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Lower Bounds for the Kirchhoff Index

Rao Li
Department of Mathematical Sciences
University of South Carolina Aiken
Aiken, SC 29801, USA
raol@usca.edu

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Abstract

Zhou and Trinajstić in [17] and Das, Güngör, and Çevik in [6] obtained lower bounds for the Kirchhoff index of a connected graph. Using some ideas in [17], [6], and other established results, we obtain new lower bounds for the Kirchhoff index of a connected graph.

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [3]. Let G be a graph of order n. We use G^c to denote the complement of G. We assume that the vertices in G are ordered such that $d_1 \geq d_2 \geq ... \geq d_n$, where d_i, $1 \leq i \leq n$, is the degree of vertex v_i in G. Grone and Merris in [9] introduced the definition of $d_i^*(G)$. They defined $d_i^*(G)$ as $|\{ v \in V(G) : d(v) \geq i \}|$. For vertex v_i, we use $N(v_i)$ to denote its neighbors. For two distinct vertices u and v in G, we define $c_{u,v}$ as $|N(u) \cap N(v)|$; if $d(u) = d_1$ and $d(v) = d_2$, we further define

$$a_{u,v} := \begin{cases} \dfrac{d(v) + 2 + \sqrt{(d(v)+2)^2 - 8d(v) + 4c_{u,v}}}{2} & \text{if } uv \in E(G)\,, \\ \dfrac{d(v) + 1 + \sqrt{(d(v)+1)^2 - 4c_{u,v}}}{2} & \text{if } uv \notin E(G)\ . \end{cases}$$

and $a := \max\{ a_{u,v} : \text{where } d(u) = d_1, d(v) = d_2, \text{ and } u \neq v \}$. We use S_n to denote the star graph $K_{1,n-1}$. We also use $K_n - e$ to denote the graph obtained by removing one edge e from K_n. Finally, we use $S_n + e$ to denote the graph obtained by adding one edge e to $K_{1,n-1}$.

The Laplacian of a graph G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of the degree sequence of G and $A(G)$ is the adjacency matrix of G.

The eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) = 0$ of $L(G)$ are called the Laplacian eigenvalues of the graph G. The notion and definition of the Kirchhoff index can be found in [13] and [2]. It was proved by Zhu, Klein, and Lukovist in [18] and Gutman and Mohar in [10] that the Kirchhoff index $Kf(G)$ of a connected graph G with order at least two can be written as $n\sum_{i=1}^{n-1}\frac{1}{\lambda_i}$.

Zhou and Trinajstić in [17] and Das, Güngör, and Çevik in [6] obtained lower bounds for the Kirchhoff index of a connected graph. Using some ideas in [17], [6], and other established results, we will present a new lower bound for the Kirchhoff index of a connected graph.

Theorem 1 Let G be a connected graph with $n \geq 2$ vertices and e edges.
(1) If $G = K_n$, then $Kf(G) = n - 1$.
(2) If $G = S_n$, then $Kf(G) = (n - 1)^2$.
(3) If $G \neq K_n$, $G \neq S_n$, and $n \geq 4$, then

$$Kf(G) \geq n\left(\frac{1}{n} + \frac{1}{n + d_2^* - d_1 - 1} + \frac{1}{d_n} + \frac{(n-4)^2}{2e - d_1 - 1 - a - d_n}\right),$$

with equality if and only if G is $K_n - e$, $K_2 \vee K_{n-2}^c$, $K_2 \vee S_{n-2}^c$, $K_1 \vee S_{n-1}^c$, $S_n + e$, or $K_1 \vee (K_1 \cup S_{n-2})$.

In order to prove Theorem 1, we need several established results. Theorem 2 below is Lemma 13.1.3 on Page 280 in [8].

Theorem 2 Let G be a graph of order n. Then $\lambda_i(G) = n - \lambda_{n-i}(G^c)$ for each i with $1 \leq i \leq n - 1$. In particular, $\lambda_1(G) \leq n$.

The following Theorem 3 was proven by Fiedler in [7].

