(2k+1)}{2}. \end{aligned}$$

Note that for our goal it will suffice to provide appropriate bounds on the Wiener indices of $Q_{n,k}$ and $\mathrm{con}(Q_{n,k})$, instead of their explicit values.

Proposition 2. $W(Q_{n,k}) \le 3n^2 + 2n(2k^2 + 7k + 2) + \dfrac{k(k+1)(2k+1)}{2}$.

Proof Clearly,

$$W(Q_{n,k}) = W(K_{n,n}) + W(C_{2k+1}) + \sum_{u \in K_{n,n}} \sum_{v \in C_{2k+1}} d(u, v \,|\, Q_{n,k}).$$

Next, the largest distance between a vertex u of $K_{n,n}$ and a vertex v of C_{2k+1} is at most $k+3$: the distance between u and a is at most two, while the distance between b and v is at most k. Hence,

$$\begin{aligned} W(Q_{n,k}) &\le W(K_{n,n}) + W(C_{2k+1}) + (n+n) \cdot (2k+1) \cdot (k+3) \\ &= 3n^2 + 2n(2k^2 + 7k + 2) + \frac{k(k+1)(2k+1)}{2}. \qquad \square \end{aligned}$$

Proposition 3. $W(\mathrm{con}(Q_{n,k})) \ge (k+2)n^2$.

Proof The graph $\mathrm{con}(Q_{n,k})$ consists of two cliques corresponding to parts X and Y of $K_{n,n}$, the odd cycle corresponding to vertices of C_{2k+1}, and the edges between $K_{n,n}$ and C_{2k+1}:

- two edges joining a to the two neighbors of b in C_{2k+1}, and
- n edges joining b to the n vertices of Y.

The shortest path in $\mathrm{con}(Q_{n,k})$ between an arbitrary vertex $y \in Y$ and an arbitrary vertex $x \in X$ then consists of an edge yb, k edges bc_2, c_2c_4, $\ldots$, $c_{2k-2}c_{2k}$, edge $c_{2k}a$, and provided that $x \ne a$, an edge ax. Therefore, $d(y, x \,|\, \mathrm{con}(Q_{n,k})) \ge k+2$ and since there is a total of n^2 pairs (y, x) with $y \in Y$, $x \in X$, it follows that

$$W(\mathrm{con}(Q_{n,k})) \ge (k+2)n^2. \qquad \square$$

Theorem 4. *For each constant $C > 0$ and each $k > 3C - 2$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$*

$$W(\mathrm{con}(Q_{n,k})) > C \cdot W(Q_{n,k}).$$

Proof Since

$$\frac{W(\mathrm{con}(Q_{n,k}))}{W(Q_{n,k})} \ge \frac{(k+2)n^2}{3n^2 + 2n(2k^2 + 7k + 2) + \frac{k(k+1)(2k+1)}{2}}$$

we have that for a fixed value of k (such that $k > 3C - 2$)

$$\lim_{n\to\infty} \frac{W(\mathrm{con}(Q_{n,k}))}{W(Q_{n,k})} = \frac{k+2}{3} > C.$$

Therefore, for $\varepsilon = \frac{1}{2}\left(\frac{k+2}{3} - C\right)$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\frac{W(\mathrm{con}(Q_{n,k}))}{W(Q_{n,k})} > \frac{k+2}{3} - \varepsilon > C. \qquad \square$$

3 Counterexamples with maximum vertex degree 3

Although the graphs $Q_{n,k}$ from previous section provide a family of counterexamples to Conjecture 1, their maximum vertex degree is equal to $n+1$. Here we present another family of counterexamples, whose maximum vertex degree is equal to three.

Let T_h denote the complete binary tree of height h, and let T'_h denote the tree obtained from T_h by subdividing all of its edges. Let $R_{h,k}$ be obtained from the disjoint union of T'_h and C_{2k+1} by adding an edge between the root of T'_h and a vertex of C_{2k+1}. An example of $R_{h,k}$ is shown in Fig. 3.

