MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **72** (2014) 215-225

ISSN 0340 - 6253

On Incidence Energy

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(Received June 22, 2013)

Abstract

The incidence energy of a graph is defined as the sum of singular values of its incidence matrix. In this paper, we establish some new bounds on the incidence energy of connected graphs.

1 Introduction

Let G be a simple connected graph with n and m edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. For $v_i \in V(G)$, the degree of the vertex v_i, denoted by d_i, is the number of vertices adjacent to v_i. Let Δ and δ be the maximum and the minimum vertex degree of G, respectively.

Let $A(G)$ be the $(0,1)$-adjacency matrix of G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues. The eigenvalues of $A(G)$ are said to be [3] the eigenvalues of G and to form its spectrum. Then the energy of the graph G is defined as [12]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

An extensive work has been done on graph energy in the literature. For more details, see [13, 22] and the references cited therein.

Nikiforov [28] extended the concept of graph energy to any matrix defining it as the sum of singular values of this matrix. Recall that the singular values of a (real) matrix M

are equal to the square roots of the eigenvalues of MM^T, where M^T denotes the transpose of M. In particular, for a graph G, $E(G) = E(A(G))$.

Let $I(G)$ be the vertex-edge incidence matrix of the graph G. For a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, the (i,j)-entry of $I(G)$ is 1 if the vertex v_i is incident with the edge e_j, and is 0 otherwise. Motivating the idea in [28], Jooyandeh et al. defined the energy of the incidence matrix $I(G)$, namely, the incidence energy as [19]

$$IE = IE(G) = \sum_{i=1}^{n} \sigma_i$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the singular values of $I(G)$. Various properties and bounds for the incidence energy were recently established in [1, 15, 16, 19, 25, 29, 31, 33, 34].

Let $D(G)$ be the diagonal matrix of vertex degrees of the graph G. Then Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$. The eigenvalues of $L(G)$ are said to be the Laplacian eigenvalues of G. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the Laplacian eigenvalues of G. It is well known that $\mu_n = 0$ and the multiplicity of zero is equal to the number of connected components of G [10]. For more information on Laplacian eigenvalues, see [26, 27].

The signless Laplacian matrix of the graph G is defined as $Q(G) = D(G) + A(G)$. Let $q_1 \geq q_2 \geq \cdots \geq q_n$ be the eigenvalues of $Q(G)$. These eigenvalues are non-negative real numbers and called signless Laplacian eigenvalues of G. For more information on signless Laplacian eigenvalues, see [4, 5, 6, 7]. As well known in graph theory, $Q(G) = I(G)\, I(G)^T$. Then

$$IE = IE(G) = \sum_{i=1}^{n} \sqrt{q_i}$$

which was discovered in [15].

Short time ago, Liu and Liu introduced the quantity LEL defined by [24]

$$LEL = LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}$$

and called it Laplacian-energy like invariant of the graph G. It was first considered [24] that LEL has properties similar to the Laplacian energy [17]. But later, it was discovered [18] that it is much more similar to the ordinary graph energy [12, 13, 22]. For survey and more details on LEL, see the review [23] and the recent papers [9, 14, 24]. Note that the Laplacian and signless Laplacian eigenvalues of bipartite graphs coincide [4, 26, 27]. Therefore, for bipartite graphs, LEL is equal to the incidence energy IE [15].

In this paper, we establish some new bounds on the incidence energy of connected graphs considering the idea in [9].

2 Lemmas

Let $t = t(G)$ be the number of spanning trees of a graph G. Let $\overline{G}$ denotes the complement of the graph G and let $G_1 \times G_2$ denotes the Cartesian product of the graphs G_1 and G_2 [3]. We now introduce the following auxiliary quantities for a graph G as

$$t_1 = t_1(G) = \frac{2t(G \times K_2)}{t(G)} \text{ and } T = T(G) = \frac{1}{2}\left[\Delta + \delta + \sqrt{(\Delta - \delta)^2 + 4\Delta}\right] \tag{1}$$

where Δ and δ are the maximum and the minimum vertex degree of G, respectively.

