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The Generalized (Terminal) Wiener Polarity Index of Generalized Bethe Trees and Coalescence of Rooted Trees*

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Abstract

Let G be a simple connected graph. The generalized polarity Wiener index of G is defined as the number of unordered pairs of vertices of G whose distance is k. The generalized terminal Wiener index of G is defined as the sum of distances between all pairs of k-degree vertices in G. The graph obtained by identifying the root vertices of two rooted trees T_1 and T_2 is called the coalescence of T_1 and T_2. We correct some formulas for computing the (terminal) Wiener index of generalized Bethe trees, recently reported in: A. Heydari, On the Wiener index and terminal Wiener index of generalized Bethe trees, *MATCH Commun. Math. Comput. Chem.* **69** (2013) 141–150. Furthermore, formulas are obtained for computing the generalized polarity Wiener index and the generalized terminal Wiener index of generalized Bethe trees and coalescence of n ($n \geq 2$) rooted trees. Formulas for the generalized terminal Wiener index of coalescence of n ($n \geq 2$) rooted trees are also given.

1 Introduction

Let G be a connected graph, with n vertices and m edges. The set $V(G) = \{v_1, ..., v_n\}$ denotes the vertex set of G while $E(G) = \{e_1, ..., e_n\}$ denotes it's edge set. The distance $d_G(v_i, v_j)$ between two vertices v_i and v_j in G is the length of a shortest path between

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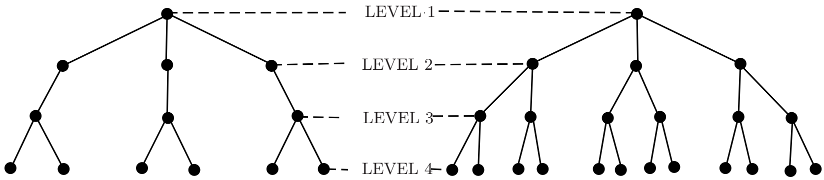
the two vertices and $deg(v)$ denotes the degree of vertex v. If $deg(v) = 1$, then v is called a *pendent vertex* of G.

A tree T, as we know, is a simple connected graph without cycles. Any vertex in a tree T can be chosen as a root vertex of T and the level of a vertex in T is exactly one more than the distance between this vertex and the root vertex.

Let B_{l+1} denote a rooted tree with $(l+1)$ levels such that all the vertices on the same level have equal degrees. Usually, B_{l+1} is called a $(l+1)$-level generalized Bethe tree [2]. As a particular case of generalized Bethe tree, a dendrimer tree [3] $T_{l,d}$ is a $(l+1)$-level generalized Bethe tree such that all its non-pendent vertices have a equal degree d. For instance, a generalized Bethe tree B_4 and a dendrimer tree $T_{3,3}$ are shown in Fig. 1.



(a)

A generalized Bethe tree B_4.

(b)

A dendrimer tree $T_{3,3}$

Figure 1: A generalized Bethe tree B_4 and a dendrimer tree $T_{3,3}$.

Suppose that T_1 and T_2 are two rooted trees. The graph obtained by identifying the root vertices of T_1 and T_2 is called the *coalescence* of T_1 and T_2, denoted by $T_1 \circ T_2$.

The Wiener index $W(G)$ and the Wiener polarity index $W_P(G)$ are, respectively, defined as [4]

$$W(G) = \sum_{\substack{1 \le i < j \le n \\ v_i, v_j \in V(G)}} d_G(v_i, v_j)$$

$$W_P(G) = \mid \{(u, v) \mid d_G(u, v) = 3, u, v \in V(G)\} \mid .$$

Motivated by the definition of $W_P(G)$, Gutman et al. [5] defined the terminal Wiener index as

$$TW(G) = \sum_{\substack{deg(u) = deg(v) = 1 \\ u, v \in V(G)}} d_G(u, v) \; .$$

Later, Ilić [6] extent the definition of Wiener polarity index and the terminal Wiener index of G, respectively, to be the generalized Wiener polarity index $W_k(G)$ and the generalized terminal Wiener index $TW_k(G)$, where

$$W_k(G) =| \{(u,v) \mid d_G(u,v) = k, u,v \in V(G)\}|$$

$$TW_k(G) = \sum_{\substack{deg(u)=deg(v)=k \\ u,v\in V(G)}} d_G(u,v) \ .$$

This paper is organized as follows. In Section 2, we correct some formulas of [1] for the (terminal) Wiener index of generalized Bethe trees. In Section 3, we obtain formulas for the generalized terminal Wiener index of the trees B_{l+1} and $T_{l,k}$. In Section 4, we establish formulas for the generalized Wiener polarity index of $T_{l,d}$ and the coalescence of $n \geq 2$ rooted trees, respectively. In Section 5, we present a formula for the generalized terminal Wiener index of the coalescence of $n \geq 2$ rooted trees.

2 The (terminal) Wiener index of the tree B_{k+1}

Suppose that B_{k+1} is a generalized Bethe tree with $k+1$ levels. The degree of the rooted vertex of B_{k+1} is equal to d_1 and the degrees of vertices on the i-th level are equal to $d_i + 1$, where $i = 2, 3, \ldots, k$.

Let n_i denote the number of vertices that lie on the i-th level of B_{k+1}. It is clear that $n_1 = 1$ and n_i $(i = 2, 3, \ldots, k)$ can be computed by the following equation:

$$n_i = \prod_{r=1}^{i-1} d_r \ .$$

Suppose that B_{k+1} contains n vertices. Then it is easy to see that

$$n = \sum_{i=1}^{k+1} n_i = 1 + \sum_{i=2}^{k+1}\prod_{j=1}^{i-1} d_j = 1 + \sum_{i=1}^{k}\prod_{j=1}^{i} d_j \ .$$

Suppose that e is an edge of B_{k+1} which is adjacent to the vertex u on the i-th level and the vertex v on the $(i+1)$-th level. The number of vertices that lie on the same side with v of the edge e is denoted by m_i . Then, $m_k = 1$ and

$$m_i = 1 + d_{i+1} + d_{i+1}d_{i+2} + \cdots + d_{i+1}\cdots d_k = 1 + \sum_{j=i+1}^{k}\prod_{r=i+1}^{j} d_r$$

where $2 \leq i < k$.