Theorem 3 Let G be a non–complete graph of order n. Then $\lambda_{n-1}(G) \leq \kappa(G) \leq \kappa'(G) \leq d_{n-1}$, where $\kappa(G)$ and $\kappa'(G)$ are the connectivity and edge–connectivity of G, respectively.

Kirkland in [12] and Li and Fan in [14] characterized the graphs with $\lambda_{n-1}(G) = \kappa(G)$. Their results are stated in Theorem 4 below.

Theorem 4 Let G be a graph of order n with $1 \leq \kappa(G) \leq n - 2$. Then $\lambda_{n-1}(G) = \kappa(G) = k$ if and only if there exists a vertex subset $S \subset V(G)$ with $|S| = k$, such that $G = G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$, $m \geq 2$, and $\kappa(G[S]) \geq 2k - n$ if $\lfloor \frac{n}{2} \rfloor < k \leq n - 2$.

Theorem 5 below was proven by Grone and Merris in [9].

Theorem 5 Let G be a graph containing at least one edge. Then $\lambda_1(G) \geq d_1(G) + 1$. Moreover, if G is connected on $n > 1$ vertices, the equality holds if and only if $d_1(G) = n - 1$.

The statement in Theorem 6 was conjectured by Grone and Merris in [9]. Bai in [1] proved that the conjecture is true.

Theorem 6 Let G be a graph of order n. Then $\sum_{i=1}^{k} \lambda_i(G) \leq \sum_{i=1}^{k} d_i^*(G)$ for each k with $1 \leq k \leq n$.

Both Theorem 7 and Theorem 8 below were proven by Das in [5].

Theorem 7 Let G be a simple connected graph of order n. Then $\lambda_2(G) = \lambda_3(G) = ... = \lambda_{n-1}(G)$ if and only if G is K_n or S_n or K_{d_1, d_1}.

Theorem 8 Let G be a simple connected graph of order n. Then $\lambda_1(G) = \lambda_2(G) = ... = \lambda_{n-2}(G)$ if and only if G is K_n or $K_n - e$.

Theorem 9 below follows from results obtained by Das in [4].

Theorem 9 Let G be a simple connected graph of order $n > 2$. If G is not S_n, then $\lambda_2 \geq a$.

Now we will prove Theorem 1.

Proof of Theorem 1. Let G be a connected graph with n vertices and e edges.

(1) It is obvious that $Kf(G) = n - 1$ if $G = K_n$.

(2) It is easy to verify that $Kf(G) = (n-1)^2$ if $G = S_n$.

(3) Suppose that $G \neq K_n$, $G \neq S_n$, and $n \geq 4$. We assume that x and y are two distinct vertices in G such that $a_{x,y} = a$ with $d(x) = d_1$ and $d(y) = d_2$. Notice first that $0 \leq c_{x,y} \leq d_2 - 1$ if $xy \in E$, and $0 \leq c_{x,y} \leq d_2$ if $xy \notin E$. By the definition of a, we can easily verify that $d_2 \leq a \leq d_2 + 1$.

From Theorem 6, we have that $\lambda_1 + \lambda_2 \leq d_1^* + d_2^* = n + d_2^*$. Theorem 5 then implies that $\lambda_2 \leq n + d_2^* - d_1 - 1$.

From Theorem 2, Theorem 9, and the AM–GM–HM inequalities, we have that

$$Kf(G) = n\sum_{i=1}^{n-1} \lambda_i \geq n\left(\frac{1}{n} + \frac{1}{n + d_2^* - d_1 - 1} + \frac{1}{\lambda_{n-1}} + \frac{(n-4)^2}{2e - \lambda_1 - \lambda_2 - \lambda_{n-1}}\right)$$

$$\geq n\left(\frac{1}{n} + \frac{1}{n + d_2^* - d_1 - 1} + \frac{1}{\lambda_{n-1}} + \frac{(n-4)^2}{2e - d_1 - 1 - a - \lambda_{n-1}}\right) .$$