Lemma 2.1. *[4, 26, 27]The spectra of $L(G)$ and $Q(G)$ coincide if and only if the graph G is bipartite.*

Lemma 2.2. *[6] If G is a connected bipartite graph of order n, then $\prod_{i=1}^{n-1} q_i = \prod_{i=1}^{n-1} \mu_i = nt(G)$. If G is a connected non-bipartite graph of order n, then $\prod_{i=1}^{n} q_i = t_1(G)$.*

Lemma 2.3. *[2, 32] Let G be a connected graph with $n \geq 3$ vertices and Δ be the maximum vertex degree of G. Then*

$$q_1 \geq T \geq \Delta + 1$$

with either equalities if and only if G is a star graph $K_{1,n-1}$.

Lemma 2.4. *[30] Let G be a simple connected graph with n vertices. Then $\rho_1 \leq 2\Delta$, with equality if and only if G is a regular graph.*

Lemma 2.5. *[4] Let G be a connected graph with diameter $d(G)$. If G has exactly k distinct signless Laplacian eigenvalues, then $d(G) + 1 \leq k$.*

Lemma 2.6. *[26] Let G be a graph with n vertices. Then $\mu_1 \leq n$, with equality if and only if $\overline{G}$ is disconnected.*

Lemma 2.7. *[8] Let G be a connected graph with n vertices. Then $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$ if and only if $G \cong K_n$.*

Lemma 2.8. *[8] Let G be a connected graph with $n \geq 3$ vertices. Then $\mu_2 = \mu_3 = \cdots = \mu_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$ or $G \cong K_{\Delta,\Delta}$.*

Lemma 2.9. *[21] Let $x_1, x_2, \ldots, x_N$ be non-negative numbers and let*

$$\alpha = \frac{1}{N}\sum_{i=1}^{N} x_i \text{ and } \gamma = \left(\prod_{i=1}^{N} x_i\right)^{1/N}$$

be their arithmetic and geometric means, respectively. Then

$$\frac{1}{N(N-1)}\sum_{i<j}\left(\sqrt{x_i}-\sqrt{x_j}\right)^2 \le \alpha - \gamma \le \frac{1}{N}\sum_{i<j}\left(\sqrt{x_i}-\sqrt{x_j}\right)^2. \tag{2}$$

Moreover, equality in (2) holds if and only if $x_1 = x_2 = \cdots = x_N$.

Lemma 2.10. *[11] For $a_1, a_2, \ldots, a_n \ge 0$ and $p_1, p_2, \ldots, p_n \ge 0$ such that $\sum_{i=1}^{n} p_i = 1$*

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \ge n\lambda\left(\frac{1}{n}\sum_{i=1}^{n} a_i - \prod_{i=1}^{n} a_i^{1/n}\right) \tag{3}$$

where $\lambda = \min\{p_1, p_2, \ldots, p_n\}$. Moreover, equality in (3) holds if and only if $a_1 = a_2 = \cdots = a_n$.

Lemma 2.11. *[20] For $a, b \ge 0$ and $\nu \in [0,1]$,*

$$(1-v)\,a + vb \ge a^{1-v}b^{v} + r\left(\sqrt{a}-\sqrt{b}\right)^2$$

where $r = \min\{\nu, 1-\nu\}$.

3 Main Results

In this section, we present some results on the incidence energy of connected bipartite graphs which improve the results obtained in [9]. We also consider the connected non-bipartite graphs and obtain some new bounds on the incidence energy of these graphs.

Theorem 3.1. *Let G be a connected graph with $n \ge 3$ vertices, m edges, maximum degree Δ and t spanning trees and let t_1 and T be given by (1).*

(i) If G is bipartite, then

$$LEL(G) = IE(G) \le \sqrt{n} + \sqrt{(n-3)(2m-T) + (n-2)\left(\frac{nt}{T}\right)^{1/(n-2)}} \tag{4}$$

and

$$LEL(G) = IE(G) \ge \sqrt{T} + \sqrt{2m - n + (n-2)(n-3)\,t^{1/(n-2)}}. \tag{5}$$

Moreover, equalities in (4) and (5) hold if and only if $G \cong K_{1,n-1}$.

(ii) If G is non-bipartite, then

$$IE\left(G\right) < \sqrt{2\Delta} + \sqrt{\left(n-2\right)\left(2m-T\right) + \left(n-1\right)\left(\frac{t_1}{T}\right)^{1/(n-1)}} \tag{6}$$

and

$$IE\left(G\right) > \sqrt{T} + \sqrt{2m - 2\Delta + \left(n-1\right)\left(n-2\right)\left(\frac{t_1}{2\Delta}\right)^{1/(n-1)}}. \tag{7}$$

Proof. Considering Lemmas 2.1–2.3,2.6,2.8 and 2.9 the proof of (i) can be easily given similar to the proof of Theorem 2.5. in [9]. Here, we only prove (ii).

