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Some Spaces Related to Cesàro Sequence Spaces and an Application to Crystallography

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Abstract This paper presents a connection between some new results from the theory of sequence spaces in functional analysis and physical chemistry and crystallography. We show that the sequence spaces a_p^r are equal to the sets of all sequences whose Cesàro means of order 1 are in ℓ_p for $1 < p < \infty$, but that this result does not hold for $p = 1$. As a consequence, we are able to considerably simplify the results concerning the dual spaces and matrix transformations involving the spaces a_p^r and $a_p^r(\Delta)$ in [1, 5], and to add the characterizations of more classes of matrix transformations. Our results also extend our recent ones in [10]. Furthermore, we demonstrate how our results can be applied to crystallography, in particular to determine the shape of Wulff's crystals which in some cases can be considered as neighborhoods in certain metrizable topologies. Finally we use our software for the graphical representation of some of the crystals and their surface energy functions.

1 Introduction, Notations and Known Results

Aydın and Başar defined the sequence spaces a_0^r, a_c^r, $a_0^r(\Delta)$ and $a_c^r(\Delta)$ for $0 < r < 1$ in [2, 3]. They determined some Schauder bases for their spaces, found the α–, β– and γ–duals, and characterized some classes of matrix transformations on them. Furthermore, various classes of compact matrix operators on the spaces a_0^r, a_c^r, $a_0^r(\Delta)$ and $a_c^r(\Delta)$ were characterized in [6, 7, 11]. In a recent paper [10], we included the sets a_∞^r and $a_\infty^r(\Delta)$ in our studies, and showed that the sets a_0^r, a_c^r and a_∞^r are equal to the matrix domains of the Cesàro matrix of order 1 in the sets c_0, c and ℓ_∞ of null, convergent and bounded sequences. Applying this result and using known results on the spaces of generalized weighted means established in [9] and [12], we were able to considerably simplify the results and their proofs in [2] and [3], and add the characterizations of some more classes of matrix transformations; in particular, the sets $a_0^r(\Delta)$, $a_c^r(\Delta)$ and $a_\infty^r(\Delta)$ reduce to simple special cases of the spaces s_α^0, s_α and $s_\alpha^{(c)}$ in [8].

Also Aydın [1], and Demiriz and Çakan [5] introduced the spaces a_p^r and $a_p^r(\Delta)$ for $0 < r < 1$ and $1 < p < \infty$. In this paper, we obtain the corresponding results to those in [10] for the spaces a_p^r and $a_p^r(\Delta)$, in particular, we are going to show that the sets a_p^r are equal to the matrix domains of the Cesàro matrix of order 1 in the sets ℓ_p for $1 < p < \infty$, and that this result does not extend to the case $p = 1$. We will also see that the spaces $a_p^r(\Delta)$ reduce to simple special cases of the spaces ℓ_α^p in [8] for $1 < p < \infty$.

Now we recall the most important notations, definitions and results needed in this paper.

A sequence $(b_n)_{n=0}^\infty$ in a linear metric space X is called a Schauder basis if, for each $x \in X$, there exists a unique sequence $(\lambda_n)_{n=0}^\infty$ of scalars such that $x = \sum_{n=0}^\infty \lambda_n b_n$.

By ω and ϕ we denote the sets of all complex sequences $x = (x_k)_{k=0}^\infty$ and all finite sequences. We write bs and cs for the sets of all bounded and convergent series; also let $\ell_p = \{x \in \omega : \sum_{k=0}^\infty |x_k|^p < \infty\}$ for $1 \leq p < \infty$. As usual, e and $e^{(n)}$ $(n = 0, 1, \dots)$ are the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

A subspace X of ω is said to be a BK space if it is a Banach space with continuous coordinates $P_n : X \to \mathbb{C}$ $(n = 0, 1, \dots)$ where $P_n(x) = x_n$ for all $x = (x_k)_{k=0}^\infty \in X$. A BK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k)_{k=0}^\infty \in X$ has a unique representation $x = \lim_{m\to\infty} x^{[m]}$ where $x^{[m]} = \sum_{n=0}^m x_n e^{(n)}$ is the m^{th} section of the sequence x.

If x and y are sequences and X and Y are subsets of ω, then we write $x \cdot y = (x_k y_k)_{k=0}^\infty$, $x^{-1} * Y = \{a \in \omega : a \cdot x \in Y\}$ and $M(X, Y) = \bigcap_{x\in X} x^{-1} * Y = \{a \in \omega : a \cdot x \in Y$ for all $x \in X\}$ for the multiplier space of X and Y; in particular, we use the notations $x^\alpha = x^{-1} * \ell_1$, $x^\beta = x^{-1} * cs$ and $x^\gamma = x^{-1} * bs$, and $X^\alpha = M(X, \ell_1)$, $X^\beta = M(X, cs)$ and $X^\gamma = M(X, bs)$ for the $\alpha-$, $\beta-$ and $\gamma-$duals of X.

Given any infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ of complex numbers and any sequence x, we write $A_n = (a_{nk})_{k=0}^{\infty}$ for the sequence in the n^{th} row of A, $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ $(n = 0, 1, \dots)$ and $Ax = (A_n x)_{n=0}^{\infty}$ for the A transform of x, provided $A_n \in x^{\beta}$ for all n. If X and Y are subsets of ω, then $X_A = \{x \in \omega : Ax \in X\}$ denotes the matrix domain of A in X and (X, Y) is the class of all infinite matrices that map X into Y; so $A \in (X, Y)$ if and only if $X \subset Y_A$. A matrix A is said to be regular, if $A \in (c, c)$ and $\lim_{n\to\infty} A_n x = \lim_{k\to\infty} x_k$ for all $x \in c$.

An infinite matrix $T = (t_{nk})_{n,k=0}^{\infty}$ is said to be a triangle if $t_{nk} = 0$ $(k > n)$ and $t_{nn} \neq 0$ for all n. We write $\mathcal{U}$ for the set of all sequences u with $u_k \neq 0$ for all k; if $u \in \mathcal{U}$ then $1/u = (1/u_k)_{k=0}^{\infty}$. Let $\mathbf{n+1} = (n+1)_{n=0}^{\infty}$. We define the matrices Σ, Δ, Δ^+ and $C^{(1)}$ by $\Sigma_{nk} = 1$ $(0 \leq k \leq n)$, $\Sigma_{nk} = 0$ $(n < k)$, $\Delta_{nn} = \Delta^+_{nn} = 1$, $\Delta_{n+1,n} = \Delta^+_{n,n+1} = -1$, $\Delta_{nk} = \Delta^+_{nk} = 0$ (otherwise) and $C^{(1)}_{nk} = (1/(n+1))\Sigma_{nk}$ for all $n, k = 0, 1, \dots$.