Now consider the function

$$f(x) = \frac{1}{x} + \frac{(n-4)^2}{2e - d_1 - 1 - a - x},$$

where $0 < x \leq d_n$. It is easy to see that

$$f'(x) = -\frac{1}{x^2} + \frac{(n-4)^2}{(2e - d_1 - 1 - a - x)^2}$$

$$= -\frac{(2e - d_1 - 1 - a - x + (n-4)x)(2e - d_1 - a - x - (n-3)x)}{x^2(2e - d_1 - 1 - a - x)^2} .$$

Next we prove that $2e-d_1-1-a-x+(n-4)x \geq 0$. Notice first that if $n \geq 5$, then $2e-d_1-1-a-x+(n-4)x \geq d_1+d_2+d_3+d_4+\cdots+d_{n-1}+d_n-d_1-1-d_2-1-d_n+(n-4)x \geq 0$.

If $n=4$, then $2e-d_1-1-a-x+(n-4)x \geq d_1+d_2+d_3+d_4-d_1-1-a-d_4 = d_2+d_3-1-a$. When $d_3 \geq 2$, then $2e-d_1-1-a-x+(n-4)x \geq d_2+d_3-1-a \geq d_2+d_3-1-1-d_2 \geq 0$. When $d_3 = 1$, then $d_4 = 1$. Also $d_2 \neq 1$ otherwise we have a contradiction that $G = K_{1,3}$. Moreover, $d_2 \neq 3$ otherwise $d_1 = 3$, which implies $d_3 \geq 2$, a contradiction. Thus $d_2 = 2$. Clearly, $d_1 \neq 3$, otherwise $d_3 \geq 2$, a contradiction again. Hence $d_1 = 2$. It is easy to see that now G is a path with 4 vertices and 3 edges. Therefore $a = 2$. So $2e-d_1-1-a-x+(n-4)x \geq d_2+d_3-1-a \geq d_2+d_3-1-2 = 0$.

Next we further prove that $2e-d_1-1-a-(n-3)x \geq 0$. If $d_3 \geq 2$, then $2e-d_1-1-a-(n-3)x \geq d_1+d_2+d_3+d_4+\cdots+d_{n-1}+d_n-d_1-1-d_2-1-(n-3)d_n \geq d_3-2 \geq 0$.

If $d_3 = 1$, then $d_3 = d_4 = \cdots = d_n = 1$. Obviously, $d_2 \neq 1$. Otherwise by the assumption that G is connected we have that $v_iv_j \notin E$, where $2 \leq i \neq j \leq n$, which implies that G is S_n, a contradiction. Now we assume that $d_2 \geq 2$. Again since G is connected, we have that $v_iv_j \notin E$, where $3 \leq i \neq j \leq n$. Hence, for each i with $3 \leq i \leq n$, v_i is adjacent to exactly one of v_1 and v_2. Since $d(v_1) = d_1 \geq d(v_2) = d_2 \geq 2 > d_3 = d_4 = \cdots = d_n = 1$, we have that $\{x,y\} = \{v_1,v_2\}$. Since G is connected, $xy \in E$. Clearly, $c_{x,y} = 0$. Hence, by the definition of a, we have that $a = d_2$. So $2e-d_1-1-a-(n-3)x \geq d_1+d_2+d_3+d_4+\cdots+d_{n-1}+d_n-d_1-1-d_2-(n-3)d_n \geq d_3-1 = 0$.

Thus $f(x)$ is decreasing when $0 < x \leq d_n$. Therefore

$$\begin{aligned}
Kf(G) &\geq n\left(\frac{1}{n}+\frac{1}{n+d_2^*-d_1-1}+f(\lambda_{n-1})\right)\\
&\geq n\left(\frac{1}{n}+\frac{1}{n+d_2^*-d_1-1}+f(d_n)\right)\\
&= n\left(\frac{1}{n}+\frac{1}{n+d_2^*-d_1-1}+\frac{1}{d_n}+\frac{(n-4)^2}{2e-d_1-1-a-d_n}\right).
\end{aligned}$$