Actually, Wiener [4] himself conceived the index only for acyclic molecules and showed that for such systems it can be calculated as

$$W(T) = \sum_{e \in E(T)} n_1(e) \cdot n_2(e) \tag{2.1}$$

where $n_1(e)$ and $n_2(e)$ are the number of vertices on the two sides of the edge e, and where the summation goes over all edges of T.

Motivated by equation (2.1), Gutman et al. [5] obtained a similar formula for $TW(T)$:

$$TW(T) = \sum_{e \in E(G)} p_1(e) \cdot p_2(e) \tag{2.2}$$

where $p_1(e)$ and $p_2(e)$ are the numbers of pendent vertices lying on each side of an edge e, and the summation goes over all edges of T.

Recently, Heydari [1] applied formulas (2.1) and (2.2) to compute $W(B_{k+1})$ and $TW(B_{k+1})$, and he stated the following:

Theorem 2.1 [1] *Let B_{k+1} be a $(k+1)$-level generalized Bethe tree. Then,*

$$W(B_{k+1}) = \sum_{i=1}^{k} (n_{i+1} - 1) \cdot m_i \cdot (n - m_i)$$

$$TW(B_{k+1}) = \prod_{r=1}^{k} d_r + k \prod_{r=1}^{k} d_r - 1 - \sum_{r=1}^{k-1} \prod_{j=1}^{r} d_{k-j+1} \, .$$

As shown by the following example, the results of Theorem 2.1 are incorrect.

Example 2.1 *Let B_4 be the generalized Bethe tree as shown in Fig.* 1. *By an elementary computation, $W(B_4) = 270$ and $TW(B_4) = 78$, whereas by applying Theorem* 2.1*, $W(B_4) = 192$ and $TW(B_4) = 41$, a contradiction.*

By checking the proof of Theorem 1 [1], we find that the author mistook the number of edges between the i-th level and the $(i+1)$-th level to be $n_i - 1$ in the computation of $W(B_{k+1})$ and he also mistook the number of pendent vertices lying on each side of e which is an edge adjacent to the vertex u on the i-th level and the vertex v on the $(i+1)$-th level in the computation of $TW(B_{k+1})$. By eliminating these mistakes, we arrive at the following corrected version of Theorem 2.1:

Theorem 2.2 *Let B_{k+1} be a $(k+1)$-level generalized Bethe tree. Then,*

$$W(B_{k+1}) = \sum_{i=1}^{k}(n_{i+1}) \cdot m_i \cdot (n - m_i)$$

$$TW(B_{k+1}) = \sum_{e \in E(G)} p_1(e) \cdot p_2(e)$$

$$= \sum_{i=1}^{k-1}(d_1 \ldots d_i)(d_{i+1} \ldots d_k)(d_1 \ldots d_k - d_{i+1} \ldots d_k) + d_1 \ldots d_k(d_1 \ldots d_k - 1)$$

$$= k\left(\prod_{r=1}^{k} d_r\right)^2 - \prod_{r=1}^{k} d_r \left(\sum_{i=1}^{k-1} \prod_{j=i+1}^{k} d_j + 1\right).$$

Let B_4 be the generalized Bethe tree as shown in Fig. 1. By Theorem 2.2, it follows that $W(B_4) = 270$ and $TW(B_4) = 78$, which agrees with the results of Example 2.1.

3 The generalized terminal Wiener indices of the trees B_{l+1} and $T_{l,k}$

In order to compute the generalized terminal Wiener indices of B_{l+1} and $T_{l,k}$, we first develop a formula similar to (2.2).

Lemma 3.1 *For any tree T,*

$$TW_k(T) = \sum_{e \in E(G)} k_1(e) \cdot k_2(e) \tag{3.1}$$

where $k_1(e)$ and $k_2(e)$are the numbers of k-degree vertices located on each side of edge e, and the summation goes over all edges of T.

Proof. Given a tree T, rather than sum up the length of the shortest path between each pair of k-degree vertices, we count how many times a certain edge e will be counted during the computing of TW_k. Actually, an edge e is counted as many times as the number of the shortest path which goes through e between two k-degree vertices, that is, $k_1(e) \cdot k_2(e)$ times. Thus, the result follows. ∎

In the following, the function $S(x)$ will play an important role to establish the main result of this section, where $S(x)$ is defined as

$$S(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}.$$

From the definition of $S(x)$, it easily follows:

Lemma 3.2 *Let v be a vertex of G. Then,*

$$S\left(\frac{deg(v)}{k}\right) = \begin{cases} 1 & deg(v) = k \\ 0 & deg(v) \neq k \end{cases}.$$

Let $n(i)$ denote the number of vertices which lie on the i-th level of B_{l+1}, and let $m_k(i)$ denote the number of k-degree children of a vertex which lie on the i-th level of B_{l+1} and N_k denote the number of k-degree vertices pertaining to B_{l+1}.

Theorem 3.1 *Let B_{l+1} be a $(l+1)$-level generalized Bethe tree. Then for any $k \geq 2$,*

$$TW_k(B_{l+1}) = \sum_{i=1}^{l} n(i+1) \cdot m_k(i) \cdot (N_k - m_k(i))$$

where

$$n(i+1) = \prod_{r=1}^{i} d_r$$

$$m_k(i) = S\Big(\frac{d_{i+1}+1}{k}\Big) + \sum_{j=i+2}^{l} S\Big(\frac{d_j+1}{k}\Big) \prod_{q=i+1}^{j-1} d_q$$

and

$$N_k = S\Big(\frac{d_1}{k}\Big) + \sum_{i=2}^{l} S\Big(\frac{d_i+1}{k}\Big) \prod_{r=1}^{i-1} d_r \ .$$

Proof. By the definition of N_k and $n(i)$, we have $n(i) = \prod_{r=1}^{i-1} d_r$ for $i \geq 2$ and

$$\begin{aligned} N_k &= S\left(\frac{d_1}{k}\right) + S\left(\frac{d_2+1}{k}\right) d_1 + S\left(\frac{d_3+1}{k}\right) d_1 d_2 + \cdots + S\left(\frac{d_l+1}{k}\right) d_1 \cdots d_{l-1} \\ &= S\left(\frac{d_1}{k}\right) + \sum_{i=2}^{l} S\left(\frac{d_i+1}{k}\right) \prod_{r=1}^{i-1} d_r \ . \end{aligned}$$