Let $u, v \in \mathcal{U}$ and X be a subset of ω. The sets $W(u, v; X) = v^{-1} * (u^{-1} * X)_{\Sigma}$ of generalized weighted means were defined and studied in [9] and [12]. In particular, $W(1/(\mathbf{n+1}), e, c_0) = (c_0)_{C^{(1)}}$, $W(1/(\mathbf{n+1}), e, c) = c_{C^{(1)}}$, $W(1/(\mathbf{n+1}), e, \ell_\infty) = (\ell_\infty)_{C^{(1)}}$ and $W(1/(\mathbf{n+1}), e, \ell_p) = (\ell_p)_{C^{(1)}}$ $(1 \leq p < \infty)$ are the spaces of all sequences that are summable to 0, summable, and bounded by the Cesàro method $C^{(1)}$ of order 1 and whose $C^{(1)}$– transforms are in ℓ_p; we write $C_0 = (c_0)_{C^{(1)}}$, $C = c_{C^{(1)}}$, $C_\infty = (\ell_\infty)_{C^{(1)}}$, and $C_p = (\ell_p)_{C^{(1)}}$, for short.

Let $0 < r < 1$ and $A^{(r)} = (a^{(r)}_{nk})_{n,k=0}^{\infty}$ be the triangle with $a^{(r)}_{nk} = (1 + r^k)/(n+1)$ $(0 \leq k \leq n;\ n = 0, 1, \dots)$. Then we have $a^r_0 = (c_0)_{A^{(r)}}$, $a^r_c = c_{A^{(r)}}$ ([2]), $a^r_\infty = (\ell_\infty)_{A^{(r)}}$ ([10]), $a^r_p = (\ell_p)_{A^{(r)}}$ for $1 < p < \infty$ ([1]) and $a^r_1 = (\ell_1)_{A^{(r)}}$; we also write $\tilde{\mathbf{r}} = (1 + r^k)_{k=0}^{\infty}$, and observe $a^r_0 = \tilde{\mathbf{r}}^{-1} * C_0 = W(1/(\mathbf{n+1}), \tilde{\mathbf{r}}, c_0)$, $a^r_c = \tilde{\mathbf{r}}^{-1} * C = W(1/(\mathbf{n+1}), \tilde{\mathbf{r}}, c)$, $a^r_\infty = \tilde{\mathbf{r}}^{-1} * C_\infty = W(1/(\mathbf{n+1}), \tilde{\mathbf{r}}, \ell_\infty)$ and $a^r_p = \tilde{\mathbf{r}}^{-1} * C_p = W(1/(\mathbf{n+1}), \tilde{\mathbf{r}}, \ell_p)$ for $1 \leq p < \infty$. The spaces $a^r_0(\Delta) = (a^r_0)_\Delta$, $a^r_c(\Delta) = (a^r_c)_\Delta$ and $a^r_\infty(\Delta) = (a^r_\infty)_\Delta$ were studied in [3] and [10], and $a^r_p(\Delta) = (a^r_p)_\Delta$ for $1 < p < \infty$ were studied in [5].

We remark that since the matrices $A^{(r)}$ and Δ are triangles, ℓ_∞, c, c_0 and ℓ_p are BK spaces with respect to their natural norms defined by $\|x\|_\infty = \sup_k |x_k|$ and $\|x\|_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$ ([14, p. 55]), c_0 is a closed subspace of c and c is a closed subspace of ℓ_∞ ([14, Corollary 4.2.4]), the spaces a^r_∞, a^r_c, a^r_0 and a^r_p are BK spaces with their natural norms defined by $\|x\|_{a^r_\infty} = \|A^{(r)}x\|_\infty = \sup_n |A^{(r)}_n x|$ and $\|x\|_{a^r_p} = \|A^{(r)}x\|_p = (\sum_{k=0}^{\infty} |A^{(r)}_n x|^p)^{1/p}$ by [14, Theorem 4.3.12], a^r_0 is a closed subspace of a^r_c, and a^r_c is a closed subspace of a^r_∞

by [14, Theorem 4.3.14]; similarly $a_\infty^r(\Delta)$, $a_c^r(\Delta)$, $a_0^r(\Delta)$ and $a_p^r(\Delta)$ are BK spaces with their natural norms defined by $\|x\|_{a_\infty^r(\Delta)} = \|\Delta x\|_{a_\infty^r}$ and $\|x\|_{a_p^r(\Delta)} = \|\Delta x\|_{a_p^r}$, $a_0^r(\Delta)$ is a closed subspace of $a_c^r(\Delta)$, and $a_c^r(\Delta)$ is a closed subspace of $a_\infty^r(\Delta)$. These results contain [2, Theorem 2.1], [3, Theorem 2.1] and [5, Theorem 2.1].

Schauder bases for a_0^r and a_c^r were determined in [2, Theorem 3.1 (a) and (b)], for $a_0^r(\Delta)$ and $a_c^r(\Delta)$ in [3, Theorem 3.1 (a) and (b)] and for $a_p^r(\Delta)$ $(1 < p < \infty)$ in [5, Theorem 3.1]. We observe that, since c_0 and ℓ_p have AK, and $(e, e^{(0)}, e^{(1)}, \dots)$ is a Schauder basis for c, the statements in [2, Theorem 3.1 (a) and (b)] are an immediate consequence of the first part of [9, Theorem 2.2], and an application of the second part of [9, Theorem 2.2] to the bases of a_0^r, a_c^r and a_p^r yields the statements in [3, Theorem 3.1 (a) and (b)] and [5, Theorem 3.1]. We remark that, since matrix multiplication is associative for triangles by [14, Corollary 1.4.5], putting $B^{(r)} = A^{(r)} \cdot \Delta$, we obtain $a_0^r(\Delta) = (c_0)_{B^{(r)}}$, $a_c^r(\Delta) = c_{B^{(r)}}$ and $a_p^r(\Delta) = (\ell_p)_{B^{(r)}}$ and [3, Theorem 3.1 (a) and (b)] and [5, Theorem 3.1] would also be immediate consequences of [11, Corollary 2.3 (a) and (b)].

Our results have an interesting and important application in crystallography, in particular in the growth of crystals. According to *Wulff's principle* [15], the shape of a crystal is uniquely determined by its surface energy function. The details of Wulff's construction can be found in [13]. A surface energy function is a real–valued function depending on a direction in space. If the surface energy function is given by a norm then the shape of the corresponding crystal is given by a neighborhood in the dual norm.

We use our own software package which, among other things, enables us to graphically represent Wulff's crystals and their corresponding surface energy functions. We emphasize that all the graphics in this paper were created with our own software.