Suppose that

$$Kf(G) = n\left(\frac{1}{n}+\frac{1}{n+d_2^*-d_1-1}+\frac{1}{d_n}+\frac{(n-4)^2}{2e-d_1-1-a-d_n}\right).$$

In review of the proof above, we have that

$$\lambda_1 = n = d_1+1, \lambda_2 = n+d_2^*-d_1-1 = d_2^* = a, \lambda_3 = \cdots = \lambda_{n-2}, \lambda_{n-1} = d_n.$$

Since $\lambda_1 \geq \lambda_2$, $\lambda_2 \geq \lambda_3$, and $\lambda_{n-2} \geq \lambda_{n-1}$, we, depending on $\lambda_1 > \lambda_2$ or $\lambda_1 = \lambda_2$, $\lambda_2 > \lambda_3$ or $\lambda_2 = \lambda_3$, and $\lambda_{n-2} > \lambda_{n-1}$ or $\lambda_{n-2} = \lambda_{n-1}$, can divide our following proofs into eight cases. The proofs for the eight cases are identical to the proofs for Cases $1-8$

in [16]. For the sake of completeness, we include the proofs for the eight cases in the Appendix.

Next we show that if G is $K_n - e$, $K_2 \vee K_{n-2}^c$, $K_2 \vee S_{n-2}^c$, $K_1 \vee S_{n-1}^c$, $S_n + e$, or $K_1 \vee (K_1 \cup S_{n-2})$, then

$$Kf(G) = n\left(\frac{1}{n} + \frac{1}{n + d_2^* - d_1 - 1} + \frac{1}{d_n} + \frac{(n-4)^2}{2e - d_1 - 1 - a - d_n}\right).$$

If G is $K_n - e$ with $n \geq 4$, then the Laplacian eigenvalues of G^c are

$$(2, \ldots, 0, 0, 0).$$

Thus the eigenvalues of G are

$$(n, n, \ldots, n, (n-2), 0).$$

Hence

$$Kf(G) = n\sum_{i=1}^{n-1} \lambda_i = n\left(\frac{n-2}{n} + \frac{1}{n-2}\right)$$

$$= n\left(\frac{1}{n} + \frac{1}{n + d_2^* - d_1 - 1} + \frac{1}{d_n} + \frac{(n-4)^2}{2e - d_1 - 1 - a - d_n}\right).$$

If G is $K_2 \vee K_{n-2}^c$ with $n \geq 2$, then the Laplacian eigenvalues of G^c are

$$((n-2), (n-2), \ldots, (n-2), 0, 0, 0).$$

Thus the eigenvalues of G are

$$(n, n, 2, 2, \ldots, 2, 0).$$

Hence

$$Kf(G) = n\sum_{i=1}^{n-1} \lambda_i = n\left(\frac{2}{n} + \frac{n-3}{2}\right)$$

$$= n\left(\frac{1}{n} + \frac{1}{n + d_2^* - d_1 - 1} + \frac{1}{d_n} + \frac{(n-4)^2}{2e - d_1 - 1 - a - d_n}\right).$$

If G is $K_2 \vee S_{n-2}^c$ with $n \geq 4$, then the Laplacian eigenvalues of G^c are

$$((n-2), 1, \ldots, 1, 0, 0, 0).$$

Thus the eigenvalues of G are

$$(n, n, (n-1), \ldots, (n-1), 2, 0).$$

Hence

$$Kf(G) = n\sum_{i=1}^{n-1}\lambda_i = n\left(\frac{2}{n} + \frac{n-4}{n-1} + \frac{1}{2}\right)$$

$$= n\left(\frac{1}{n} + \frac{1}{n+d_2^*-d_1-1} + \frac{1}{d_n} + \frac{(n-4)^2}{2e-d_1-1-a-d_n}\right) .$$

