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More Trees with All Degrees Odd Having Extremal Wiener Index

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Abstract

Continuing the preceding work [*MATCH Commun. Math. Comput. Chem.* **70** (2013) 000–000], we determine the first few trees whose all degrees are odd, having smallest and greatest Wiener index.

In the preceding paper [1], the trees extremal with respect to the Wiener index were determined in the class $\mathcal{T}_{2n}$ of trees of order $2n$ whose all vertices have odd degrees. In this note we communicate the extension of this result.

A vertex of degree 1 is said to be a leaf.

Let $K_{1,2n-1}$ be the $(2n)$-vertex star, consisting of a central vertex of degree $2n-1$ and $2n-1$ leaves. Let F_{2n} be the tree with $n+1$ leaves and $n-1$ vertices of degree 3, such that after deleting all its leaves, an $(n-1)$-vertex path graph remains. Evidently, both $K_{1,2n-1}$ and F_{2n} belong to $\mathcal{T}_{2n}$.

In [1] it was proven that if $T \in \mathcal{T}_{2n} \setminus \{K_{1,2n-1}, F_{2n}\}$, then

$$W(K_{1,2n-1}) < W(T) < W(F_{2n})\,. \tag{1}$$

In what follows, a tree with second minimal Wiener index in the class $\mathcal{T}_{2n}$ will be referred to as the second minimal tree. In the same sense we shall speak of the second maximal, third minimal, third maximal, etc trees.

Finding the second minimal tree is easy. In view of Eq. (1), this must be the tree obtained by attaching two new vertices to a leaf of $K_{1,2n-3}$. The third minimal tree is obtained either (a) by attaching four new vertices to a leaf of $K_{1,2n-5}$ or (b) by attaching two pairs of new vertices to two leaves of $K_{1,2n-5}$. Easy calculation reveals that construction (a) yields a lower value for the Wiener index.

Continuing this reasoning we arrive at the result stated as Theorem 1.

Let $T(p,q)$ be the tree of order $(p+q+2)$, depicted in Fig. 1. This tree is usually referred to as a *double star*. If p and q are even integers, and $p+q+2 = 2n$, then $T(p,q) \in \mathcal{T}_{2n}$. Note that if $q = 0$, then $T(p,q) \equiv K_{1,2n-1}$. Note also that if $n = 3$, then $T(2,2) \equiv F_{2n}$.

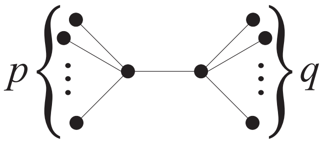

Fig. 1. The double star $T(p,q)$. By Theorem 1, trees of this kind have the few smallest Wiener indices in the class $\mathcal{T}_{2n}$.

Theorem 1. If $n = 4$, then $T(4,2)$ is second minimal in $\mathcal{T}_8$. If $n = 5$, then $T(6,2)$ and $T(4,4)$ are, respectively, second and third minimal in $\mathcal{T}_{10}$. If $n = 6$, then $T(8,2)$ and $T(6,4)$ are, respectively, second and third minimal in $\mathcal{T}_{12}$. If $n \geq 7$, then for $q = 2, 4, \ldots, 2\lfloor n/4 \rfloor + 2$, and $p = 2n - q - 2$, the trees $T(p,q)$ are, respectively, second minimal, third minimal, ..., $(\lfloor n/4 \rfloor + 1)$-th minimal in $\mathcal{T}_{2n}$.

In order to identify the second maximal, third maximal, etc. trees, we first need to recognize that these must possess only leaves and vertices of degree 3. Let T be such a tree of order $2n$. By deleting all the leaves from T, an $(n-1)$-vertex tree will remain, which we call the *skeleton* of T and denote by $S(T)$. Based on similar earlier results [2], it can be demonstrated [3] that there is a linear relation between $W(T)$ and $W(S(T))$. This implies that the ordering of the first few elements of $\mathcal{T}_{2n}$ by decreasing Wiener index will be the same as the ordering of their skeletons by decreasing Wiener index. This latter problem was earlier resolved by Deng [4] and Liu, Liu, and Li [5]. We thus can state the following theorem, whose complete proof will be reported elsewhere [3].

Let F_{2n}, F_{2n}^{*}, and F_{2n}^{**} be trees of order $2n$ whose structure is depicted in Fig. 2.

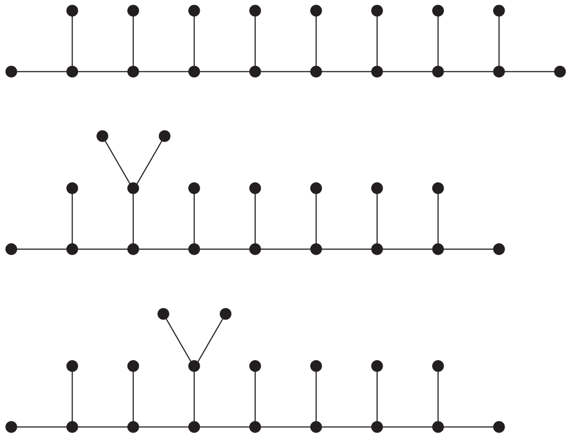

Fig. 2. The trees F_{2n} (top), F_{2n}^{*} (middle), and F_{2n}^{**} (bottom) of order $2n$ (in this example $2n = 18$) with all degrees odd, having maximal, second maximal, and third maximal Wiener index.

Theorem 2. If $n = 5$, then F_{10}^{*} is second maximal in $\mathcal{T}_{10}$. If $n = 6$, then F_{12}^{*} is second maximal in $\mathcal{T}_{12}$. If $n \geq 7$, then F_{2n}^{*} and F_{2n}^{**} are, respectively, second and third maximal in $\mathcal{T}_{2n}$. If $n \geq 8$, then the fourth and higher (up to eighteenth) maximal elements of $\mathcal{T}_{2n}$ are trees whose skeletons are specified in Theorem 1.1 of Liu, Liu, and Li [5].

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