2 The Main Results

First we show that $a_p^r = C_p$ for all $r \in (0,1)$ and $1 < p < \infty$; we also show that this result does not extend to $p = 1$.

If y is any sequence then we write

$$\sigma_n(y) = C_n^{(1)} y = \frac{1}{n+1} \sum_{k=0}^{n} y_k \ (n = 0, 1, \dots)$$

for the n^{th} $C^{(1)}$ mean of the sequence y.

Theorem 2.1 *We have*

$$C_p \underset{\neq}{\subset} \left(\frac{1}{\mathbf{n+1}}\right)^{-1} * \ell_p \text{ for } 1 \le p < \infty \tag{2.1}$$

and

$$C_p = a_p^r \text{ for all } r \in (0,1) \text{ and } 1 < p < \infty. \tag{2.2}$$

Proof: First we show the inclusion in (2.1).

Let $x \in C_p$ be given. Then we have $\sum_{n=0}^{\infty} |\sigma_n(x)|^p < \infty$, and

$$\frac{x_n}{n+1} = \sigma_n(x) - \frac{n}{n+1} \cdot \sigma_{n-1}(x) \text{ for } n = 0, 1, \dots$$

implies

$$\left(\sum_{n=0}^{\infty}\left(\frac{|x_n|}{n+1}\right)^p\right)^{1/p} \le \left(\sum_{n=0}^{\infty}|\sigma_n(x)|^p\right)^{1/p} + \left(\sum_{n=0}^{\infty}|\sigma_{n-1}(x)|^p\right)^{1/p} < \infty,$$

that is, $(x_n/(n+1))_{n=0}^{\infty} \in \ell_p$ and so $x \in (1/(\mathbf{n+1}))^{-1} * \ell_p$. Thus we have shown the inclusion in (2.1).

Now we show that the inclusion in (2.1) is strict.

If $1 < p < \infty$, then obviously $e \in ((1/(\mathbf{n+1}))^{-1} * \ell_p) \setminus C_p$. If $p = 1$, then we choose the sequence $x = (1/\sqrt{k+1})_{k=0}^{\infty}$, and obtain $x_k/(k+1) = (k+1)^{-3/2}$ for $k = 0, 1, \dots$, hence $x \in (1/(\mathbf{n+1}))^{-1} * \ell_1$, but

$$\sigma_n(x) = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{\sqrt{k+1}} \ge \frac{1}{n+1} \cdot (n+1) \cdot \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{n+1}} \text{ for } n = 1, 2, \dots,$$

and so $(\sigma_n(x))_{n=0}^{\infty} \notin \ell_1$, that is, $x \notin C_1$. Thus we have shown that the inclusion in (2.1) is strict for $1 \le p < \infty$.

Now we show

$$C_p \subset a_p^r \text{ for } 1 < p < \infty. \tag{2.3}$$

Let $x \in C_p$. Then we have $\sum_{n=0}^{\infty} |\sigma_n(x)|^p = M^p < \infty$ for some real number M, and so

$$\begin{aligned}\left(\sum_{n=0}^{\infty}|\sigma_n(x \cdot \tilde{\mathbf{r}})|^p\right)^{1/p} &= \left(\sum_{n=0}^{\infty}\left|\sigma_n\left(((1+r^k)x_k)_{k=0}^{\infty}\right)\right|^p\right) \\ &\le \left(\sum_{n=0}^{\infty}|\sigma_n(x)|^p\right)^{1/p} + \left(\sum_{n=0}^{\infty}\left|\sigma_n\left((r^k x_k)_{k=0}^{\infty}\right)\right|^p\right)^{1/p} \\ &\le M + S_1, \text{ where } S_1 = \left(\sum_{n=0}^{\infty}\left|\sigma_n\left((r^k x_k)_{k=0}^{\infty}\right)\right|^p\right)^{1/p}.\end{aligned}$$

We also have $K = (\sum_{n=0}^{\infty}(n+1)^{-p})^{1/p} < \infty$ and $x \in (1/(\mathbf{n+1}))^{-1} * \ell_p$ by (2.1) which implies $D = \sup_k(|x_k|/(k+1)) < \infty$. So Minkowski's inequality yields

$$\begin{aligned} S_1 &= \left(\sum_{n=0}^{\infty}\left|\frac{1}{n+1}\sum_{k=0}^{n} r^k x_k\right|^p\right)^{1/p} \leq \sum_{k=0}^{\infty}|r^k x_k| \cdot \left(\sum_{n=k}^{\infty}\left(\frac{1}{n+1}\right)^p\right)^{1/p} \\ &\leq K \cdot \sum_{k=0}^{\infty} r^k(k+1)\frac{|x_k|}{k+1} \leq K \cdot D \cdot \sum_{k=0}^{\infty} r^k(k+1) < \infty. \end{aligned}$$

Therefore we have $\sum_{k=0}^{\infty}|\sigma_n(x \cdot \tilde{\mathbf{r}})|^p < \infty$, and so $x \in a_p^r$. Thus we have shown the inclusion in (2.3).

Finally we show

$$C_p \supset a_p^r \text{ for } 1 < p < \infty. \tag{2.4}$$

Let $x \in a_p^r$. Then we have $y = \tilde{\mathbf{r}}^{-1} \cdot x \in C_p$ and it follows similarly as above that

$$x = (1/\tilde{\mathbf{r}})^{-1} \cdot y = \left(\left(1 - \frac{r^k}{1+r^k}\right) y_k\right)_{k=0}^{\infty} \in a_p^r.$$

Finally, the inclusions in (2.3) and (2.4) yield the identity in (2.2). ∎

We need the following result to show that the identity in (2.2) does not hold for $p = 1$.

Proposition 2.2 *Let X and Y be arbitrary subsets of ω and A and B be triangles. Then we have $X_A \subset Y_B$ if and only if $BA^{-1} \in (X, Y)$, where A^{-1} denotes the inverse of A.*

Proof: We use several times in the proof that matrix multiplication of triangles is associative ([14, Corollary 1.4.5]). We also note that every triangle T has a unique inverse S which also is a triangle ([4, Remark 22 (a)] and [14, 1.4.8]).

First we show that $X_A \subset Y_B$ implies $BA^{-1} \in (X, Y)$. Let $x \in X$ be given. Then $x = (AA^{-1})x = A(A^{-1}x) \in X$ implies $A^{-1}x \in X_A \subset Y_B$, hence $(BA^{-1})x = B(A^{-1}x) \in Y$, that is, $BA^{-1} \in (X, Y)$.