If G is $K_1 \vee S_{n-1}^c$ with $n \geq 4$, then the Laplacian eigenvalues of G^c are

$$((n-1), 1, \ldots, 1, 0, 0).$$

Thus the eigenvalues of G are

$$(n, (n-1), \ldots, (n-1), 1, 0).$$

Hence

$$Kf(G) = n\sum_{i=1}^{n-1}\lambda_i = n\left(\frac{1}{n} + \frac{n-3}{n-1} + \frac{1}{1}\right)$$

$$= n\left(\frac{1}{n} + \frac{1}{n+d_2^*-d_1-1} + \frac{1}{d_n} + \frac{(n-4)^2}{2e-d_1-1-a-d_n}\right) .$$

If G is $S_n + e$ with $n \geq 4$, then the Laplacian eigenvalues of G^c are

$$((n-1), (n-1), \ldots, (n-1), (n-3), 0, 0).$$

Thus the eigenvalues of G are

$$(n, 3, 1, \ldots, 1, 1, 0).$$

Hence

$$Kf(G) = n\sum_{i=1}^{n-1}\lambda_i = n\left(\frac{1}{n} + \frac{1}{3} + \frac{n-3}{1}\right)$$

$$= n\left(\frac{1}{n} + \frac{1}{n+d_2^*-d_1-1} + \frac{1}{d_n} + \frac{(n-4)^2}{2e-d_1-1-a-d_n}\right) .$$

If G is $K_1 \vee (K_1 \cup S_{n-2})$ with $n \geq 4$, from the above proof of Case 8, we have that the eigenvalues of $G - \{x, v_n\}$ are

$$((\lambda_2 - 1), (\lambda_3 - 1), \ldots, (\lambda_{n-2} - 1), 0).$$

Since $G - \{x, v_n\}$ is S_{n-2}, $\lambda_2 - 1 = n - 2$ and $\lambda_3 - 1 = \cdots = \lambda_{n-2} - 1 = 1$. Thus $\lambda_1 = n$, $\lambda_2 = n - 1$, $\lambda_3 = \cdots = \lambda_{n-2} = 2$, and $\lambda_{n-1} = 1$. Hence

$$Kf(G) = n\sum_{i=1}^{n-1}\lambda_i = n\left(\frac{1}{n} + \frac{1}{n-1} + \frac{n-4}{2} + \frac{1}{1}\right)$$

$$= n\left(\frac{1}{n} + \frac{1}{n+d_2^*-d_1-1} + \frac{1}{d_n} + \frac{(n-4)^2}{2e-d_1-1-a-d_n}\right) .$$

Hence we complete the proof of Theorem 1.

Remark 1. Recall that Das, Güngör, and Çevik in [6] proved that if G is a non-complete graph of order n, then

$$Kf(G) \geq 1 + \frac{n}{d_n} + \frac{n(n-3)^2}{2e - d_1 - d_n - 1} \tag{1}$$

with equality if and only if G is S_n, $K_n - e$, or $K_1 \vee S_{n-1}^c$. Therefore if G is $K_2 \vee K_{n-2}^c$, $K_2 \vee S_{n-2}^c$, $S_n + e$, or $K_1 \vee (K_1 \cup S_{n-2})$, then

$$\begin{aligned} Kf(G) &= n\left(\frac{1}{n} + \frac{1}{n + d_2^* - d_1 - 1} + \frac{1}{d_n} + \frac{(n-4)^2}{2e - d_1 - 1 - a - d_n}\right) \\ &> \left(1 + \frac{n}{d_n} + \frac{n(n-3)^2}{2e - d_1 - d_n - 1}\right) . \end{aligned}$$

On the other hand, if G is a cycle C_n with $n \geq 5$ vertices and edges, then $d_1 = d_2 = \cdots = d_n$, $e = n$, $d_2^* = n$, and $2 \leq a \leq 3$. Therefore

$$\begin{aligned} &\left(1 + \frac{n}{d_n} + \frac{n(n-3)^2}{2e - d_1 - d_n - 1}\right) - n\left(\frac{1}{n} + \frac{1}{n + d_2^* - d_1 - 1} + \frac{1}{d_n} + \frac{(n-4)^2}{2e - d_1 - 1 - a - d_n}\right) \\ &= n\left(\frac{(n-3)^2}{2n-5} - \frac{(n-4)^2}{2n-5-a} - \frac{1}{2n-3}\right) > n\left(\frac{(n-3)^2}{2n-5} - \frac{(n-4)^2}{2n-8} - \frac{1}{2n-5}\right) \\ &= \frac{n(n-4)^2}{(2n-5)(2n-8)} > 0 . \end{aligned}$$

Hence the lower bound in Theorem 1 and the lower bound in (1) are not comparable.