Furthermore, from the notation of $m_k(i)$, it follows that

$$\begin{aligned} m_k(i) &= S\Big(\frac{d_{i+1}+1}{k}\Big) + S\Big(\frac{d_{i+2}+1}{k}\Big) d_{i+1} + S\Big(\frac{d_{i+3}+1}{k}\Big) d_{i+1} d_{i+2} \\ &\quad + \cdots + S\Big(\frac{d_l+1}{k}\Big) d_{i+1} \cdots d_{l-1} \\ &= S\Big(\frac{d_{i+1}+1}{k}\Big) + \sum_{j=i+2}^{l} S\Big(\frac{d_j+1}{k}\Big) \prod_{q=i+1}^{j-1} d_q \ . \end{aligned}$$

Suppose that e is an edge, which is adjacent to the vertex u on the i-th level and the vertex v on the $(i+1)$-th level of B_{l+1}. Then, there are exactly $m_k(i)$ and $(n_k - m_k(i))$

k-degree vertices lie on different sides of e, respectively. Since there are $n(i+1)$ edges that adjacent the vertices in the i-th level and the vertices in the $(i+1)$-th level of B_{l+1}, by Lemma 3.1,

$$TW_k(T) = \sum_{e\in E(G)} k_1(e)\cdot k_2(e) = \sum_{i=1}^{l} n(i+1)\cdot m_k(i)\cdot (N_k - m_k(i)) .$$

This completes the proof. ■

By applying Theorem 3.1 to the $(l+1)$-level dendrimer tree $T_{l,k}$, we arrive at:

Corollary 3.1 *Let $T_{l,k}$ be a $(l+1)$-level dendrimer tree, whose degrees of all the non-pendent vertices are equal to $k(k\geq 3)$. Then,*

$$\begin{aligned} TW_k(T_{l,k}) &= \frac{k}{(k-2)^3}\Big((l-1)(k-1)^{2l} - 2(k-1)^{2l-1} \\ &- l(k-1)^{2l-2} + 2(k-1)^{l} + 2(k-1)^{l-1} - 1\Big) . \end{aligned}$$

Proof. Since the degrees of all the non-pendent vertices of $T_{l,k}$ are equal to k, the value of $n(i+1)$, $m_k(i)$ and N_k can be calculated as

$$\begin{aligned} n(i+1) &= k(k-1)^{i-1}, i\geq 1 \quad ; \\ m_k(i) &= (1+(k-1)+\cdots+(k-1)^{l-i-1} = \frac{\big((k-1)^{l-i}-1\big)}{k-2}, 1\leq i\leq l-1 \\ m_k(l) &= 0 ; \\ N_k &= (1+k+k(k-1)+\cdots+k(k-1)^{l-2} = 1+\frac{k\big((k-1)^{l-1}-1\big)}{k-2} . \end{aligned}$$

Thus, by Theorem 3.1,

$$\begin{aligned} & TW_k(T_{l,k}) = \sum_{i=1}^{l-1} n(i+1)\cdot m_k(i)\cdot (N_k - m_k(i)) \\ = & \sum_{i=1}^{l-1} k(k-1)^{i-1}\frac{((k-1)^{l-i}-1)}{k-2}\left(1+\frac{k((k-1)^{l-1}-1)}{k-2} - \frac{((k-1)^{l-i}-1)}{k-2}\right) \\ = & \frac{k}{(k-2)^2}\sum_{i=1}^{l-1}\left((k-1)^{l-1}-(k-1)^{i-1}\right)\left(k(k-1)^{l-1}-(k-1)^{l-i}-1\right) \\ = & \frac{k}{(k-2)^2}\left((l-1)k(k-1)^{2l-2} + \frac{(k-1)^{l-1}-1}{k-2}\left(1-(k-1)^{l}-k(k-1)^{l-1}\right)\right) \end{aligned}$$

$$= \frac{k}{(k-2)^3}\big((l-1)(k-1)^{2l} - 2(k-1)^{2l-1}$$

$$- \quad l(k-1)^{2l-2} + 2(k-1)^{l} + 2(k-1)^{l-1} - 1\big)\,. \qquad \blacksquare$$

4 The generalized Wiener polarity index of the coalescence of rooted trees and $T_{l,d}$

In this section, we consider the generalized Wiener polarity index of the coalescence of $n \geq 2$ rooted trees. We first consider the case of the coalescence of two rooted trees.

As before, we use the symbol $T_1 \circ T_2$ to define the coalescence of T_1 and T_2, and we use w to denote the common root vertex of T_1 and T_2 in $T_1 \circ T_2$. Let $N^i_{T_1}(w)$ (respectively, $N^i_{T_2}(w)$) denote the number of vertices whose distance to w is equal to i in T_1 (respectively, T_2).

Theorem 4.1 *Suppose that T_1 and T_2 are two rooted trees, and $k \geq 2$. Then,*

$$W_k\big(T_1 \circ T_2\big) = W_k\big(T_1\big) + W_k\big(T_2\big) + \sum_{i=1}^{k-1} N^i_{T_1}(w) \cdot N^{k-i}_{T_2}(w)$$

where w is the common root vertex of T_1 and T_2 in $T_1 \circ T_2$.

Proof. Let $T = T_1 \circ T_2$. Suppose that $u,\ v \in V(T)$ with $d_T(u,v) = k$. Then, either $u, v \in T_i$ $(1 \leq i \leq 2)$ or $u \in T_1$, $v \in T_2$. It is clear that

$$|\ \{(u,v) \mid d_T(u,v) = k, u, v \in V(T_i)\}\ | = W_k(T_i),\ \ i = 1, 2,$$

$$|\ \{(u,v) \mid d_T(u,v) = k, u \in V(T_1), v \in V(T_2)\}\ | = \sum_{i=1}^{k-1} N^i_{T_1}(w) \cdot N^{k-i}_{T_2}(w)\ .$$

This completes the proof. $\blacksquare$

Theorem 4.2 *Suppose that T is a coalescence of n rooted trees $T_1, \ldots, T_n$ and $n \geq 2$. Then,*

$$W_k\big(T\big) = \sum_{i=1}^{n} W_k\big(T_i\big) + \sum_{1 \leq p < q \leq n} \sum_{i=1}^{k-1} N^i_{T_p}(w) \cdot N^{k-i}_{T_q}(w)$$

where w is the common root vertex of $T_1, \ldots, T_n$.