Taking $N = n-1$ and $x_i = \rho_i$, $i = 2, \ldots, n$ in Lemma 2.9 and using Lemma 2.2, we obtain

$$\frac{\sum_{2\le i<j\le n}\left(\sqrt{\rho_i} - \sqrt{\rho_j}\right)^2}{\left(n-1\right)\left(n-2\right)} \le \frac{2m-\rho_1}{n-1} - \left(\frac{t_1}{\rho_1}\right)^{1/(n-1)} \le \frac{\sum_{2\le i<j\le n}\left(\sqrt{\rho_i} - \sqrt{\rho_j}\right)^2}{n-1}.$$

Since $\sum_{i=1}^{n}\rho_i = 2m$, we have

$$\begin{aligned}
\sum\nolimits_{2\le i<j\le n}\left(\sqrt{\rho_i} - \sqrt{\rho_j}\right)^2 &= \left(n-2\right)\sum_{i=2}^{n}\rho_i - 2\sum\nolimits_{2\le i<j\le n}\sqrt{\rho_i\rho_j} \\
&= \left(n-2\right)\left(2m-\rho_1\right) - \left(\sum_{i=2}^{n}\sqrt{\rho_i}\right)^2 + \sum_{i=2}^{n}\rho_i \\
&= \left(n-1\right)\left(2m-\rho_1\right) - \left(IE - \sqrt{\rho_1}\right)^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\left(n-1\right)\left(2m-\rho_1\right) - \left(IE - \sqrt{\rho_1}\right)^2}{\left(n-1\right)\left(n-2\right)} &\le \frac{2m-\rho_1}{n-1} - \left(\frac{t_1}{\rho_1}\right)^{1/(n-1)} \\
&\le \frac{\left(n-1\right)\left(2m-\rho_1\right) - \left(IE - \sqrt{\rho_1}\right)^2}{n-1}.
\end{aligned}$$

This implies that

$$IE \le \sqrt{\rho_1} + \sqrt{\left(n-2\right)\left(2m-\rho_1\right) + \left(n-1\right)\left(\frac{t_1}{\rho_1}\right)^{1/(n-1)}} \tag{8}$$

and

$$IE \ge \sqrt{\rho_1} + \sqrt{2m - \rho_1 + \left(n-1\right)\left(n-2\right)\left(\frac{t_1}{\rho_1}\right)^{1/(n-1)}}. \tag{9}$$

Consider the function

$$f\left(x\right) = \sqrt{\left(n-2\right)\left(2m-x\right) + \left(n-1\right)\left(\frac{t_1}{x}\right)^{1/(n-1)}}.$$

It can be easliy seen that $f(x)$ is decreasing for $x \geq \Delta+1$. By Lemma 2.3, we have $\rho_1 \geq T \geq \Delta+1$. Therefore we obtain

$$f(x) \leq f(T) = \sqrt{(n-2)(2m-T)+(n-1)\left(\frac{t_1}{T}\right)^{1/(n-1)}}.$$

Considering this, Eq. (8) and Lemma,2.4, we get

$$\begin{aligned} IE(G) &\leq \sqrt{\rho_1} + \sqrt{(n-2)(2m-T)+(n-1)\left(\frac{t_1}{T}\right)^{1/(n-1)}} \\ &\leq \sqrt{2\Delta} + \sqrt{(n-2)(2m-T)+(n-1)\left(\frac{t_1}{T}\right)^{1/(n-1)}}. \end{aligned}$$

Hence the inequality (6) holds. Similar to the above manner, let us consider the function

$$g(x) = \sqrt{2m-x+(n-1)(n-2)\left(\frac{t_1}{x}\right)^{1/(n-1)}}.$$

It can be easliy seen that $g(x)$ is decreasing for $x \leq 2\Delta$. Therefore

$$g(x) \geq g(2\Delta) = \sqrt{2m-2\Delta+(n-1)(n-2)\left(\frac{t_1}{2\Delta}\right)^{1/(n-1)}}.$$

Considering this, Eq. (9) and Lemma 2.3, we get

$$\begin{aligned} IE &\geq \sqrt{\rho_1} + \sqrt{2m-2\Delta+(n-1)(n-2)\left(\frac{t_1}{2\Delta}\right)^{1/(n-1)}} \\ &\geq \sqrt{T} + \sqrt{2m-2\Delta+(n-1)(n-2)\left(\frac{t_1}{2\Delta}\right)^{1/(n-1)}}. \end{aligned}$$

