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Inequalities Between Vertex-Degree-Based Topological Indices

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Abstract

In this work, we obtain several inequalities between some vertex-degree-based topological indices, such as Randić index, aton–bond connectivity (ABC) index, sum–connectivity index and harmonic index. Sharp lower bounds for the harmonic index of graphs are also presented.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Randić index $R(G)$, proposed by Randić [25] in 1975, is defined as the sum of the weights $\frac{1}{\sqrt{d_u\, d_v}}$ over all edges uv of G, that is,

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u\, d_v}}$$

where d_u denotes the degree of a vertex u of G. The Randić index is one of the most successful vertex-degree-based molecular descriptors (topological indices) in structure-property and structure-activity relationship studies [20, 24]. Mathematical properties of this descriptor have also been studied extensively, as summarized in [13, 21].

A few years ago, Estrada et al. [8] introduced a new vertex-degree-based topological index, nowadays known as the aton–bond connectivity (ABC) index. This index is defined as

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u\, d_v}}\,.$$

It displays an excellent correlation with the heat of formation of alkanes [8], and a basically topological approach was developed on the basis of the ABC index to explain the differences in the energy of linear and branched alkanes both qualitatively and quantitatively [7]. The mathematical properties of this index were reported in [3, 11, 14, 15, 18].

The sum–connectivity index $X(G)$ was recently proposed by Zhou and Trinajstić in [34] and defined as

$$X(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}\,.$$

It has been found that the sum–connectivity index and the Randić index correlate well between themselves and with the π-electronic energy of benzenoid hydrocarbons [22, 23]. Some mathematical properties of the sum–connectivity index were given in [5, 27, 28, 34, 36].

The harmonic index $H(G)$ is another vertex-degree-based topological index. This index first appeared in [9], and was defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}\,.$$

Favaron, Mahéo and Saclé [10] considered the relation between the harmonic index and the eigenvalues of graphs. See [4, 19, 29–32] for more information of this index. Note that both the sum–connectivity index and the harmonic index can be viewed as particular cases of the general sum–connectivity index proposed by Zhou and Trinajstić in [35] (see also in [6, 26]).

In this paper, we first present sharp lower bounds for the harmonic index of graphs, and characterize graphs for which these bounds are best possible, and then we obtain several inequalities between the Randić index, ABC index, sum–connectivity index and harmonic index.

2 Lower bounds for the harmonic index

In this section, we present several sharp lower bounds for the harmonic index of graphs, and characterize the corresponding extremal graphs.

The first Zagreb index [12, 16] of a graph G is defined as

$$M_1(G) = \sum_{v \in V(G)} d_v^2 \ .$$

This index is also an important vertex-degree-based topological index, and it can be rewritten as $M_1(G) = \sum_{uv \in E(G)} (d_u + d_v)$. Ilić [19] and Xu [29] independently found the following relation between the harmonic index and the first Zagreb index.

Lemma 2.1 *Let G be a graph with $m \geq 1$ edges. Then*

$$H(G) \geq \frac{2m^2}{M_1(G)}$$

with equality if and only if $d_u + d_v$ is a constant for every edge uv of G.

There are many upper bounds for the first Zagreb index, from which we may deduce lower bounds for the harmonic index by Lemma 2.1. We give three examples in (a)–(c). We use S_n , P_n , and K_n to denote the star, the path and the complete graph with n vertices, respectively.

(a) Let G be a graph with $m \geq 1$ edges containing no isolated vertices. For each edge uv of G, we have $d_u + d_v \leq m + 1$ with equality if and only if every other edge of G is adjacent to the edge uv. Then

$$M_1(G) \leq \sum_{uv \in E(G)} (m+1) = m(m+1)$$

and thus

$$H(G) \geq \frac{2m}{m+1}$$

with equality if and only if G has no two independent edges, i.e., $G \cong S_{m+1}$ or $G \cong K_3$.