Proof. Let $T = T_1 \circ T_2 \circ \cdots \circ T_n$. Suppose that $u, v \in V(T)$ with $d_T(u,v) = k$. Then, either $u, v \in T_i$ $(1 \le i \le 2)$ or $u \in T_p$, $v \in T_q$ $(1 \le p < q \le n)$. Thus,

$$| \{(u,v) \mid d_T(u,v) = k, u, v \in V(T_i)\} |= W_k(T_i),\ 1 \le i \le n$$

$$| \{(u,v) \mid d_T(u,v) = k, u \in V(T_i), v \in V(T_j)\} |= \sum_{i=1}^{k-1} N^i_{T_p}(w) \cdot N^{k-i}_{T_q}(w),\ 1 \le p < q \le n\ .$$

We prove this theorem by induction on n. First, Theorem 4.1 implies that the result holds for $n = 2$. Now, we suppose that the theorem already holds for $n = m$, and consider the case of $n = m + 1$. Let $T' = T_1 \circ \cdots \circ T_m$. Then,

$$\begin{aligned}
W_k(T) &= W_k(T' \circ T_{m+1}) \\
&= W_k(T') + W_k(T_{m+1}) + \sum_{i=1}^{k-1} N^i_{T'}(w) \cdot N^{k-i}_{T_{m+1}}(w) \\
&= \sum_{i=1}^{m} W_k(T_i) + \sum_{1\le p<q\le m} \sum_{i=1}^{k-1} N^i_{T_p}(w) \cdot N^{k-i}_{T_q}(w) + W_k(T_{m+1}) \\
&+ \sum_{i=1}^{k-1} \Big(\sum_{d=1}^{l} N^i_{T_d}(w) \cdot N^{k-i}_{T_{m+1}}(w) \Big) \\
&= \sum_{i=1}^{m+1} W_k(T_i) + \sum_{1\le p<q\le m+1} \sum_{i=1}^{k-1} N^i_{T_p}(w) \cdot N^{k-i}_{T_q}(w)\ .
\end{aligned}$$

Hence, the theorem also holds for $n = m + 1$. So, the result holds. ■

A $(l + 1)$-level bronchial tree $L_{l,d}$ is obtained from $T_{l,d}$ by deleting all the children of a vertex, which lies on the 2-level of $T_{l,d}$. For detail see Fig. 2 presenting an illustrative example.

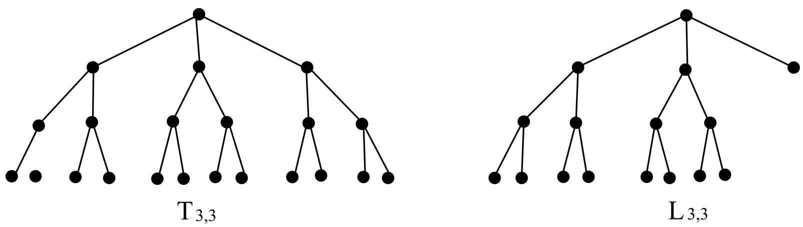


Figure 2: A dendrimer tree $T_{3,3}$ and a bronchia tree $L_{3,3}$.

Lemma 4.1 *Let $L_{k-2,d}$ be a $(k-1)$-level bronchial tree, where $k \geq 3$. Then*

$$W_k(L_{k-2,d}) = \begin{cases} \frac{(d-1)^{k-1}}{2(d-2)}\Big((d-1)^{\frac{k}{2}} + (d-1)^{\frac{k-2}{2}} - (k-1)(d-1) + k - 3\Big) & k \text{ is even} \\ \frac{(d-1)^{k-1}}{2(d-2)}\Big(2(d-1)^{\frac{k-1}{2}} - (k-1)(d-1) + k - 3\Big) & k \text{ is odd}. \end{cases}$$

Proof. Let $(d-1)\cdot L_{k-3,d}$ denote the coalescence of $(d-1)$ identical $(k-2)$-level bronchial trees $L_{k-3,d}$. Since $L_{k-2,d} = (d-1)\cdot L_{k-3,d} \circ P_2$, we have

$$\sum_{i=1}^{k-1} N^i_{L_{k-3,d}}(w)\cdot N^{k-i}_{L_{k-3,d}}(w) = (k-3)(d-1)^{k-2}$$

$$W_k\big(P_2\big) = 0, \quad \text{and} \quad \sum_{i=1}^{k-1} N^i_{L_{k-2,d}}(w)\cdot N^{k-i}_{P_2}(w) = 0\ .$$

By Theorem 4.2, it follows that

$$\begin{aligned} W_k\big(L_{k-2,d}\big) &= W_k\big((d-1)\cdot L_{k-3,d}\circ P_2\big) \\ &= (d-1)W_k\big(L_{k-3,d}\big) + \binom{d-1}{2}(k-3)(d-1)^{k-2} \\ W_k\big(L_{k-3,d}\big) &= (d-1)W_k\big(L_{k-4,d}\big) + \binom{d-1}{2}(k-5)(d-1)^{k-2} \\ &\vdots \end{aligned}$$

$$\begin{cases} W_k\big(L_{\frac{k+2}{2},d}\big) = (d-1)\binom{d-1}{2}(d-1)^{k-2} + \binom{d-1}{2}3(d-1)^{k-2}, & k \text{ is even}, \\ W_k\big(L_{\frac{k+3}{2},d}\big) = (d-1)2\binom{d-1}{2}(d-1)^{k-2} + \binom{d-1}{2}4(d-1)^{k-2}, & k \text{ is odd}. \end{cases}$$