Now we show that $BA^{-1} \in (X, Y)$ implies $X_A \subset Y_B$. We assume $BA^{-1} \in (X, Y)$. Let $x \in X_A$ be given. Then we have $Ax \in X$, and so $(BA^{-1})(Ax) = B(AA^{-1})x = Bx \in Y$, that is, $x \in Y_B$. Thus we have shown $X_A \subset Y_B$. ∎

Theorem 2.3 *We have* $a_1^r \not\subset C_1$ *and* $C_1 \not\subset a_1^r$.

Proof: Let y be a sequence and $D(y)$ denote the diagonal matrix with y on the main diagonal. We denote the inverse of the Cesàro matrix $C^{(1)}$ by $B = (b_{nk})_{n,k=0}^{\infty}$, that is, $b_{nn} = n+1$, $b_{n-1,k} = -n$ and $b_{nk} = 0$ otherwise, and put $M(y) = C^{(1)}D(y)B$. Then we have

$$M(\tilde{\mathbf{r}}) = C^{(1)}D(\tilde{\mathbf{r}})B = A^{(r)}(C^{(1)})^{-1}$$

and

$$M(1/\tilde{\mathbf{r}}) = C^{(1)}D(1/\tilde{\mathbf{r}})B = C^{(1)}\left(C^{(1)}D(\tilde{\mathbf{r}})\right)^{-1} = C^{(1)}\left(A^{(r)}\right)^{-1}$$

and the statement of the theorem will follow from Proposition 2.2, once we have shown $M(\tilde{\mathbf{r}}), M(1/\tilde{\mathbf{r}}) \notin (\ell_1, \ell_1)$.

Since

$$\left(C^{(1)}D(y)\right)_{nj} = \sum_{l=0}^{\infty} c_{nl}^{(1)} d_{lj}(y) = \sum_{l=0}^{n} \frac{d_{lj}}{n+1} = \begin{cases} \dfrac{y_j}{n+1} & (0 \le j \le n) \\ 0 & (j > n) \end{cases} \qquad (n = 0, 1, \dots),$$

we obtain

$$\begin{aligned} (M(y))_{nk} &= \sum_{j=0}^{\infty} \left(C^{(1)}D(y)\right)_{nj} b_{jk} = \sum_{j=0}^{n} \frac{y_j b_{jk}}{n+1} \\ &= \begin{cases} y_n & (k = n) \\ \dfrac{k+1}{n+1} \cdot (y_k - y_{k+1}) & (0 \le k \le n-1) \\ 0 & (k > n) \end{cases} \qquad (n = 0, 1, \dots). \end{aligned}$$

It follows from

$$\sup_k \sum_{n=0}^{\infty} |(M(y))_{nk}| \ge \sum_{n=0}^{\infty} |(M(y))_{n0}| = |y_1 - y_0| \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$$

and by [14, Example 8.4.1D] that $M(y) \notin (\ell_1, \ell_1)$ for $y = \tilde{\mathbf{r}}, 1/\tilde{\mathbf{r}}$. ∎

We write $bv^p = (\ell_p)_\Delta = \{x \in \omega : \sum_{k=0}^{\infty} |x_k - x_{k-1}|^p < \infty\}$ for $1 \le p < \infty$.

Part (b) of the next result is related to [5, Theorem 2.3].

Theorem 2.4 *We have*

(a) $\ell_p \not\subset C_p$ *and* $C_p \not\subset \ell_p$ *for* $1 \le p < \infty$;

(b) $bv^p \not\subset (1/(\mathbf{n}+\mathbf{1}))^{-1} * \ell_p$ *and* $(1/(\mathbf{n}+\mathbf{1}))^{-1} * \ell_p \not\subset bv^p$ *for* $1 \le p < \infty$.

Proof: (a) If $p = 1$ then we obviously have $e^{(0)} \in \ell_1 \setminus C_1$, that is, $\ell_1 \not\subset C_1$.

If $1 < p < \infty$, we define the sequence x by

$$x_k = \begin{cases} \dfrac{1}{\nu+1} & (k = 2^\nu) \\ 0 & (k \neq 2^\nu) \end{cases} \quad (\nu = 0, 1, \dots).$$

Then it follows that

$$\sum_{k=0}^{\infty} |x_k|^p = \sum_{\nu=0}^{\infty} \frac{1}{(\nu+1)^p} < \infty, \text{ that is, } x \in \ell_p,$$

but we have for $n \in [2^\nu, 2^{\nu+1} - 1]$

$$\sigma_n(x) = \frac{1}{n+1} \sum_{\mu=0}^{\nu} \frac{1}{\mu+1} \geq \frac{1}{2^{\nu+1}} \cdot \frac{\nu+1}{\nu+1} = \frac{1}{2^{\nu+1}},$$

hence

$$\sum_{n=2^\nu}^{2^{\nu+1}-1} \sigma_n(x) \geq \frac{1}{2} \text{ for all } \nu = 0, 1, \dots,$$

and so $x \notin C_p$. Thus we have shown $x \in \ell_p \setminus C_p$, that is, $\ell_p \not\subset C_p$.

To show $C_p \not\subset \ell_p$, we define the sequence x by

$$x_k = \begin{cases} 1 & (k = 2^\nu) \\ -1 & (k = 2^\nu + 1) \\ 0 & (\text{otherwise}) \end{cases} \quad (\nu = 1, 2, \dots). \tag{2.5}$$

Then it follows for $n \in [2^\nu, 2^{\nu+1} - 1]$ that

$$\sigma_n(x) = \frac{1}{n+1} \sum_{\mu=0}^{\nu-1} \sum_{k=2^\mu}^{2^{\mu+1}-1} x_k + \frac{1}{n+1} \sum_{k=2^\nu}^{n} x_k = \begin{cases} \dfrac{1}{n+1} & (n = 2^\nu) \\ 0 & (n \neq 2^\nu) \end{cases} \quad (\nu = 1, 2, \dots),$$

so

$$\sum_{n=0}^{\infty} |\sigma_n(x)|^p = \sum_{\nu=1}^{\infty} \left(\frac{1}{2^\nu}\right)^p < \infty, \text{ that is, } x \in C_p,$$

but $x \notin \ell_p$, since $x \notin c_0$. Thus we have $x \in C_p \setminus \ell_p$, and so $C_p \not\subset \ell_p$.

(b) Part (b) is an immediate consequence of Part (a), since $y \in \ell_p$ if and only if $x = \Sigma y \in bv^p = (\ell_p)_\Delta$, and $y \in C_p$ if and only if $x = \Sigma y \in (C_p)_\Delta = (1/(\mathbf{n}+\mathbf{1}))^{-1} * \ell_p$. ■

Now we determine Schauder bases for the spaces C_p and $(1/(\mathbf{n}+\mathbf{1}))^{-1} * \ell_p$ for $1 \leq p < \infty$.