(b) Let G be a triangle- and quadrangle-free graph with n vertices and $m \geq 1$ edges. Then $M_1(G) \leq n(n-1)$ with equality if and only if $G \cong S_n$ or G is a Moore graph of diameter 2 [33], and thus $H(G) \geq \frac{2m^2}{n(n-1)}$ with equality if and only if $G \cong S_n$ or G is a Moore graph of diameter 2.

(c) Let G be a graph with n vertices, $m \geq 1$ edges, maximum degree Δ and minimum degree δ. Then [2] $M_1(G) \leq 2m(\Delta + \delta) - n\Delta\,\delta$ with equality if and only if G has only two types of degrees Δ and δ, and thus

$$H(G) \geq \frac{2m^2}{2m(\Delta + \delta) - n\Delta\,\delta}$$

with equality if and only if one vertex has degree Δ and the other vertex has degree δ for every edge of G. ∎

Zhong [30] proved that S_n is the unique extremal graph with the minimum harmonic index among all connected graphs with n vertices. Using item (a), we can generalize this result to graphs with n vertices containing no isolated vertices, and we show that the extremal graph is still S_n . This also implies a shorter proof than the proof in [30].

Theorem 2.2 *Let G be a graph with n vertices containing no isolated vertices. Then*

$$H(G) \geq \frac{2(n-1)}{n}$$

with equality if and only if $G \cong S_n$.

Proof. First suppose that G is a connected graph. Let m be the number of edges of G, then $m \geq n-1$. Since $\frac{2m}{m+1}$ is strictly monotonically increasing in $m \geq 1$, by item (a) above, we have $H(G) \geq \frac{2m}{m+1} \geq \frac{2(n-1)}{n}$ with equalities if and only if $G \cong S_{m+1}$ and $m = n-1$, i.e., $G \cong S_n$.

So we may assume that G is disconnected. Let $G_1, G_2, \ldots, G_k$ $(k \geq 2)$ be the connected components of G with $|V(G_i)| = n_i$ for each $1 \leq i \leq k$. Since G contains no isolated vertices, we have $n_i \geq 2$ and $\sum_{i=1}^{k} n_i = n$. Then

$$H(G) = \sum_{i=1}^{k} H(G_i) \geq \sum_{i=1}^{k} \frac{2(n_i - 1)}{n_i} \ .$$

Since

$$\frac{2(n_1-1)}{n_1} + \frac{2(n_2-1)}{n_2} - \frac{2(n_1+n_2-1)}{n_1+n_2} = \frac{2(n_1^2 n_2 + n_1 n_2^2 - n_1^2 - n_2^2 - n_1 n_2)}{n_1 n_2 (n_1+n_2)}$$

$$= \frac{2n_1^2(n_2-2) + 2n_2^2(n_1-2) + 2(n_1^2 - n_1 n_2 + n_2^2)}{n_1 n_2 (n_1+n_2)} > 0$$

we conclude that

$$H(G) > \frac{2(n_1+n_2-1)}{n_1+n_2} + \sum_{i=3}^{k} \frac{2(n_i-1)}{n_i} > \frac{2(n_1+n_2+n_3-1)}{n_1+n_2+n_3} + \sum_{i=4}^{k} \frac{2(n_i-1)}{n_i}$$

$$> \cdots > \frac{2(n_1+n_2+\cdots+n_k-1)}{n_1+n_2+\cdots+n_k} = \frac{2(n-1)}{n} \ .$$

This completes the proof of the theorem. ∎

3 Inequalities between Randić index and harmonic indices

In this section, we consider the relations between the Randić index and the harmonic index. Nordhaus–Gaddum–type results for the harmonic index are also given.

Theorem 3.1 *Let G be a graph with n vertices, then*

$$\frac{2\sqrt{n-1}}{n} R(G) \leq H(G) \leq R(G) \ . \tag{3.1}$$

The lower bound is attained if and only if $G \cong S_n$, and the upper bound is attained if and only if all connected components of G are regular.