From the above equations, we deduce that

$$W_k\big(L_{k-2,d}\big) = \begin{cases} (d-1)^{k-2}\binom{d-1}{2}\Big((d-1)^{\frac{k-4}{2}} + 3(d-1)^{\frac{k-6}{2}} + \cdots + k - 3\Big), & k \text{ is even} \\ (d-1)^{k-2}\binom{d-1}{2}\Big(2(d-1)^{\frac{k-5}{2}} + 4(d-1)^{\frac{k-7}{2}} + \cdots + k - 3\Big), & k \text{ is odd} \end{cases}$$

$$= \begin{cases} (d-1)^{k-2}\binom{d-1}{2}\left(\frac{(d-1)^{\frac{k-2}{2}}-k+3}{d-2} + \frac{2(d-1)((d-1)^{\frac{k-4}{2}}-1)}{(d-2)^2}\right), & k \text{ is even} \\ (d-1)^{k-2}\binom{d-1}{2}\left(\frac{2(d-1)^{\frac{k-3}{2}}-k+3}{d-2} + \frac{2(d-1)((d-1)^{\frac{k-5}{2}}-1)}{(d-2)^2}\right), & k \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{(d-1)^{k-2}\binom{d-1}{2}}{(d-2)^2}\Big((d-2)\big((d-1)^{\frac{k-2}{2}} - k + 3\big) + 2(d-1)\big((d-1)^{\frac{k-4}{2}} - 1\big)\Big), & k \text{ is even} \\ \frac{(d-1)^{k-2}\binom{d-1}{2}}{(d-2)^2}\Big((d-2)\big(2(d-1)^{\frac{k-3}{2}} - k + 3\big) + 2(d-1)\big((d-1)^{\frac{k-5}{2}} - 1\big)\Big), & k \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{(d-1)^{k-2}\binom{d-1}{2}}{(d-2)^2}\Big((d-1)^{\frac{k}{2}} + (d-1)^{\frac{k-2}{2}} - (k-1)(d-1) + k - 3\Big), & k \text{ is even} \\ \frac{(d-1)^{k-2}\binom{d-1}{2}}{(d-2)^2}\Big(2(d-1)^{\frac{k-1}{2}} - (k-1)(d-1) + k - 3\Big), & k \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{(d-1)^{k-1}}{2(d-2)}\left((d-1)^{\frac{k}{2}} + (d-1)^{\frac{k-2}{2}} - (k-1)(d-1) + k - 3\right), & k \text{ is even} \\ \frac{(d-1)^{k-1}}{2(d-2)}\left(2(d-1)^{\frac{k-1}{2}} - (k-1)(d-1) + k - 3\right), & k \text{ is odd}. \end{cases}$$

This completes the proof of Lemma 4.1. ■

Lemma 4.2 *Let $L_{l,d}$ be a $(l+1)$-level bronchial tree, where $l \geq k-1$. Then,*

$$\begin{aligned} W_k\big(L_{l,d}\big) &= g(l,k,d) + \frac{(d-1)^{k-1}}{2(d-2)}\Big[(k-1)(d-1)^{l-k+3} \\ &- (k-3)(d-1)^{l-k+2} - (k-1)(d-1) + k - 3\Big] \end{aligned}$$

where $g(l,k,d) = (d-1)^{l-k+2} W_k\big(L_{k-2,d}\big)$ and $k \geq 3$.

Proof. Let $(d-1) \cdot L_{l-1,d}$ denote the coalescence of $(d-1)$ identical $(l-1)$-level bronchial trees $L_{l-1,d}$. Since $L_{l,d} = (d-1) \cdot L_{l-1,d} \circ P_2$, it follows that

$$\sum_{i=1}^{k-1} N^i_{L_{l-1,d}}(w) \cdot N^{k-i}_{L_{l-1,d}}(w) = (k-1)(d-1)^{k-2}$$

$$W_k\big(P_2\big) = 0, \quad \text{and} \quad \sum_{i=1}^{k-1} N^i_{L_{k-2,d}}(w) \cdot N^{k-i}_{P_2}(v) = (d-1)^{k-1}.$$

Since $L_{t,d} = (d-1) \cdot L_{t-1,d} \circ P_2$, by Theorem 4.2, we have

$$W_k\big(L_{t,d}\big) = (d-1) W_k\big(L_{t-1,d}\big) + \binom{d-1}{2}(k-1)(d-1)^{k-2} + (d-1)^{k-1}. \quad (4.1)$$

Now, equation (4.1) implies that

$$\begin{aligned} W_k\big(L_{l,d}\big) &= g(l,k,d) \\ &+ \left(\binom{d-1}{2}(k-1)(d-1)^{k-2} + (d-1)^{k-1}\right)\left((d-1)^{l-k+1} + \cdots + 1\right) \\ &= g(l,k,d) + \left(\binom{d-1}{2}(k-1)(d-1)^{k-2} + (d-1)^{k-1}\right)\left(\frac{(d-1)^{l-k+2} - 1}{d-2}\right) \\ &= g(l,k,d) + \frac{(d-1)^{k-1}}{2(d-2)}\left((k-1)(d-1) - (k-3)\right)\left((d-1)^{l-k+2} - 1\right) \\ &= g(l,k,d) + \frac{(d-1)^{k-1}}{2(d-2)}\Big((k-1)(d-1)^{l-k+3} \\ &- (k-3)(d-1)^{l-k+2} - (k-1)(d-1) + k - 3\Big). \end{aligned}$$

This completes the proof of this lemma. ■

We are now ready to present the main result of this section:

Theorem 4.3 *Let $T_{l,d}$ be a $(l+1)$-level dendrimer tree, where $l \geq k \geq 3$. Then*

$$W_k\big(T_{l,d}\big) = df(l,k,d) + \frac{1}{2}(k-1)d(d-1)^{k-1}.$$

where

$$\begin{aligned} f(l,k,d) &= (d-1)^{l-k+1}h(l,k,d) + \frac{(d-1)^{k-1}}{2(d-2)}\big((k-1)(d-1)^{l-k+2} - (k-3)(d-1)^{l-k+1} \\ &- (k-1)(d-1) + k - 3\big) \end{aligned}$$

and

$$h(l,k,d) = \begin{cases} \frac{(d-1)^{k-1}}{2(d-2)}\Big((d-1)^{\frac{k}{2}} + (d-1)^{\frac{k-2}{2}} - (k-1)(d-1) + k - 3\Big), & k \text{ is even} \\ \frac{(d-1)^{k-1}}{2(d-2)}\Big(2(d-1)^{\frac{k-1}{2}} - (k-1)(d-1) + k - 3\Big), & k \text{ is odd.} \end{cases}$$

Proof. Let $d \cdot L_{l-1,d}$ denote the coalescence of d identical l-level bronchia trees $L_{l-1,d}$. Since $T_{l,d} = d \cdot L_{l-1,d}$,

$$\sum_{i=1}^{k-1} N^{i}_{L_{l-1,d}}(w) \cdot N^{k-i}_{L_{l-1,d}}(w) = (k-1)(d-1)^{k-2}.$$

By Theorem 4.2, it follows that

$$\begin{aligned} W_k\big(T_{l,d}\big) &= W_k\big(d \cdot L_{l-1,d}\big) \\ &= dW_k\big(L_{l-1,d}\big) + \binom{d}{2}(k-1)(d-1)^{k-2} \\ &= dW_k\big(L_{l-1,d}\big) + \frac{1}{2}(k-1)d(d-1)^{k-1}\ . \end{aligned}$$

By the above equation and Lemmas 4.1 and 4.2, the result follows. ■

5 The generalized terminal Wiener index of the coalescence of rooted trees

In this section, we consider the generalized terminal Wiener index of coalescence of $n \geq 2$ rooted trees. Let $d^k_{T_i}(w)$ denote the sum of distances between the common root vertex w of $T_1 \circ T_2 \circ \cdots \circ T_n$ and every k-degree vertex of T_i, and let $N_k(T_i)$ denote the number of k-degree vertices of T_i.