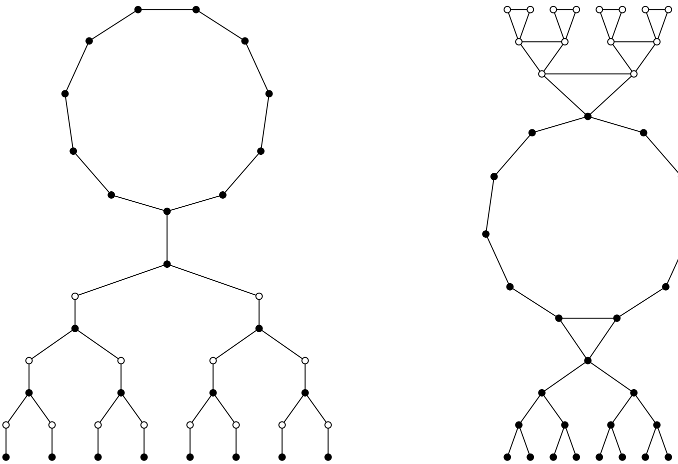

Figure 3: The graphs $R_{3,5}$ (left) and $\mathrm{con}(R_{3,5})$ (right).

Proposition 5. $W(R_{h,k}) \leq h\,4^{h+2} + k\binom{2k+1}{2} + (2h+k+1)\left(2^{h+2} - 3\right)(2k+1)$.

Proof The upper bound on $W(R_{h,k})$ follows from the observation that in $R_{h,k}$:

- the distance between any two of $2^{h+2}-3$ vertices of T'_h is at most $2h$;
- the distance between any two of $2k+1$ vertices of C_{2k+1} is at most k;
- the distance between any vertex of T'_h and any vertex of C_{2k+1} is at most $2h+k+1$,

and the inequality $2\binom{2^{h+2}-3}{2} < 4^{h+2}$. $\square$

Proposition 6. $W(\mathrm{con}(R_{h,k})) \geq (k+2h+1)4^h$.

Proof Let X be the set of all 2^h leaves of T'_h, and let Y be the set of 2^h vertices that have a neighbor in X (i.e., Y is the set of vertices at distance $2h-1$ from the root of T'_h). The shortest path in $\mathrm{con}(R_{h,k})$ between an arbitrary vertex $x \in X$ and an arbitrary vertex $y \in Y$ consists of:

- h edges from x to the root of T'_h via black vertices of T'_h (see Fig. 3),
- $k+1$ edges from the root of T'_h to its neighbor on C_{2k+1} via vertices of C_{2k+1} and
- h edges to y via white vertices of T'_h.

Therefore, since $x \in X$ and $y \in Y$ are chosen arbitrarily,

$$W(\mathrm{con}(R_{h,k})) \geq 2^h \cdot 2^h \cdot (k+2h+1). \qquad \square$$

Theorem 7. *For each constant $C > 0$, there exists $h_0 \in \mathbb{N}$ such that for all $h \geq h_0$*

$$W(\mathrm{con}(R_{h,h^2})) > C \cdot W(R_{h,h^2}).$$

Proof Since

$$\frac{W(\mathrm{con}(R_{h,k}))}{W(R_{h,k})} \geq \frac{(k+2h+1)4^h}{h\,4^{h+2} + k\binom{2k+1}{2} + (2h+k+1)\,(2^{h+2}-3)\,(2k+1)},$$

it follows that for $k = h^2$

$$\lim_{h\to\infty} \frac{W(\mathrm{con}(R_{h,h^2}))}{W(R_{h,h^2})} = \infty.$$

Therefore, there exists h_0 such that for all $h \geq h_0$,

$$\frac{W(\mathrm{con}(R_{h,h^2}))}{W(R_{h,h^2})} > C. \qquad \square$$

References

[1] M. Knor, B. Lužar, R. Škrekovski, I. Gutman, On Wiener Index of Common Neighborhood Graphs, *MATCH Commun. Math. Comput. Chem.* **72** (2014), 000–000.