Proof. Let uv be an edge of G. By the symmetry between u and v, we may assume that $1 \le d_u \le d_v \le n-1$. To prove (3.1), we consider the function

$$f(x,y) = \frac{\frac{2}{x+y}}{\frac{1}{\sqrt{xy}}} = \frac{2\sqrt{xy}}{x+y}$$

with $1 \le x \le y \le n-1$. Since

$$\frac{\partial f(x,y)}{\partial x} = \frac{\sqrt{y}(y-x)}{\sqrt{x}(x+y)^2} \ge 0 \qquad \text{and} \qquad \frac{\partial f(x,y)}{\partial y} = \frac{\sqrt{x}(x-y)}{\sqrt{y}(x+y)^2} \le 0$$

we have $f(x,y)$ is strictly monotonically increasing in x and monotonically decreasing in y. Hence the minimum value of $f(x,y)$ is attained for $(x,y) = (1, n-1)$, and the maximum value of $f(x,y)$ is attained for $x = y$ (for each $1 \le x \le n-1$). In other words,

$$\frac{2\sqrt{n-1}}{n} = f(1, n-1) \le f(x,y) \le f(x,x) = 1 \ .$$

Consequently,

$$\frac{2\sqrt{n-1}}{n} \le \frac{H(G)}{R(G)} \le 1$$

with the left equality if and only if $(d_u, d_v) = (1, n-1)$ for every edge uv of G, and the right equality if and only if $d_u = d_v$ for every edge uv of G. This proves the theorem. ∎

Caporossi et al. [1] showed that among all graphs with n vertices, the graphs containing no isolated vertices, in which all connected components are regular, have the maximum value $n/2$ for the Randić index. Then by Theorem 3.1, we know that these graphs are also the extremal graphs with the maximum harmonic index. This implies the following Nordhaus–Gaddum–type results for the harmonic index.

Theorem 3.2 *Let G be a graph with n vertices, then*

$$\frac{n}{2} \le H(G) + H(\overline{G}) \le n \ . \tag{3.2}$$

The lower bound is attained if and only if $G \cong K_n$ or $\overline{G} \cong K_n$, and the upper bound is attained if and only if G is a k-regular graph with $1 \le k \le n-2$.

Proof. Let m and $\overline{m}$ be the number of edges in G and $\overline{G}$, respectively. Then

$$\begin{aligned} H(G)+H(\overline{G}) &= \sum_{uv\in E(G)} \frac{2}{d_u+d_v} + \sum_{uv\in E(\overline{G})} \frac{2}{(n-1-d_u)+(n-1-d_v)} \\ &\geq \sum_{uv\in E(G)} \frac{2}{2n-2} + \sum_{uv\in E(\overline{G})} \frac{2}{2n-2} \\ &= \frac{2}{2n-2}(m+\overline{m}) = \frac{2}{2n-2}\cdot\frac{n(n-1)}{2} = \frac{n}{2} \end{aligned}$$

with equality if and only if either $d_u = d_v = n-1$ for every edge uv of G or $E(G) = \emptyset$, i.e., $G \cong K_n$ or $\overline{G} \cong K_n$. So the lower bound of (3.2) holds.

We now prove the upper bound of (3.2). By Theorem 3.1, we have

$$H(G)+H(\overline{G}) \leq R(G)+R(\overline{G}) \leq \frac{n}{2}+\frac{n}{2} = n$$

with equalities if and only if both G and $\overline{G}$ contain no isolated vertices (i.e., $1 \leq \delta(G) \leq \Delta(G) \leq n-2$) and all connected components of G and $\overline{G}$ are regular. We claim that G must be a regular graph. For otherwise, there exist two vertices u, v in G such that $d_u \neq d_v$. Then u and v are contained in two different connected components of G, and hence $uv \in E(\overline{G})$. But this forces u and v lie in the same component of $\overline{G}$, a contradiction. So Theorem 3.2 holds. ∎

4 Inequalities between sum–connectivity and harmonic indices

In this section, we present some inequalities between the sum–connectivity index and the harmonic index. Combining with Theorem 3.1, we also obtain some relations between the Randić index and the sum–connectivity index.