Theorem 5.1 *Let T be a coalescence of n rooted trees $T_1, \ldots, T_n$, where $n \geq 2$. Then*

$$TW_k(T) = \sum_{i=1}^{n} TW_k(T_i) + \sum_{j=1}^{n}\sum_{i=1}^{n} d^k_{T_j}(w)(N_k(T_i) + \alpha) - \sum_{i=1}^{n} d^k_{T_i}(w)(N_k(T_i) + \beta)$$

where w is the common root vertex of $T_1, \ldots, T_n$ and α, β are two variable quantities such that $\alpha = -\frac{l}{n}, \beta = 0$ if w is a k-degree vertex of l $(0 \leq l \leq n)$ trees of $\{T_1, \ldots, T_n\}$ but not a k-degree vertex of T and $\alpha = 0, \beta = -1$ if w is not a k-degree vertex of any root tree of $\{T_1, \ldots, T_n\}$ but a k-degree vertex of T.

Proof. For any two vertices $u, v \in V(T)$ with $deg(u) = deg(v) = k$, either $u, v \in V(T_i)$ $(1 \leq i \leq n)$ or $u \in V(T_i)$, $v \in V(T_j)$ $(1 \leq i < j \leq n)$. Hence,

$$TW_k(T) = \sum_{i=1}^{n} \sum_{\substack{deg(u)=deg(v)=k \\ u,v \in V(T_i)}} d_T(u,v) + \sum_{1 \leq i < j \leq n} \sum_{\substack{deg(u)=deg(v)=k \\ u \in V(T_i), v \in V(T_j)}} d_T(u,v)\ .$$

(1) Suppose that the root vertex w is not a k-degree vertex of T but a k-degree vertex of l trees of $\{T_1, \ldots, T_n\}$, where $0 \leq l \leq n$.

If $l > 0$,without loss of generality, we assume that w is a k-degree vertex of the previous l roots trees, i.e., $T_1, \ldots, T_l$. In this case

$$\sum_{\substack{deg(u)=deg(v)=k \\ u,v \in V(T_i)}} d_T(u,v) = \begin{cases} TW_k(T_i) - d^k_{T_i}(w), & i \leq l \\ TW_k(T_i), & l+1 \leq i \leq n \end{cases}$$

$$\sum_{\substack{deg(u)=deg(v)=k \\ u \in V(T_i), v \in V(T_j)}} d_T(u,v) = \begin{cases} d^k_{T_i}(w)N_k(T_j) + d^k_{T_j}(w)(N_k(T_i) - 1), & i \leq l, l+1 \leq j \leq n \\ d^k_{T_i}(w)(N_k(T_j) - 1) + d^k_{T_j}(w)(N_k(T_i) - 1), & i < j \leq l \\ d^k_{T_i}(w)N_k(T_j) + d^k_{T_j}(w)N_k(T_i), & l+1 \leq i < j \leq n\ . \end{cases}$$

We prove this result by induction on n. We first consider the case of $n = 2$, and divide the proof into the following three cases.

Case 1. $l = 0$.

Since w is not a k-degree vertex of T_1 or T_2, we have

$$\begin{aligned} TW_k(T) &= \sum_{i=1}^{2} \sum_{\substack{deg(u)=deg(v)=k \\ u,v \in V(T_i)}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k \\ u \in V(T_1), v \in V(T_2)}} d_T(u,v) \\ &= \sum_{i=1}^{2} TW_k(T_i) + \left(d^k_{T_1}(w)N_k(T_2) + d^k_{T_2}(w)N_k(T_1)\right) \\ &= \sum_{i=1}^{2} TW_k(T_i) + [d^k_{T_1}(w)N_k(T_1) - d^k_{T_1}(w)N_k(T_1)] + d^k_{T_1}(w)N_k(T_2) \end{aligned}$$

$$+ \quad d_{T_2}^k(w)N_k(T_1) + [d_{T_2}^k(w)N_k(T_2) - d_{T_2}^k(w)N_k(T_2)]$$

$$= \sum_{i=1}^{2} TW_k(T_i) + \sum_{j=1}^{2}\sum_{i=1}^{2} d_{T_j}^k(w)N_k(T_i) - \sum_{i=1}^{2} d_{T_i}^k(w)N_k(T_i) \ .$$

Case 2. $l = 1$.

$$TW_k(T) = \sum_{i=1}^{2} \sum_{\substack{deg(u)=deg(v)=k \\ u,v\in V(T_i)}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k \\ u\in V(T_1),v\in V(T_2)}} d_T(u,v)$$

$$= TW_k(T_1) - d_{T_1}^k(w) + TW_k(T_2) + d_{T_1}^k(w)N_k(T_2) + d_{T_2}^k(w)(N_k(T_1) - 1)$$

$$= TW_k(T_1) + TW_k(T_2) + d_{T_1}^k(w)N_k(T_2) + d_{T_2}^k(w)N_k(T_1) - d_{T_1}^k(w) - d_{T_2}^k(w)$$

$$= \sum_{i=1}^{2} TW_k(T_i) + [d_{T_1}^k(w)N_k(T_1) - \frac{1}{2}d_{T_1}^k(w) - d_{T_1}^k(w)N_k(T_1)] + d_{T_1}^k(w)N_k(T_2)$$

$$- \quad \frac{1}{2}d_{T_1}^k(w) + d_{T_2}^k(w)N_k(T_1) - \frac{1}{2}d_{T_2}^k(w) + [d_{T_2}^k(w)N_k(T_2) - \frac{1}{2}d_{T_2}^k(w) - d_{T_2}^k(w)N_k(T_2)]$$

$$= \sum_{i=1}^{2} TW_k(T_i) + \sum_{j=1}^{2}\sum_{i=1}^{2} d_{T_j}^k(w)(N_k(T_i) - \frac{1}{2}) - \sum_{i=1}^{2} d_{T_i}^k(w)N_k(T_i).$$