Theorem 2.5 *(a) Every sequence* $x = (x_k)_{k=0}^{\infty} \in C_p$ *has a unique representation*

$$x = \sum_{n=0}^{\infty} \sigma_n(x)(n+1)\left(e^{(n)} - e^{(n+1)}\right).$$

(b) The spaces $(1/(\mathbf{n}+\mathbf{1}))^{-1} * \ell_p$ $(1 \leq p < \infty)$ *have AK.*

Proof: (a) Since ℓ_p $(1 \leq p < \infty)$ has AK, it follows from [11, Corollary 2.3 (a)] that every sequence $x \in C_p$ has a unique representation

$$x = \sum_{n=0} \sigma_n(x) d^{(n)}, \text{ where } d^{(n)} = \left(C^{(1)}\right)^{-1} e^{(n)} = (n+1)\left(e^{(n)} - e^{(n+1)}\right) \text{ for } n = 0, 1, \ldots.$$

(b) Since ℓ_p $(1 \leq p < \infty)$ has AK, it follows by [14, Theorem 4.3.6] that $(1/(\mathbf{n}+\mathbf{1}))^{-1} * \ell_p$ has AK. ■

We showed in [10, Theorem 2.1] that a_0^r has AK. This result does not extend to the spaces a_p^r for $1 \leq p < \infty$.

Remark 2.6 The spaces C_p $(1 \leq p < \infty)$ *do not have AK.*

Proof: Since $e^{(0)} \notin C_1$, the space C_1 does not have AK.
Let $1 < p < \infty$ and x be the sequence defined in (2.5). We know from Part (b) of the proof of Theorem 2.4 that $x \in C_p$. Let $\nu \in \mathbb{N}$ be given. Then we have

$$\sigma_n\left(x - x^{[2^\nu]}\right) = \frac{1}{n+1} \text{ for all } n \in [2^\nu + 1, 2^{\nu+1} - 1],$$

hence

$$\left| \sum_{n=2^\nu+1}^{2^{\nu+1}} \sigma_n\left(x - x^{[2^\nu]}\right) \right| \geq \frac{2^\nu - 1}{2^{\nu+1}},$$

and so C_p does not have AK. ■

Now we determine the α–, β– and γ–duals of the spaces a_p^r for $1 \leq p < \infty$. We need the following known result which we state here for the reader's convenience.

Proposition 2.7 ([9, Theorem 3.1]) *Let* $u, v \in \mathcal{U}$. *We write* $b = (1/u) \cdot \Delta^+(a/v)$ *for* $a \in \omega$, q *for the conjugate number of* p, *that is,* $q = \infty$ *for* $p = 1$ *and* $q = p/(p-1)$ *for* $1 < p < \infty$, *and* $S_q(u,v) = \{a \in \omega : b \in \ell_q\}$. *Then we have*
(a) $(W(u,v,\ell_p))^\alpha = S_q(u,v) \cap ((1/(u \cdot v))^{-1} * \ell_q)$;
(b) $(W(u,v,\ell_p))^\beta = W(u,v,\ell_p))^\gamma = S_q(u,v) \cap ((1/(u \cdot v))^{-1} * \ell_\infty)$.

Proposition 2.7 yields the following corollary.

Corollary 2.8 *We use the notations of Proposition 2.7 and put*

$$S_q = S_q(1/(\mathbf{n+1}), e) = \left\{a \in \omega : (\mathbf{n+1}) \cdot \Delta^+ a = \left((n+1)\Delta_n^+ a\right)_{n=0}^{\infty} \in \ell_q\right\}.$$

Then we have

$$C_p^\alpha = (\mathbf{n+1})^{-1} * \ell_q \text{ for } 1 \leq p < \infty; \tag{a}$$

$$C_p^\beta = C_p^\gamma = \begin{cases} C_1^\alpha & (p=1) \\ S_q \cap (\mathbf{n+1})^{-1} * \ell_\infty & (1 < p < \infty). \end{cases} \tag{b}$$

Proof: (a) It follows from Proposition 2.7 (a) with $u = 1/(\mathbf{n+1})$ and $v = e$ that $C_p^\alpha = S_q \cap (\mathbf{n+1})^{-1} * \ell_q$, and obviously $S_q \supset (\mathbf{n+1})^{-1} * \ell_q$ for $1 < q \leq \infty$, that is, $C_p^\alpha = (\mathbf{n+1})^{-1} * \ell_q$ for $1 \leq p < \infty$.
(b) It follows from Proposition 2.7 (b) with $u = 1/(\mathbf{n+1})$ and $v = e$ that $C_p^\beta = C_p^\gamma = S_q \cap (\mathbf{n+1})^{-1} * \ell_\infty$.
If $p = 1$ then $q = \infty$, and $C_1^\beta = S_\infty \cap (\mathbf{n+1})^{-1} * \ell_\infty = (\mathbf{n+1})^{-1} * \ell_\infty = C_1^\alpha$ by Part (a).
If $1 < p < \infty$ then obviously $e \in S_q \setminus ((\mathbf{n+1})^{-1} * \ell_\infty)$ and

$$a = \left(\frac{(-1)^k}{k+1}\right)_{k=0}^{\infty} \in (\mathbf{n+1})^{-1} * \ell_\infty,$$

but

$$\left|(k+1)\Delta_k^+ a_k\right| = (k+1)\left(\frac{1}{k+1} + \frac{1}{k+2}\right) \geq 2 \cdot \frac{k+1}{k+2} \not\to 0 \text{ for } k \to \infty,$$

hence $a \notin S_q$. So we have $a \in ((\mathbf{n+1})^{-1} * \ell_\infty) \setminus S_q$. ∎

We use our results to give the duals of the sets a_p^r for $1 \leq p < \infty$ and $a_p^r(\Delta)$ for $1 < p < \infty$.

Remark 2.9 Let † denote any of the symbols α, β or γ.
(a) In view of (2.2), we have $(a_p^r)^\dagger = C_p^\dagger$ for $1 < p < \infty$. Also since $a_1^r = \tilde{\mathbf{r}}^{-1} * C_1$, we trivially have

$$(a_1^r)^\dagger = \left(\tilde{\mathbf{r}}^{-1} * C_1\right)^\dagger = (1/\tilde{\mathbf{r}})^{-1} * C_1^\dagger,$$

and so by Corollary 2.8 (a) and (b)

$$(a_1^r)^\dagger = ((\mathbf{n+1})/\tilde{\mathbf{r}})^{-1} * \ell_\infty = (\mathbf{n+1})^{-1} * \ell_\infty.$$

These results retrieve [1, Theorems 2.11 and 2.12].

