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Harary Index and Hamiltonian Property of Graphs

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Abstract

The Harary index is defined as the sum of reciprocal distances between all pairs of vertices in a nontrivial connected graph. All established results on Harary index mainly deal with bounds and extremal properties of Harary index. In this paper, we give a new sufficient condition, in terms of Harary index, for a connected bipartite graph to be Hamiltonian.

1 Introduction

Let G be a simple connected graph whose vertex set and edge set are $V(G)$ and $E(G)$, respectively. For a graph G, we use $d_G(v)$ to denote the degree of a vertex v in G and use $d_G(u, v)$ to denote the distance between two vertices u and v in G.

The *Harary index* of a graph G, denoted by $H(G)$, has been introduced independently by Plavšić et al. [7] and by Ivanciuc et al. [5] in 1993. It has been named in honor of Professor Frank Harary on the occasion of his 70th birthday. The Harary index of a simple connected graph G is defined as follows:

$$H(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d_G(u, v)}$$

where the summation goes over all pairs of vertices $\{u, v\}$ of G.

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For any given vertex v in G, if we let $\widehat{D}_G(v) = \sum\limits_{u \in V(G)} \frac{1}{d_G(u,v)}$, then the Harary index of G can be rewritten via

$$H(G) = \frac{1}{2} \sum_{v \in V(G)} \widehat{D}_G(v). \tag{1}$$

Up to now, there are many established results dealing with bounds and extremal properties of Harary index, published both in mathematical and in mathematical chemistry literature, see [2, 4, 6, 8–10] and the references quoted therein.

A connected graph is said to be *traceable* if it possesses a Hamiltonian path, that is, a path passing through all vertices of the underlying graph. A connected graph is said to be *Hamiltonian* if it possesses a Hamiltonian cycle, that is, a cycle passing through all vertices of the underlying graph. Concerning the existence of Hamiltonian cycle, there are many famous sufficient conditions such as Dirac's condition, Ore's condition, Fan's condition and so on.

More recently, Hua and Wang [3] use Harary index to give a new sufficient condition for a connected graph to be traceable. In this short paper, we establish a new sufficient condition, in terms of Harary index, for a connected bipartite graph to be Hamiltonian.

To arrive at our result, we need the following terminology and notation. The *eccentricity* of a vertex v in a graph G is defined to be $ec_G(v) = \max\{d_G(u, v)|u \in V(G)\}$. Denote by, as usual, K_n, P_n and C_n, the complete graph, the path and the cycle on n vertices, respectively. Let $K_{n,n-1}$ be a complete bipartite graph whose two bipartite sets being X and Y such that $|X| = n$ and $|Y| = n-1$. Let $K_{n,n-1}^{2n}$ be a bipartite graph of order $2n$ obtained from $K_{n,n-1}$ by introducing a pendent edge between an isolated vertex and a vertex in X.

For the sake of clarity, we will write d_i and $\widehat{D}_i$ instead of $d_G(v_i)$ and $\widehat{D}_G(v_i)$, respectively, in the next section. For other notation and terminology not defined here, the readers are referred to [1].

2 Harary index condition for connected bipartite graphs to be Hamiltonian

We first cite here a result, which gives a sufficient condition for a bipartite connected graph to be Hamiltonian.

Lemma 2.1 ([1])**.** *Let $G := G[X,Y]$ be a connected bipartite graph with $|X| = |Y| = n \geq 2$ and degree sequence $(d_1, d_2, \ldots, d_{2n})$, where $d_1 \leq d_2 \leq \ldots \leq d_{2n}$. Suppose that there exists no integers $k \leq \frac{n}{2}$ such that $d_k \leq k$ and $d_n \leq n-k$. Then G is Hamiltonian.*

Now, we use Lemma 2.1 to give a new sufficient condition, in terms of Harary index, for connected bipartite graphs to be Hamiltonian.

Theorem 2.2. *Let $G := G[X,Y]$ be a connected bipartite graph with $|X| = |Y| = n \geq 2$. If*

$$H(G) \geq \frac{3}{2}n^2 - n + \frac{1}{2},$$

then G is Hamiltonian.

Proof. By contradiction. Suppose that G is not a Hamiltonian connected bipartite graph whose degree sequence is $(d_1, d_2, \ldots, d_{2n})$ such that $d_1 \leq d_2 \leq \ldots \leq d_{2n}$. By Lemma 2.1, there exists an integer $k \leq \frac{n}{2}$ such that $d_k \leq k$ and $d_n \leq n-k$. Note that for any i in G, $i = 1, 2, \ldots, 2n$, we have $\widehat{D}_i \leq d_i + \frac{1}{2}(2n-1-d_i)$. By (1), we obtain

$$\begin{aligned}
H(G) &= \frac{1}{2}\sum_{i=1}^{2n}\widehat{D}_i \\
&\leq \frac{1}{2}\sum_{i=1}^{2n}\left[d_i + \frac{1}{2}(2n-1-d_i)\right] && (2)\\
&= \frac{1}{2}\sum_{i=1}^{2n}\left[\frac{2n-1}{2} + \frac{1}{2}d_i\right] \\
&= \frac{n(2n-1)}{2} + \frac{1}{4}\sum_{i=1}^{2n} d_i \\
&\leq \frac{n(2n-1)}{2} + \frac{1}{4}[k^2 + (n-k)^2 + n^2] \quad (\text{as } d_k \leq k \text{ and } d_n \leq n-k) && (3)\\
&= \frac{n(2n-1)}{2} + \frac{1}{2}(n^2 - n + 1) - \frac{1}{2}(k-1)(n-k-1) \\
&\leq \frac{n(2n-1)}{2} + \frac{1}{2}(n^2 - n + 1) && (4)\\
&= \frac{3}{2}n^2 - n + \frac{1}{2}.
\end{aligned}$$

Combining this fact and our assumption, we get $H(G) = \frac{3}{2}n^2 - n + \frac{1}{2}$. Therefore, all inequalities in (2), (3) and (4) should be equalities. Thus,

(a). From (2), we know that the diameter of G is no more than two.

(b). From (3), we know that $d_1 = \ldots = d_k = k$, $d_{k+1} = \ldots = d_n = n-k$ and $d_{n+1} = \ldots = d_{2n} = n$.

(c). From (4), we know that $k = 1$ or $k = n-1$.

If $k = n-1$, then by our assumption that $k \leq \frac{n}{2}$ and $n \geq 2$, we must have $n - 1 = \frac{n}{2}$, that is, $n = 2$. Since G is bipartite, we must have $G \cong P_4$ or C_4. If $G \cong P_4$, then the diameter of G is equal to three, a contradiction to (a). If $G \cong C_4$, then G is Hamiltonian, a contradiction to our choice of G.

So, we may suppose that $k \neq n-1$. By (c), we thus have $k = 1$. Combining this fact and (b), we can deduce that $G \cong K^{2n}_{n,n-1}$. But then, the unique pendent vertex in G has eccentricity three, a contradiction to (a).

This completes the proof. ■

3 Concluding remarks

In Section 2, we have given a new sufficient condition, in terms of Harary index, for a connected bipartite graph to be Hamiltonian. That is, so long as Harary index is bounded from below by a required value, one can conclude that the underlying graph is Hamiltonian. A problem relevant to our present study is: among all Hamiltonian graphs, which graphs attain the lower or upper bound for Harary index? This problem seems to be an easy one. Because the removal of edges from an underlying graph will decrease its Harary index, and noting that the underlying graph is Hamiltonian, we conclude that the cycle C_n and the complete graph K_n attain, respectively, the lower and upper bound for Harary index among all Hamiltonian graphs of order n.

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