Theorem 4.1 *Let G be a connected graph with $n \geq 3$ vertices, then*

$$\sqrt{\frac{2}{n-1}}\,X(G) \leq H(G) \leq \frac{2}{\sqrt{3}}\,X(G)\ . \tag{4.1}$$

The lower bound is attained if and only if $G \cong K_n$, and the upper bound is attained if and only if $G \cong P_3$.

Proof. Let uv be an edge of G. By the symmetry between u and v, we may assume that $1 \leq d_u \leq d_v \leq n-1$. Since G is a connected graph with $n \geq 3$ vertices, we have $d_v \geq 2$. We define a function

$$f(x,y) = \frac{\frac{2}{x+y}}{\frac{1}{\sqrt{x+y}}} = \frac{2}{\sqrt{x+y}}$$

with $1 \leq x \leq y \leq n-1$ and $y \geq 2$. It is easy to see that $f(x,y)$ is strictly monotonically decreasing in both x and y. Therefore the minimum value of $f(x,y)$ is $f(n-1,n-1)$, and the maximum value of $f(x,y)$ is $f(1,2)$. That is to say,

$$\sqrt{\frac{2}{n-1}} = f(n-1,n-1) \leq f(x,y) \leq f(1,2) = \frac{2}{\sqrt{3}}$$

and thus

$$\sqrt{\frac{2}{n-1}} \leq \frac{H(G)}{X(G)} \leq \frac{2}{\sqrt{3}} \, .$$

The left equality holds if and only if $(d_u, d_v) = (n-1, n-1)$ for every edge uv of G, and the right equality holds if and only if $(d_u, d_v) = (1, 2)$ for every edge uv of G. This finishes the proof of the theorem. ∎

If the graph G has minimum degree at least $k \geq 2$, then the upper bound of (4.1) can be further improved.

Corollary 4.2 *Let G be a connected graph with minimum degree at least $k \geq 2$. Then*

$$H(G) \leq \sqrt{\frac{2}{k}}\, X(G)$$

with equality if and only if G is a k-regular graph.

Zhou and Trinajstić [36] recently proved that if G is a connected graph with $n \geq 3$ vertices, then $\sqrt{\frac{2}{3}}\, R(G) \leq X(G)$ with equality if and only if $G \cong P_3$. In fact, a similar argument shows that if the graph G has minimum degree at least $k \geq 2$, then $\sqrt{\frac{k}{2}}\, R(G) \leq X(G)$ with equality if and only if G is a k-regular graph. We further have the following.

Corollary 4.3 *Let G be a connected graph with $n \geq 3$ vertices. Then*

$$X(G) \leq \sqrt{\frac{n-1}{2}}\, R(G)$$

with equality if and only if $G \cong K_n$.

Proof. By Theorem 3.1 and Theorem 4.1, we have

$$X(G) \leq \sqrt{\frac{n-1}{2}}\, H(G) \leq \sqrt{\frac{n-1}{2}}\, R(G)$$

with the first equality if and only if $G \cong K_n$, and the second equality if and only if G is a regular graph. This proves the corollary. ∎

5 Inequalities between Randić and ABC indices

Theorem 5.1 *Let G be a connected graph with $n \geq 3$ vertices, then*

$$R(G) \leq ABC(G) \leq \sqrt{2n-4}\, R(G) \ . \tag{5.1}$$

The lower bound is attained if and only if $G \cong P_3$, and the upper bound is attained if and only if $G \cong K_n$.