(b) We also have for $1 < p < \infty$

$$a_p^r(\Delta) = (C_p)_\Delta = (1/(\mathbf{n}+\mathbf{1}))^{-1} * \ell_p = \ell_\alpha^p, \text{ where } \alpha = \mathbf{n}+\mathbf{1} \text{ ([8])}, \tag{2.6}$$

hence trivially

$$\left(a_p^r(\Delta)\right)^\dagger = (\mathbf{n}+\mathbf{1})^{-1} * \ell_q \text{ for } 1 < p < \infty. \tag{2.7}$$

In view of (2.6), it suffices to compare (2.7) with [5, Theorems 4.5, 4.6 and 4.7] for $\tilde{\mathbf{r}}$ *replace by* e*, that is,* $r = 0$*.*

By [5, Theorems 4.5], we have $a \in (a_p^0(\Delta))^\alpha$ *if and only if*

$$\sup_{\substack{K \subset \mathbb{N}_0 \\ K \text{ finite}}} \sum_{k=0}^\infty \left| \sum_{n \in K} c_{nk}^0 \right|^q < \infty \tag{2.8}$$

where $(c_{nk}^0)_{n,k=0}^\infty$ *is the diagonal matrix with* $c_{nn}^0 = (1+n)a_n$ $(n = 0, 1, \dots)$*, and clearly the condition in (2.8) is equivalent to* $a \in (\mathbf{n}+\mathbf{1})^{-1} * \ell_q$*.*

By [5, Theorems 4.7], we have $a \in (a_p^0(\Delta))^\gamma$ *if and only if*

$$\sup_n \sum_{k=0}^\infty |e_{nk}^0|^q < \infty \tag{2.9}$$

where $(e_{nk}^0)_{n,k=0}^\infty$ *is the triangle with* $e_{nk}^0 = (1+k)a_k$ *for* $0 \le k \le n$ *and* $n = 0, 1, \dots,$ *so the condition in (2.9) is equivalent to* $a \in (\mathbf{n}+\mathbf{1})^{-1} * \ell_q$*.*

Also the $\alpha-$*,* $\beta-$ *and* $\gamma-$*duals coincide in each case.*

It is worthwhile to take a closer look at [5, Theorem 4.6]. The following results are useful in our analysis.

Lemma 2.10 *Let* a *be a sequence and* $E^r = (e_{nk}^r)_{n,k=0}^\infty$ *be the triangle with*

$$e_{nk}^r = \begin{cases} (k+1)\left(\dfrac{a_k}{1+r^k} + \left(\dfrac{1}{1+r^k} - \dfrac{1}{1+r^{k+1}}\right)\sum\limits_{j=k+1}^n a_j\right) & (0 \le k \le n-1) \\ \dfrac{1+n}{1+r^n} \cdot a_n & (k = n) \end{cases}$$

for $n = 0, 1, \dots$*. We consider the conditions*

$$M = \sup_n \|E_n^r\|_q = \sup_n \left(\sum_{k=0}^n |e_{nk}^r|^q\right)^{1/q} < \infty, \tag{2.10}$$

$$\alpha_k = \lim_{n\to\infty} e_{nk}^r \text{ exists for each } k, \tag{2.11}$$

$$a \in cs \tag{2.12}$$

and

$$a \in ((\mathbf{n+1})/\tilde{\mathbf{r}})^{-1} * cs.. \tag{2.13}$$

Then we have

(a) *(2.11) is equivalent to (2.12);*

(b) *(2.10) implies (2.12);*

(c) *(2.10) does not imply (2.13), in general.*

Proof: (a) It is clear from the definition of the matrix E^r that (2.12) implies (2.11). Conversely, we assume that (2.11) is satisfied. We fix $k \in \mathbb{N}_0$, write

$$s_k = \frac{(1+r^k)(1+r^{k+1})}{(k+1)(r^{k+1}-r^k)} \text{ and } t_k = -\frac{1+r^{k+1}}{r^{k+1}-r^k},$$

and obtain by (2.11)

$$\sum_{j=k+1}^{n} a_j = s_k \cdot e^r_{nk} + t_k \cdot a_k \to s_k \cdot \alpha_k + t_k \cdot a_k \ (n \to \infty),$$

that is, (2.12) holds.

(b) First it follows from (2.10) that

$$|e^r_{n0}|^q = \left| \frac{a_0}{2} + \left(\frac{1}{2} - \frac{1}{1+r} \right) \sum_{j=1}^{n} a_j \right|^q \leq M^q \text{ for all } n,$$

that is, $a \in bs$. Furthermore, we obtain by Minkowski's inequality and from (2.10) with $\tilde{M} = \sup_{k,n} |\sum_{j=k+1}^{n} a_j|$ and $L = (1-r)\sum_{k=0}^{\infty}(k+1)r^k < \infty$, since $(1+r^k)(1+r^{k+1}) > 1$ for all k and $1/q < 1$,

$$\left(\sum_{k=0}^{n-1} \left((k+1) \frac{|a_k|}{1+r^k} \right)^q \right)^{1/q} \leq$$

$$\left(\sum_{k=0}^{n-1} |e^r_{nk}|^q \right)^{1/q} + \left(\sum_{k=0}^{n-1} \left((k+1) \left| \frac{1}{1+r^k} - \frac{1}{1+r^{k+1}} \right| \left| \sum_{j=k+1}^{n} a_j \right| \right)^q \right)^{1/q} \leq$$

$$M + \tilde{M} \left(\sum_{k=0}^{n-1} \left((k+1) \frac{r^k(1-r)}{(1+r^k)(1+r^{k+1})} \right)^q \right)^{1/q} \leq$$

$$M + (1-r)\tilde{M} \sum_{k=0}^{\infty} (k+1) r^k = M + L \tag{2.14}$$

for all n (the last inequality is a consequence of the Jensen's inequality since $q > 1$), hence $a \in ((\mathbf{n+1})/\tilde{\mathbf{r}})^{-1} * \ell_q$, and so $a \in (\mathbf{n+1})^{-1} * \ell_q$. Finally, we have by Hölder's inequality

$$\sum_{k=0}^{\infty} |a_k| \le \left(\sum_{k=0}^{\infty} ((k+1)|a_k|)^q \right)^{1/q} \cdot \left(\sum_{k=0}^{\infty} \frac{1}{(k+1)^p} \right)^{1/p} < \infty,$$

that is $a \in \ell_1$, and so $a \in cs$.