Case 3. $l = 2$.

$$TW_k(T) = \sum_{i=1}^{2} \sum_{\substack{deg(u)=deg(v)=k \\ u,v\in V(T_i)}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k \\ u\in V(T_1),v\in V(T_2)}} d_T(u,v)$$

$$= TW_k(T_1) - d_{T_1}^k(w) + TW_k(T_2) - d_{T_2}^k(w) + d_{T_1}^k(w)(N_k(T_2) - 1)$$

$$+ \quad d_{T_2}^k(w)(N_k(T_1) - 1)$$

$$= TW_k(T_1) + TW_k(T_2) + d_{T_1}^k(w)N_k(T_2) + d_{T_2}^k(w)N_k(T_1) - 2d_{T_1}^k(w) - 2d_{T_2}^k(w)$$

$$= \sum_{i=1}^{2} TW_k(T_i) + [d_{T_1}^k(w)N_k(T_1) - d_{T_1}^k(w) - d_{T_1}^k(w)N_k(T_1)] + d_{T_1}^k(w)N_k(T_2)$$

$$- \quad d_{T_1}^k(w) + d_{T_2}^k(w)N_k(T_1) - d_{T_2}^k(w) + [d_{T_2}^k(w)N_k(T_2) - d_{T_2}^k(w) - d_{T_2}^k(w)N_k(T_2)]$$

$$= \sum_{i=1}^{2} TW_k(T_i) + \sum_{j=1}^{2}\sum_{i=1}^{2} d_{T_j}^k(w)(N_k(T_i) - 1) - \sum_{i=1}^{2} d_{T_i}^k(w)N_k(T_i) \ .$$

By combining the above three cases, the theorem holds for $n = 2$.

Now, we suppose that the theorem is true for $n = m$, and we consider the case of $n = m + 1$ in the sequel. Let $T' = T_1 \circ \cdots \circ T_m$. We divide the proof into the following three cases.

Case 1. $l = 0$.

$$\begin{aligned}
TW_k(T) &= TW_k(T' \circ T_{m+1}) \\
&= \sum_{\substack{deg(u)=deg(v)=k \\ u,v \in V(T')}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k \\ u,v \in V(T_{m+1})}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k \\ u \in V(T'), v \in V(T_{m+1})}} d_T(u,v) \\
&= TW_k(T') + TW_k(T_{m+1}) + \left(d^k_{T'}(v)N_k(T_{m+1}) + d^k_{T_{m+1}}(v)N_k(T')\right) \\
&= \sum_{i=1}^{m} TW_k(T_i) + \sum_{j=1}^{m}\sum_{i=1}^{m} d^k_{T_j}(w)N_k(T_i) - \sum_{i=1}^{m} d^k_{T_i}(w)N_k(T_i) + TW_k(T_{m+1}) \\
&+ \left(\sum_{i=1}^{m} d^k_{T_i}(w)\right) N_k(T_{m+1}) + d^k_{T_{m+1}}(w)\left(\sum_{i=1}^{m} N_k(T_i)\right) + d^k_{T_{m+1}}(w)N_k(T_{m+1}) \\
&- d^k_{T_{m+1}}(w)N_k(T_{m+1}) \\
&= \sum_{i=1}^{m+1} TW_k(T_i) + \sum_{j=1}^{m+1}\sum_{i=1}^{m+1} d^k_{T_j}(w)N_k(T_i) - \sum_{i=1}^{m+1} d^k_{T_i}(w)N_k(T_i)\ .
\end{aligned}$$

Case 2. $1 \leq l \leq m$.

$$\begin{aligned}
TW_k(T) &= TW_k(T' \circ T_{m+1}) \\
&= \sum_{\substack{deg(u)=deg(v)=k \\ u,v \in V(T')}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k \\ u,v \in V(T_{m+1})}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k \\ u \in V(T'), v \in V(T_{m+1})}} d_T(u,v) \\
&= TW_k(T') + TW_k(T_{m+1}) + d^k_{T'}(w)N_k(T_{m+1}) + d^k_{T_{m+1}}(w)\left(\sum_{i=1}^{l} N_k(T_i) - 1)\right) \\
&+ d^k_{T_{m+1}}(w)\left(\sum_{i=l+1}^{m} N_k(T_i)\right) \\
&= \sum_{i=1}^{m} TW_k(T_i) + \sum_{j=1}^{m}\sum_{i=1}^{m} d^k_{T_j}(w)\left(N_k(T_i) - \frac{l}{m}\right) - \sum_{i=1}^{m} d^k_{T_i}(w)N_k(T_i) + TW_k(T_{m+1}) \\
&+ \left(\sum_{i=1}^{m} d^k_{T_i}(w)\right) N_k(T_{m+1}) + d^k_{T_{m+1}}(w)\sum_{i=1}^{l}(N_k(T_i) - 1) + d^k_{T_{m+1}}(w)\left(\sum_{i=l+1}^{m} N_k(T_i)\right) \\
&+ \left(d^k_{T_{m+1}}(w)N_k(T_{m+1}) - d^k_{T_{m+1}}(w)N_k(T_{m+1})\right)
\end{aligned}$$

$$= \sum_{i=1}^{m+1} TW_k(T_i) + \sum_{j=1}^{m+1}\sum_{i=1}^{m+1} d^k_{T_j}(w)\left(N_k(T_i) - \frac{l}{m+1}\right) - \sum_{i=1}^{m+1} d^k_{T_i}(w) N_k(T_i)\ .$$

Case 3. $l = m + 1$.