Recall that Li and Pan in [15] proved that $\lambda_2(G) \geq d_2(G)$ for a connected graph of order $n \geq 3$ and Guo in [11] proved that $\lambda_3(G) \geq d_3(G) - 1$ for a connected graph of order $n \geq 4$. By Theorem 6, we have that $\lambda_3 = \lambda_1 + \lambda_2 + \lambda_3 - \lambda_1 - \lambda_2 \leq d_1^* + d_2^* + d_3^* - d_1 - 1 - d_2 = n + d_2^* + d_3^* - d_1 - d_2 - 1$.

Using arguments similar to those in the proof of Theorem 1, we can obtain another lower bound for a connected graph of order $n \geq 5$.

Theorem 10 Let G be a non–complete connected graph with $n \geq 5$ vertices and e edges. Then

$$\begin{aligned} Kf(G) \geq 1 + \frac{n}{d_n} &+ \frac{n}{n + d_2^* + d_3^* - d_1 - d_2 - 1} \\ &+ \frac{n}{n + d_2^* - d_1 - 1} + \frac{n(n-5)^2}{2e - d_1 - d_2 - d_3 - d_n} . \end{aligned}$$

The characterization of the extremal graphs for an equality in Theorem 10 is unknown.

Remark 2. If G is $K_2 \vee K_{n-2}^c$ with $n \geq 14$, then $d_1 = d_2 = (n-1)$, $d_3 = d_4 = \cdots = d_n = 2$, $e = 2n-3$, $d_2^* = n$, and $d_3^* = 2$. Therefore

$$\left(1 + \frac{n}{d_n} + \frac{n}{n + d_2^* + d_3^* - d_1 - d_2 - 1} + \frac{n}{n + d_2^* - d_1 - 1} + \frac{n(n-5)^2}{2e - d_1 - d_2 - d_3 - d_n}\right)$$

$$- \left(1 + \frac{n}{d_n} + \frac{n(n-3)^2}{2e - d_1 - d_n - 1}\right) = n\left(\frac{1}{3} + \frac{1}{n} + \frac{(n-5)^2}{2n-8} - \frac{(n-3)^2}{3n-8}\right)$$

$$> n\left(\frac{11}{3n-18} + \frac{2}{3n-18} + \frac{3(n-5)^2}{2(3n-18)} - \frac{(n-3)^2}{3n-8}\right) = \frac{n((n-9)^2+2)}{2(3n-18)} > 0\ .$$

On the other hand, if G is a cycle C_n with $n \geq 5$ vertices and edges, then $d_1 = d_2 = \cdots = d_n$, $e = n$, $d_2^* = n$, and $d_3^* = 0$. Therefore

$$\left(1 + \frac{n}{d_n} + \frac{n}{n + d_2^* + d_3^* - d_1 - d_2 - 1} + \frac{n}{n + d_2^* - d_1 - 1} + \frac{n(n-5)^2}{2e - d_1 - d_2 - d_3 - d_n}\right)$$

$$- \left(1 + \frac{n}{d_n} + \frac{n(n-3)^2}{2e - d_1 - d_n - 1}\right) = n\left(\frac{1}{2n-3} + \frac{1}{2n-5} + \frac{(n-5)^2}{2n-8} - \frac{(n-3)^2}{2n-5}\right)$$

$$< n\left(\frac{1}{2n-5} + \frac{1}{2n-5} + \frac{(n-5)^2}{2n-8} - \frac{(n-3)^2}{2n-5}\right) = -\frac{(n-3)(5n-23)}{(2n-5)(2n-8)} < 0\ .$$

Hence the lower bound in Theorem 10 and the lower bound in (1) are not comparable.