Hence the inequality (7) holds. Either equality in (6) and (7) holds if and only if $\rho_1 = T$, $\rho_2 = \rho_3 = \cdots = \rho_n$ and $\rho_1 = 2\Delta$. From the conditions $\rho_1 = 2\Delta$ and $\rho_2 = \rho_3 = \cdots = \rho_n$, we conclude that $G \cong K_n$. However $\rho_1(K_n) = 2(n-1)$ which differs from $T(K_n) = n-1+\sqrt{n-1}$. Thus, (6) and (7) cannot become an equality. □

Remark 3.2. *From Lemmas 2.1 and 2.3, we have $\mu_1 = \rho_1 \geq T \geq \Delta+1$ for bipartite graphs. Then by the proof of Theorem 2.5 in [9], one may conclude that the bounds (4) and (5) improve the bounds of Theorem 2.5 in [9] for bipartite graphs.*

Theorem 3.3. *Let G be a connected graph with $n \geq 3$ vertices and t spanning trees and let t_1 and T be given by (1).*

(i) If G is bipartite, then

$$LEL(G) = IE(G) > \sqrt{T} + (n-2)(nt)^{1/[2(n-1)]}\left(\frac{2(nt)^{1/[4(n-1)(n-2)]}}{T^{1/[4(n-2)]}} - 1\right). \qquad (10)$$

(ii) If G is non-bipartite, then

$$IE\left(G\right) > \sqrt{T} + \left(n-1\right)\left(t_1\right)^{1/(2n)}\left(\frac{2\left(t_1\right)^{1/[4n(n-1)]}}{T^{1/[4(n-1)]}} - 1\right). \tag{11}$$

Proof. Considering Lemmas 2.1–2.3,2.7 and 2.10 one can prove the inequality (10) similar to proof of Theorem 2.12 in [9]. Here, we only prove the inequality (11).

Taking $a_i = \sqrt{\rho_i}$, $i = 1, 2, \ldots, n$, $p_1 = \frac{1}{2n}$, $p_i = \frac{2n-1}{2n(n-1)}$, $i = 2, 3, \ldots, n$ in Lemma 2.10, we obtain

$$\begin{aligned} & \frac{\sqrt{\rho_1}}{2n} + \frac{2n-1}{2n(n-1)}\sum_{i=2}^{n}\sqrt{\rho_i} - \rho_1^{1/(4n)} \times \prod_{i=2}^{n}\rho_i^{(2n-1)/[4n(n-1)]} \\ \geq\ & \frac{1}{2n}\sum_{i=1}^{n}\sqrt{\rho_i} - \frac{1}{2}\prod_{i=1}^{n}\rho_i^{1/(2n)}. \end{aligned}$$

Then by Lemma 2.2, we get

$$\begin{aligned} & \frac{\sqrt{\rho_1}}{2n} + \frac{2n-1}{2n(n-1)}\left(IE - \sqrt{\rho_1}\right) - \rho_1^{-1/[4(n-1)]} \times \left(t_1\right)^{(2n-1)/[4n(n-1)]} \\ \geq\ & \frac{1}{2n}IE - \frac{1}{2}\left(t_1\right)^{1/(2n)} \end{aligned}$$

and

$$IE \geq 2(n-1)\left[\frac{\left(t_1\right)^{(2n-1)/[4n(n-1)]}}{\rho_1^{1/[4(n-1)]}} - \frac{1}{2}\left(t_1\right)^{1/(2n)} + \frac{\sqrt{\rho_1}}{2(n-1)}\right]. \tag{12}$$

Let us consider the function

$$f(x) = \frac{\left(t_1\right)^{(2n-1)/[4n(n-1)]}}{x^{1/[4(n-1)]}} + \frac{\sqrt{x}}{2(n-1)}.$$

It can be easily seen that $f\left(x\right)$ is increasing for $x > \left(t_1\right)^{1/n}$. By Lemma 2.3, we have

$$\rho_1 \geq T \geq \Delta + 1 > \Delta \geq \frac{2m}{n}.$$

Using Arithmetic-Geometric Mean Inequality and Lemma 2.2, we get

$$\frac{2m}{n} = \frac{\sum_{i=1}^{n}\rho_i}{n} \geq \left(\prod_{i=1}^{n}\rho_i\right)^{1/n} = \left(t_1\right)^{1/n}.$$