Proof. Let uv be an edge of G. By the symmetry between u and v, we may assume that $1 \leq d_u \leq d_v \leq n-1$. Since G is a connected graph with $n \geq 3$ vertices, we have $d_v \geq 2$. In order to prove (5.1), we consider the function

$$f(x,y) = \frac{\sqrt{\frac{x+y-2}{xy}}}{\frac{1}{\sqrt{xy}}} = \sqrt{x+y-2}$$

with $1 \leq x \leq y \leq n-1$ and $y \geq 2$. Obviously, $f(x,y)$ is strictly monotonically increasing in both x and y. Hence the minimum value of $f(x,y)$ is attained for $(x,y) = (1,2)$, and the maximum value of $f(x,y)$ is attained for $(x,y) = (n-1, n-1)$, i.e.,

$$1 = f(1,2) \leq f(x,y) \leq f(n-1,n-1) = \sqrt{2n-4} \ .$$

Then $1 \leq ABC(G)/R(G) \leq \sqrt{2n-4}$ with the left equality if and only if $(d_u, d_v) = (1,2)$ for every edge uv of G, and the right equality if and only if $(d_u, d_v) = (n-1, n-1)$ for every edge uv of G. This completes the proof of the theorem. ∎

A similar argument shows that the following result holds if we assume the graph G has minimum degree at least $k \geq 2$.

Corollary 5.2 *Let G be a connected graph with minimum degree at least $k \geq 2$. Then $\sqrt{2k-2}\, R(G) \leq ABC(G)$ with equality if and only if G is a k-regular graph.*

6 Inequalities between ABC and sum–connectivity indices

Theorem 6.1 *Let G be a connected graph with $n \geq 3$ vertices. Then*

(i) $\sqrt{\frac{3}{2}}\, X(G) \leq ABC(G)$ *with equality if and only if* $G \cong P_3$ *;*

(ii) $ABC(G) \leq \sqrt{2}\, X(G)$ *if* $n = 3$*, with equality if and only if* $G \cong K_3$ *;*
$ABC(G) \leq \sqrt{\frac{8}{3}} X(G)$ *if* $n = 4$*, with equality if and only if* $G \cong K_4$ *or* $G \cong S_4$ *;*
$ABC(G) \leq \sqrt{\frac{n(n-2)}{n-1}}\, X(G)$ *if* $n \geq 5$*, with equality if and only if* $G \cong S_n$ *.*

Proof. Let uv be an edge of G. By the symmetry between u and v, we may assume that $1 \leq d_u \leq d_v \leq n-1$. Since G is a connected graph with $n \geq 3$ vertices, we have $d_v \geq 2$. We consider the function

$$f(x,y) = \left(\frac{\sqrt{\frac{x+y-2}{xy}}}{\frac{1}{\sqrt{x+y}}} \right)^2 = \frac{(x+y)(x+y-2)}{xy}$$

with $1 \leq x \leq y \leq n-1$ and $y \geq 2$. Since

$$\frac{\partial f(x,y)}{\partial y} = \frac{(y^2 - x^2) + 2x}{xy^2} > 0$$

we see that $f(x,y)$ is strictly monotonically increasing in y. Therefore the minimum value of $f(x,y)$ is either $f(x,2)$ for some $1 \le x \le 2$ or $f(x,x)$ for some $2 \le x \le n-1$, and the maximum value of $f(x,y)$ is $f(x,n-1)$ for some $1 \le x \le n-1$.

Since $f(x,2) = \frac{x+2}{2}$ is strictly monotonically increasing in $1 \le x \le 2$ and $f(x,x) = \frac{4(x-1)}{x}$ is strictly monotonically increasing in $2 \le x \le n-1$, we conclude that the minimum value of $f(x,y)$ is $f(1,2) = 3/2$. Hence

$$\sqrt{\frac{3}{2}} \le \frac{ABC(G)}{X(G)}$$

with equality if and only if $(d_u, d_v) = (1,2)$ for every edge uv of G. So (i) holds.