Appendix

Case 1. $\lambda_1 = \lambda_2$, $\lambda_2 = \lambda_3$, and $\lambda_{n-2} = \lambda_{n-1}$.

This case is impossible since otherwise we have the contradiction $d_1 + 1 = \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = d_n$.

Case 2. $\lambda_1 = \lambda_2$, $\lambda_2 = \lambda_3$, and $\lambda_{n-2} > \lambda_{n-1}$.

By Theorem 8 and the assumption that $G \neq K_n$, we have that G is $K_n - e$ with $n \geq 4$.

Case 3. $\lambda_1 = \lambda_2$, $\lambda_2 > \lambda_3$, and $\lambda_{n-2} = \lambda_{n-1}$.

From $d_2 + 1 \geq a = \lambda_2 = \lambda_1 = d_1 + 1$, we have that $(a-1) = d_1 = d_2 = (n-1)$. Thus $d(x) = d(y) = (n-1)$. Hence $N(x) = V - \{x\}$ and $N(y) = V - \{y\}$. In particular, $xy \in E$.

Since $G \neq K_n$ and $\lambda_{n-1} = d_n$, Theorem 4 implies that G is the same as the graph $G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$, where $|S| = d_n$ and $m \geq 2$. In fact, from the proof of Theorem 1 in [14], we can see that S is a cutset of G and $G_1, G_2, \ldots, G_m$ are components of $G[V-S]$. Since $d(x) = d(y) = (n-1)$, both x and y are in S.

We claim that $|S| = 2$. Suppose, to the contrary, that $|S| \geq 3$. Since $N(x) = V - \{x\}$ and $N(y) = V - \{y\}$, G^c has at least four components. Notice that the eigenvalues of G

are

$$(n, n, \lambda_3, \ldots, \lambda_{n-1}, 0).$$

Thus the eigenvalues of G^c are

$$((n - \lambda_{n-1}), (n - \lambda_{n-2}), \ldots, (n - \lambda_3), 0, 0, 0)$$

and G^c has three components, a contradiction. Hence $|S| = 2$.

Now we have that $\lambda_3 = \cdots = \lambda_{n-1} = d_n = |S| = 2$. Therefore $G^c - \{x, y\}$ has eigenvalues

$$((n - 2), (n - 2), \ldots, (n - 2), 0).$$

By Theorem 8, we have that $G^c - \{x, y\}$ is K_{n-2} or $K_{n-2} - e$. If $n = 4$ and $G^c - \{x, y\} = K_{n-2} - e$, then $G = K_4$, a contradiction. If $n \geq 5$, since $G^c - \{x, y\}$ and $K_{n-2} - e$ have different sets of eigenvalues, $G^c - \{x, y\} \neq K_{n-2} - e$. Thus $G^c - \{x, y\} = K_{n-2}$. Hence G is $K_2 \vee K_{n-2}^c$ with $n \geq 4$.

Case 4. $\lambda_1 = \lambda_2$, $\lambda_2 > \lambda_3$, and $\lambda_{n-2} > \lambda_{n-1}$.

Using arguments similar to those in Case 3, we have that $(a - 1) = d_1 = d(x) = d_2 = d(y) = (n - 1)$, $N(x) = V - \{x\}$, $N(y) = V - \{y\}$, and $xy \in E$. Moreover, $S = \{x, y\}$ is a cutset of G and G is the same as the graph $G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$, where $|S| = d_n = \lambda_{n-1} = 2$ and $m \geq 2$.