(c) Let $a = (a_k)_{k=0}^{\infty}$ be the sequence with $a_k = (1+r^k)/(k+1)^2$ for $k = 0, 1, \ldots$. Then we have for all n

$$S_n^{(1)} = \left(\sum_{k=0}^{n} \left| (k+1) \cdot \frac{a_k}{1+r^k} \right|^q \right)^{1/q} = \left(\sum_{k=0}^{n} \frac{1}{(k+1)^q} \right)^{1/q} \le \|1/(\mathbf{n+1})\|_q < \infty \quad (2.15)$$

and with $\tilde{N} = \sum_{k=0}^{\infty} a_k < \infty$ and $\tilde{L} = (1-r)\tilde{N} \sum_{k=0}^{\infty} (k+1) r^k < \infty$ as in (2.14)

$$S_n^{(2)} = \left(\sum_{k=0}^{n-1} \left| (k+1) \left(\frac{1}{1+r^k} - \frac{1}{1+r^{k+1}} \right) \sum_{j=k+1}^{n} a_j \right|^q \right)^{1/q} \le \tilde{L}, \quad (2.16)$$

hence by Minkowski's inequality and (2.15) and (2.16)

$$\sup_n \|E_n^r\|_q \le \sup_n \left(S_n^{(1)} + S_n^{(2)} \right) \le \|1/(\mathbf{n+1})\|_q + \tilde{L} < \infty,$$

that is, the sequence a satisfies (2.10). But, since $\sum_{k=0}^{\infty} (k+1)/(1+r^k) \cdot a_k = \sum_{k=0}^{\infty} 1/(k+1) = \infty$, the sequence a does not satisfy (2.13). ■

Remark 2.11 (a) It would follow by the argument in the proof of [5, Theorem 4.6] that $a \in (a_p^r(\Delta))^\beta$ *if and only if the conditions in (2.10) and (2.11) hold. By Lemma 2.10 (a) and (b), this is the case if and only if the condition in (2.10) holds; this is consistent with Remark 2.6.*

However, [5, Theorem 4.6] states that $a \in (a_p^r(\Delta))^\beta$ *if and only if the conditions in (2.10), (2.12) and (2.13) are satisfied, but it is not clear where (2.13) comes from, and it seems that this condition should be dropped by Lemma 2.10. We already mentioned that the condition in (2.12) is redundant.*

(b) The β*–dual of* $a_1^r(\Delta)$ *could be obtained from [11, theorem 3.2 or Lemma 3.8].*

We close with a few remarks on some characterizations of matrix transformations.

Remark 2.12 (a) The necessary and sufficient conditions for $A \in (W(u,v,X),Y)$ for arbitrary sequences $u,v \in \mathcal{U}$ were given in [9, Theorem 3.3] when $X = \ell_p$ $(1 \leq p \leq \infty)$, $X = c_0$ or $X = c$ and $Y = \ell_\infty, c, c_0$ or $Y = \ell_1$ $((\ell_p,\ell_1)$ with $1 < p < \infty)$. In particular, putting $u = 1/(\mathbf{n}+\mathbf{1})$ and $v = e$ and applying Theorem 2.1, we observe that the characterizations of the classes (a_p^r,ℓ_∞), (a_p^r,c) and (a_p^r,c_0) in [1, Theorems 3.1, 3.2 and 3.4] would be special cases of [9, Theorem 3.3 (1.), (3.) and (2.)]. In the same way, we would obtain the characterizations of (a_p^r,ℓ_1) for $1 < p < \infty$ from [9, Theorem 3.3 (17.)]. Furthermore, by [9, Remark 3.1], the necessary and sufficient conditions for A to map a_p^r into the matrix domains of a triangle T in ℓ_∞ or c can be obtained from [9, Theorem 3.3 (1.) and (3.)] by replacing the entries of A by $c_{nk} = \sum_{j=k}^n t_{nj}a_{jk}$ $(n,k = 0,1,\dots)$ in the relevant conditions. This would yield [1, Theorem 3.3 and Theorems 3.5–3.10].

(b) Let $u,v \in \mathcal{U}$. Then it is clear that, for arbitrary subsets X and Y of ω,

$$A \in (u^{-1} * X, v^{-1} * Y) \text{ if and only if } B \in (X,Y) \text{ where } b_{nk} = \frac{v_n a_{nk}}{u_k} \text{ for all } n,k.$$

Applying this result with $u = 1/(\mathbf{n}+\mathbf{1})$ and $v = e$, and (2.6), we immediately obtain the characterizations of the classes $(a_p^r(\Delta),\ell_\infty)$, $(a_p^r(\Delta),c)$, $(a_p^r(\Delta),c_0)$ and $(a_p^r(\Delta),\ell_1)$ for $1 < p < \infty$ from the well-known classical results, for instance in [14, Example 8.4.5D] for the first three classes and [14, Example 8.4.8B] for (ℓ_p,ℓ_1).

3 Graphical Representations of Neighbourhoods and Wulff's Crystals

Here we use our own software to illustrate some of our results by representing neighborhoods in the norms of our spaces and their duals. This also has an important application in crystallography related to the shape of Wulff's crystals.

We represent those neighborhoods of points in a metric space (X,d) which are open balls $B(x_0,r) = \{x \in X : d(x,x_0)\}$ of radius $r > 0$ and centre x_0 by their boundaries $\partial B(x_0,r)$. For instance, in three–dimensional space dimensional space they would be surfaces given by the equation $d(x,x_0) = r$.

According to Wulff's principle [15], the shape of a crystal is uniquely determined by its surface energy function, which is a real–valued function depending on a direction in space.

Let S^n denote the unit sphere in euclidean $\mathbb{R}$, and let $F : S^n \to \mathbb{R}$ be a surface energy function. Then we may consider the set $PS = \{\vec{x} = F(\vec{e}) \in \mathbb{R}^{n+1} : \vec{e} \in S^n\}$ as a natural representation of the function F.

If $n = 2$, then $\vec{e} = \vec{e}(u_1, u_2) = (\cos u_1 \cos u_2, \cos u_1 \sin u_2, \sin u_1)$ for $(u_1, u_2) \in R = (-\pi/2, \pi/2) \times (0, 2\pi)$, and obtain a potential surface with a parametric representation

$$PS = \{\vec{x} = f(u_1, u_2)(\cos u_1 \cos u_2, \cos u_1 \sin u_2, \sin u_1) : (u_1, u_2) \in R\}$$
$$\text{where } f(u_1, u_2) = F(\vec{e}(u_1, u_2)). \quad (3.1)$$

The principles of Wulff's construction of crystals were discussed in detail in [13], in particular, it was shown that if F is equal to a norm then the boundary of the corresponding Wulff's crystal is given by the dual norm. More precisely, we have

Corollary 3.1 ([13, Corollary 5.5]) *Let $\|\cdot\|$ be a norm on $\mathbb{R}^{n+1}$ and, for each $\vec{w} = (w_1, \ldots, w_{n+1}) \in S^n$, let $\phi_{\vec{w}} : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined by*

$$\phi_{\vec{w}}(\vec{x}) = \vec{w} \bullet \vec{x} = \sum_{k=1}^{n+1} w_k x_k \text{ for all } \vec{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}.$$

Then the boundary $\partial C_{\|\cdot\|}$ of Wulff's crystal corresponding to $F = \|\cdot\|$ is given by

$$\partial C_{\|\cdot\|} = \left\{\vec{x} = \left(\|\phi_{\vec{e}}\|^*\right)^{-1} \cdot \vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in S^n\right\}, \quad (3.2)$$

where $\|\phi_{\vec{e}}\|^*$ is the norm of the functional $\phi_{\vec{e}}$, that is, the dual norm of $\|\cdot\|$.