On the other hand, we have

$$f(x,n-1) = \frac{(x+n-1)(x+n-3)}{(n-1)x}$$

and

$$\frac{df(x,n-1)}{dx} = \frac{x^2-(n^2-4n+3)}{(n-1)x^2} = \frac{(x+\sqrt{n^2-4n+3})(x-\sqrt{n^2-4n+3})}{(n-1)x^2} \, .$$

If $n=3$, then $df(x,n-1)/dx \ge 0$. This implies that $f(x,n-1)$ is strictly monotonically increasing in x, and hence the maximum value of $f(x,y)$ is $f(n-1,n-1) = f(2,2) = 2$. If $n \ge 4$, then it is easy to check that $f(x,n-1)$ is strictly monotonically decreasing in $1 \le x \le \sqrt{n^2-4n+3}$, and strictly monotonically increasing in $\sqrt{n^2-4n+3} \le x \le n-1$. So the maximum value of $f(x,n-1)$ is

$$\max\{f(1,n-1), f(n-1,n-1)\} = \max\left\{\frac{n(n-2)}{n-1}, \frac{4(n-2)}{n-1}\right\}$$

$$= \begin{cases} f(1,3) = f(3,3) = \frac{8}{3} & \text{if } n = 4 \\ f(1,n-1) = \frac{n(n-2)}{n-1} & \text{if } n \ge 5 \, . \end{cases}$$

Hence if $n=3$, then $ABC(G)/X(G) \le \sqrt{2}$ with equality if and only if $(d_u,d_v) = (2,2)$ for every edge uv of G. If $n=4$, then $ABC(G)/X(G) \le \sqrt{8/3}$ with equality if and only if $(d_u,d_v) = (1,3)$ or $(d_u,d_v) = (3,3)$ for every edge uv of G. If $n \ge 5$, then $ABC(G)/X(G) \le \sqrt{\frac{n(n-2)}{n-1}}$ with equality if and only if $(d_u,d_v) = (1,n-1)$ for every edge uv of G. This proves (ii), and hence completes the proof of the theorem. ∎

If the graph G has minimum degree at least $k \geq 2$, then the bound in Theorem 6.1(i) can be improved as follows.

Corollary 6.2 *Let G be a connected graph with minimum degree at least $k \geq 2$. Then $\sqrt{\frac{4(k-1)}{k}}\, X(G) \leq ABC(G)$ with equality if and only if G is a k-regular graph.*

7 Inequalities between ABC and harmonic indices

Theorem 7.1 *Let G be a connected graph with $n \geq 3$ vertices. Then*

(i) $\frac{3\sqrt{2}}{4}\, H(G) \leq ABC(G)$ *with equality if and only if* $G \cong P_3$;

(ii) $ABC(G) \leq \sqrt{2n-4}\, H(G)$ *if* $3 \leq n \leq 6$, *with equality if and only if* $G \cong K_n$;
$ABC(G) \leq \frac{n}{2}\sqrt{\frac{n-2}{n-1}}\, H(G)$ *if* $n \geq 7$, *with equality if and only if* $G \cong S_n$.

Proof. By Theorem 4.1 and Theorem 6.1,

$$H(G) \leq \frac{2}{\sqrt{3}}\, X(G) \leq \frac{2}{\sqrt{3}} \cdot \sqrt{\frac{2}{3}}\, ABG(G) = \frac{4}{3\sqrt{2}}\, ABC(G)$$

i.e., $\frac{3\sqrt{2}}{4}\, H(G) \leq ABC(G)$ with equalities if and only if $G \cong P_3$. This proves (i).

We now prove (ii). Let uv be an edge of G. By the symmetry between u and v, we may assume that $1 \leq d_u \leq d_v \leq n-1$. Since G is a connected graph with $n \geq 3$ vertices, we have $d_v \geq 2$. We define a function

$$f(x,y) = \left(\frac{\sqrt{\frac{x+y-2}{xy}}}{\frac{2}{x+y}} \right)^2 = \frac{(x+y)^2(x+y-2)}{4xy}$$

with $1 \leq x \leq y \leq n-1$ and $y \geq 2$. Since

$$\frac{\partial f(x,y)}{\partial y} = \frac{(x+y)[x(y+2)+(y^2-x^2)+y(y-2)]}{4xy^2} > 0$$