Therefore

$$f\left(x\right) \geq f\left(T\right) = \frac{\left(t_1\right)^{(2n-1)/[4n(n-1)]}}{T^{1/[4(n-1)]}} + \frac{\sqrt{T}}{2(n-1)}.$$

Combining this with (12), we get the inequality (11). Now we assume that the equality in (11) holds. Then all inequalities in the above arguments must be equalities. Then, by Lemmas 2.3 and 2.10, $\rho_1 = T$ and $\rho_1 = \rho_2 = \cdots = \rho_n = \frac{2m}{n}$. Thus, we have $\rho_1 = \frac{2m}{n} \leq \Delta < \Delta + 1 \leq T$, which is a contradiction. Hence we conclude that (11) cannot become an equality. $\square$

Remark 3.4. *From Lemmas 2.1 and 2.3, we have $\mu_1 = \rho_1 \geq T \geq \Delta + 1$ for bipartite graphs. Then by the proof of Theorem 2.12 in [9], one may conclude that the bound (10) improves the bound of Theorem 2.12 in [9] for bipartite graphs.*

Theorem 3.5. *Let G be a connected graph with $n \geq 3$ vertices, m edges and t spanning trees and let t_1 be defined by (1).*

(İ) If G is bipartite, then

$$LEL\,(G) = IE\,(G) > \sqrt{\left(\frac{n-1}{n-2}\right)\left[(n-1)^2\,(nt)^{1/(n-1)} - 2m\right]}. \tag{13}$$

(ii) If G is non-bipartite, then

$$IE\,(G) > \sqrt{\left(\frac{n}{n-1}\right)\left[n^2\,(t_1)^{1/n} - 2m\right]}. \tag{14}$$

Proof. Inequality (13) has been established by Das et al. [9]. Therefore, its proof will be omitted. Here, we only prove the inequality (14).

By Arithmetic-Geometric Mean Inequality, we have

$$\sum_{i=1}^{n} \rho_i^{(n-1)/(2n)} \geq n \left(\prod_{i=1}^{n} \rho_i\right)^{(n-1)/(2n^2)} \tag{15}$$

and

$$\sum_{j=1}^{n} \rho_j^{(n+1)/(2n)} \geq n \left(\prod_{j=1}^{n} \rho_j\right)^{(n+1)/(2n^2)}. \tag{16}$$

Taking $\nu = \frac{n+1}{2n}$, $a = \rho_i$, $b = \rho_j$ and $r = \frac{n-1}{2n}$ in Lemma 2.11, we get

$$\frac{n-1}{2n}\rho_i + \frac{n+1}{2n}\rho_j \geq \rho_i^{(n-1)/(2n)}\rho_j^{(n+1)/(2n)} + \frac{n-1}{2n}\left(\rho_i + \rho_j - 2\sqrt{\rho_i\rho_j}\right).$$

By summation over i and j yields

$$\begin{aligned} &\frac{n-1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_i + \frac{n+1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_j \\ \geq\ & \sum_{i=1}^{n}\rho_i^{(n-1)/(2n)}\sum_{j=1}^{n}\rho_j^{(n+1)/(2n)} + \frac{n-1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\rho_i + \rho_j - 2\sqrt{\rho_i\rho_j}\right). \end{aligned}$$

Then

$$2nm \geq n^2\left(\prod_{i=1}^{n}\rho_i\right)^{(n-1)/(2n^2)}\left(\prod_{j=1}^{n}\rho_j\right)^{(n+1)/(2n^2)} + 2m(n-1) - \left(\frac{n-1}{n}\right)IE^2$$

by (15) and (16). Considering Lemma 2.2, we get

$$IE^2 \geq \left(\frac{n}{n-1}\right)\left[n^2\left(t_1\right)^{1/n} - 2m\right].$$

Hence the inequality (14) holds. Now, we assume that the equality in (14) holds. Then, by Arithmetic-Geometric Mean Inequality, it must be $\rho_1 = \rho_2 = \cdots = \rho_n$. Thus by Lemma 2.5, $d\left(G\right) = 0$ which is a contradiction, since G is connected. Hence, (14) cannot become an equality. □

Acknowledgement: The authors are partially supported by TÜBİTAK and the Office of Selçuk University Scientific Research Project (BAP).

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