$$\begin{aligned}
TW_k(T) &= TW_k(T' \circ T_{m+1}) \\
&= \sum_{\substack{deg(u)=deg(v)=k\\ u,v\in V(T')}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k\\ u,v\in V(T_{m+1})}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k\\ u\in V(T'),v\in V(T_{m+1})}} d_T(u,v) \\
&= TW_k(T') + TW_k(T_{m+1}) - d^k_{T_{m+1}}(w) + d^k_{T'}(w)\left(N_k(T_{m+1}) - 1\right) \\
&+ d^k_{T_{m+1}}(w)\left(\sum_{i=1}^{m} N_k(T_i) - 1)\right) \\
&= \sum_{i=1}^{m} TW_k(T_i) + \sum_{j=1}^{m}\sum_{i=1}^{m} d^k_{T_j}(w)\left(N_k(T_i) - 1\right) - \sum_{i=1}^{m} d^k_{T_i}(w)N_k(T_i) + TW_k(T_{k+1}) \\
&- d^k_{T_{m+1}}(w) + \left(\sum_{i=1}^{m} d^k_{T_i}(w)\right)\left(N_k(T_{m+1}) - 1\right) + d^k_{T_{m+1}}(w)\sum_{i=1}^{m}\left(N_k(T_i) - 1\right) \\
&+ d^k_{T_{m+1}}(w)N_k(T_{m+1}) - d^k_{T_{m+1}}(w)N_k(T_{m+1}) \\
&= \sum_{i=1}^{m+1} TW_k(T_i) + \sum_{j=1}^{m+1}\sum_{i=1}^{m+1} d^k_{T_j}(w)\left(N_k(T_i) - 1\right) - \sum_{i=1}^{m+1} d^k_{T_i}(w)N_k(T_i)\ .
\end{aligned}$$

Hence, the result also holds for $n = m + 1$. Thus, the result holds.

(2) Suppose that the root vertex w is a k-degree vertex of T, but not a k-degree vertex of any root tree of $\{T_1, \ldots, T_n\}$. Then

$$\sum_{\substack{deg(u)=deg(v)=k\\ u,v\in V(T_i)}} d_T(u,v) = TW_k(T_i) - d^k_{T_i}(w), \quad 1 \le i \le n$$

$$\sum_{\substack{deg(u)=deg(v)=k\\ u\in V(T_i),v\in V(T_j)}} d_T(u,v) = d^k_{T_i}(w)N_k(T_j) + d^k_{T_j}(w)N_k(T_i), \quad 1 \le i < j \le n\ .$$

We shall prove this result by induction on n. When $n = 2$, we have

$$\begin{aligned}
TW_k(T) &= \sum_{i=1}^{2}\sum_{\substack{deg(u)=deg(v)=k\\ u,v\in V(T_i)}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k\\ u\in V(T_1),v\in V(T_2)}} d_T(u,v) \\
&= TW_k(T_1) + d^k_{T_1}(w) + TW_k(T_2) + d^k_{T_2}(w) + d^k_{T_1}(w)N_k(T_2) + d^k_{T_2}(w)N_k(T_1)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} TW_k(T_i) + [d^k_{T_1}(w)N_k(T_1) - d^k_{T_1}(w)N_k(T_1)] + d^k_{T_1}(w)N_k(T_2) + d^k_{T_1}(w) \\
&+ d^k_{T_2}(w)N_k(T_1) + [d^k_{T_2}(w)N_k(T_2) - d^k_{T_2}(w)N_k(T_2)] + d^k_{T_2}(w) \\
&= \sum_{i=1}^{2} TW_k(T_i) + d^k_{T_1}(w)N_k(T_1) + d^k_{T_1}(w)N_k(T_2) - [d^k_{T_1}(w)N_k(T_1) - d^k_{T_1}(w)] \\
&+ d^k_{T_2}(w)N_k(T_1) + d^k_{T_2}(w)N_k(T_2) - [d^k_{T_2}(w)N_k(T_2) - d^k_{T_2}(w)] \\
&= \sum_{i=1}^{2} TW_k(T_i) + \sum_{j=1}^{2}\sum_{i=1}^{2} d^k_{T_j}(w)N_k(T_i) - \sum_{i=1}^{2} d^k_{T_i}(w)(N_k(T_i) - 1)\ .
\end{aligned}
$$

Therefore, the result holds for $n = 2$. We now suppose that the equation holds for $n = m$, and consider the case of $n = m + 1$. Let $T' = T_1 \circ \cdots \circ T_m$.

$$
\begin{aligned}
TW_k(T) &= TW_k(T' \circ T_{m+1}) \\
&= \sum_{\substack{deg(u)=deg(v)=k \\ u,v\in V(T')}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k \\ u,v\in V(T_{m+1})}} d_T(u,v) + \sum_{\substack{deg(u)=deg(v)=k \\ u\in V(T'),v\in V(T_{m+1})}} d_T(u,v) \\
&= TW_k(T') + d^k_{T'}(w) + TW_k(T_{m+1}) + d^k_{T_{m+1}}(w) + d^k_{T'}(w)N_k(T_{m+1}) \\
&+ d^k_{T_{m+1}}(w)N_k(T') \\
&= \sum_{i=1}^{m} TW_k(T_i) + \sum_{j=1}^{m}\sum_{i=1}^{m} d^k_{T_j}(w)N_k(T_i) - \sum_{i=1}^{m} d^k_{T_i}(w)N_k(T_i) + TW_k(T_{m+1}) \\
&= \sum_{i=1}^{m} d^k_{T_i}(w) + d^k_{T_{m+1}}(w) + \left(\sum_{i=1}^{m} d^k_{T_i}(w)\right) N_k(T_{m+1}) + d^k_{T_{m+1}}(w)\sum_{i=1}^{m} N_k(T_i) \\
&+ [d^k_{T_{m+1}}(w)N_k(T_{m+1}) - d^k_{T_{m+1}}(w)N_k(T_{m+1})] \\
&= \sum_{i=1}^{m+1} TW_k(T_i) + \sum_{j=1}^{m+1}\sum_{i=1}^{m+1} d^k_{T_j}(w)N_k(T_i) - \sum_{i=1}^{m+1} d^k_{T_i}(w)\left(N_k(T_i) - 1\right).
\end{aligned}
$$

Hence, the result also holds for $n = m + 1$. Thus, the result holds. ■

Let $n \cdot T_{l,k}$ denote the coalescence of n identical dendrimer tree $T_{l,k}$. From Theorem 5.1 we obtain:

Corollary 5.1 *Let $T_{l,k}$ be a $(l+1)$-level dendrimer tree, where $k \geq 2$. Then,*

$$TW_k(n \cdot T_{l,k}) = nTW_k(T_{l,k}) + 2\binom{n}{2} d^k_{T_{l,k}}(w)N_k(T_{l,k}) - n^2 d^k_{T_{l,k}}(w)\ .$$

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