we know that $f(x,y)$ is strictly monotonically increasing in y. Hence the maximum value of $f(x,y)$ is $f(x,n-1)$ for some $1 \leq x \leq n-1$. We consider the function

$$f(x,n-1) = \frac{(x+n-1)^2(x+n-3)}{4(n-1)x} \, .$$

Then

$$\frac{df(x,n-1)}{dx} = \frac{(x+n-1)[2x^2+(n-3)x-(n-1)(n-3)]}{4(n-1)x^2}$$

$$= \frac{(x+n-1)\left(x-\frac{-(n-3)-\sqrt{(n-3)(9n-11)}}{4}\right)\left(x-\frac{-(n-3)+\sqrt{(n-3)(9n-11)}}{4}\right)}{2(n-1)x^2}\ .$$

If $n=3$ or $n=4$, then we have $df(x,n-1)/dx \geq 0$. This implies that $f(x,n-1)$ is strictly monotonically increasing in x, and hence the maximum value of $f(x,y)$ is $f(n-1,n-1)=2n-4$. If $n \geq 5$, then it is easy to calculate that $f(x,n-1)$ is strictly monotonically decreasing in $1 \leq x \leq \frac{-(n-3)+\sqrt{(n-3)(9n-11)}}{4}$, and strictly monotonically increasing in $\frac{-(n-3)+\sqrt{(n-3)(9n-11)}}{4} \leq x \leq n-1$. Then the maximum value of $f(x,n-1)$ is

$$\max\{f(1,n-1), f(n-1,n-1)\} = \max\left\{\frac{n^2(n-2)}{4(n-1)}, 2n-4\right\}$$

$$= \begin{cases} f(n-1,n-1)=2n-4 & \text{if } n=5 \text{ or } n=6 \\ f(1,n-1)=\frac{n^2(n-2)}{4(n-1)} & \text{if } n \geq 7\ . \end{cases}$$

Therefore if $3 \leq n \leq 6$, then $ABC(G)/H(G) \leq \sqrt{2n-4}$ with equality if and only if $(d_u, d_v) = (n-1, n-1)$ for every edge uv of G; if $n \geq 7$, then

$$\frac{ABC(G)}{H(G)} \leq \frac{n}{2}\sqrt{\frac{n-2}{n-1}}$$

with equality if and only if $(d_u, d_v) = (1, n-1)$ for every edge uv of G. This completes the proof of the theorem. ∎

Similarly, we can improve the bound in Theorem 7.1(i) by Corollary 4.2 and Corollary 6.2.

Corollary 7.2 *Let G be a connected graph with minimum degree at least $k \geq 2$. Then $\sqrt{2k-2}\,H(G) \leq ABC(G)$ with equality if and only if G is a k-regular graph.*

Corollary 7.3 *Let G be a connected graph with minimum degree at least 2. Then $H(G) \leq R(G) \leq X(G) < ABC(G)$ with the first equality if and only if G is a regular graph, and the second equality if and only if G is a cycle.*

Deng et al. [4] recently considered the relation between the harmonic index and the chromatic number $\chi(G)$ and proved that $\chi(G) \leq 2H(G)$ for every connected graph G with equality if and only if G is a complete graph. It strengthens a conjecture relating the Randić index and the chromatic number which is based on the system AutoGraphiX and proved by Hansen and Vukičević [17]. We now present two sharp upper bounds of $\chi(G)$ in terms of the ABC index and the sum–connectivity index.

Corollary 7.4 *Let G be a connected graph with n vertices and minimum degree at least $k \geq 2$. Then*

$$\chi(G) \leq \sqrt{\frac{2}{k-1}}\, ABC(G) \qquad \text{and} \qquad \chi(G) \leq \sqrt{\frac{8}{k}}\, X(G)$$

with equalities if and only if $k = n-1$, i.e., $G \cong K_n$.

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