Remark 3.2 ([13, Remark 5.6]) *In the special case of $n = 2$, we obtain from (3.2) and (3.1) the following parametric representation for the boundary of Wulff's crystal corresponding to a norm $\|\cdot\|$ in $\mathbb{R}^{n+1}$*

$$\vec{x}(u_1, u_2) = \left(\|\phi_{\vec{e}}\|^*\right)^{-1} \cdot \vec{e}(u_1, u_2) \text{ for } (u_1, u_2) \in R;$$

the potential surface of has a parametric representation

$$\vec{y}(u_1, u_2) = \|\vec{e}(u_1, u_2)\| \cdot \vec{e}(u_1, u_2) \text{ for } (u_1, u_2) \in R.$$

Example 3.3 Finally we represent some potential surfaces and the corresponding Wulff's crystals for some of our spaces.

We identify the sequences $x = (x_k)_{k=0}^{\infty}$ with the three–dimensional vectors $\vec{x} = (x_1, x_2, x_3)$, and use Remark 3.2.

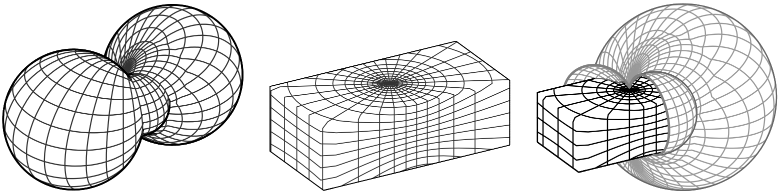

Figure 1: Potential surface given by the norm of C_1 and corresponding Wulff's crystal

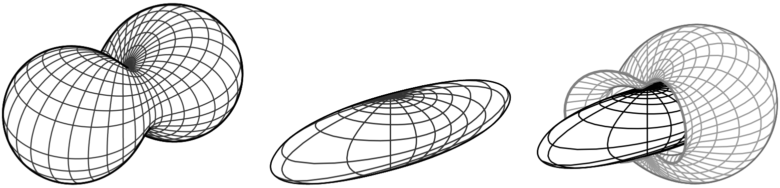

Figure 2: Potential surface given by the norm of C_p for $p = 1.8$ and corresponding Wulff's crystal

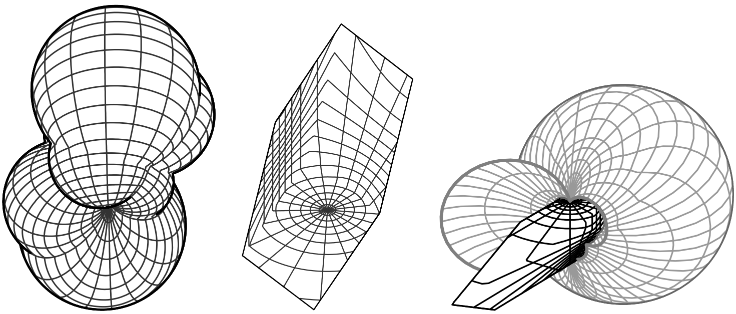

Figure 3: Potential surface given by the norm of $W(u, v, \ell_1)$ for $u = (6, 5, 3)$ and $v = (6, 2, 4)$ and corresponding Wulff's crystal

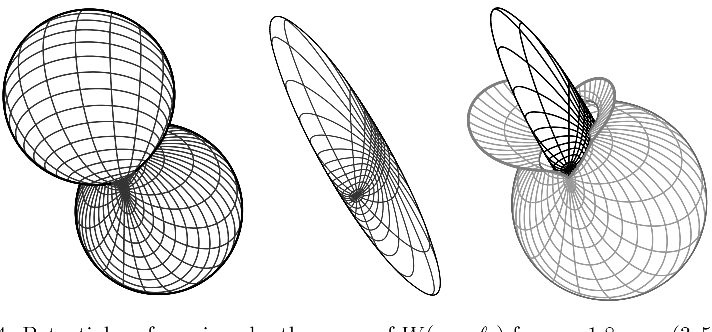

Figure 4: Potential surface given by the norm of $W(u, v, \ell_p)$ for $p = 1.8$, $u = (3, 5, 3)$ and $v = (6, 2, 4)$ and corresponding Wulff's crystal

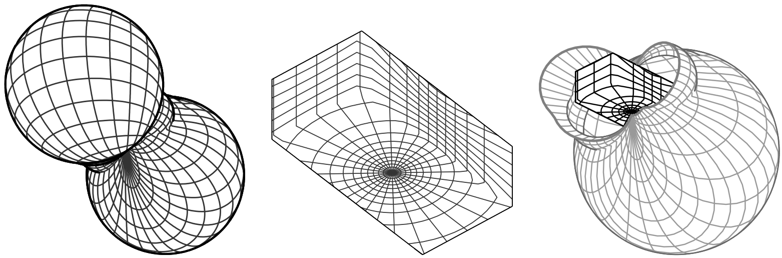

Figure 5: Potential surface given by the norm of a_1^r for $r = 0.5$ and corresponding Wulff's crystal

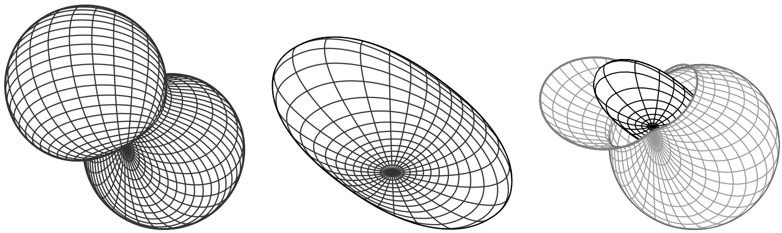

Figure 6: Potential surface given by the norm of a_p^r for $p = 1.8$ and $r = 0.5$, and corresponding Wulff's crystal

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