Therefore $G^c - \{x, y\}$ has eigenvalues

$$((n - \lambda_{n-1}), (n - \lambda_{n-2}), \ldots, (n - \lambda_3), 0).$$

By Theorem 7, we have that $G^c - \{x, y\}$ is K_{n-2}, S_{n-2}, or $K_{\frac{n-2}{2}, \frac{n-2}{2}}$. Since $(n - \lambda_{n-1}) > (n - \lambda_{n-2})$, $G^c - \{x, y\}$ cannot be K_{n-2} if $n \geq 5$. If $G^c - \{x, y\}$ is $K_{\frac{n-2}{2}, \frac{n-2}{2}}$, the minimum degree of G is $\frac{n}{2}$, which is not equal to $d_n = 2$ when $n \geq 5$. Thus $G^c - \{x, y\}$ cannot be $K_{\frac{n-2}{2}, \frac{n-2}{2}}$ if $n \geq 5$. Notice that K_{n-2} and $K_{\frac{n-2}{2}, \frac{n-2}{2}}$ are K_2 when $n = 4$. Therefore $G^c - \{x, y\} = K_2$ or S_{n-2}. Notice again that $K_2 \vee K_2^c = K_2 \vee S_{n-2}^c$ when $n = 4$. Hence G is $K_2 \vee S_{n-2}^c$ with $n \geq 4$.

Case 5. $\lambda_1 > \lambda_2$, $\lambda_2 = \lambda_3$, and $\lambda_{n-2} = \lambda_{n-1}$.

By Theorem 7 and the assumptions that $G \neq K_n$ and $G \neq S_n$, we have that G is K_{d_1, d_1}. Since $d_1 = (n - 1)$, G must be K_2, contradicting the assumption that $n \geq 4$. Hence this case is impossible.

Case 6. $\lambda_1 > \lambda_2$, $\lambda_2 = \lambda_3$, and $\lambda_{n-2} > \lambda_{n-1}$.

From $d_1 = (n - 1)$, we have that $d(x) = (n - 1)$ and $N(x) = V - \{x\}$. Since G is not K_n and $\lambda_{n-1} = d_n$, Theorem 4 implies that G is the same as the graph $G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$, where $|S| = d_n$ and $m \geq 2$. Clearly, x is in S.

We claim that $|S| = 1$. Suppose, to the contrary, that $|S| \geq 2$. Since $N(x) = V - \{x\}$, G^c has at least three components. Notice that the eigenvalues of G are

$$(n, \lambda_2, \lambda_3, \ldots, \lambda_{n-1}, 0).$$

Thus the eigenvalues of G^c are

$$((n - \lambda_{n-1}), (n - \lambda_{n-2}), \ldots, (n - \lambda_2), 0, 0)$$

and therefore G^c has two components, a contradiction. Hence $|S| = 1$.

Now we have that $\lambda_{n-1} = d_n = |S| = 1$. Therefore $G^c - \{x\}$ has eigenvalues

$$((n - 1), (n - \lambda_{n-2}), \ldots, (n - \lambda_2), 0).$$

By Theorem 7, we have that $G^c - \{x\}$ is K_{n-1}, S_{n-1}, or $K_{\frac{n-1}{2}, \frac{n-1}{2}}$. Since $(n-1) = (n - \lambda_{n-1}) > (n - \lambda_{n-2})$, $G^c - \{x\}$ cannot be K_{n-1}. If $G^c - \{x\}$ is $K_{\frac{n-1}{2}, \frac{n-1}{2}}$, the minimum degree of G is $\frac{n-1}{2}$, which is not equal to $d_n = 1$ when $n \geq 4$. Thus $G^c - \{x\}$ cannot be $K_{\frac{n-1}{2}, \frac{n-1}{2}}$. Therefore $G^c - \{x\} = S_{n-1}$. Hence G is $K_1 \vee S_{n-1}^c$with $n \geq 4$.

Case 7. $\lambda_1 > \lambda_2$, $\lambda_2 > \lambda_3$, and $\lambda_{n-2} = \lambda_{n-1}$.

Again, since $d_1 = (n-1)$, we have that $d(x) = (n-1)$ and $N(x) = V - \{x\}$. Since G is not K_n and $\lambda_{n-1} = d_n$, Theorem 4 implies that G is the same as the graph $G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$, where $|S| = d_n$ and $m \geq 2$. Clearly, x is in S.

We claim again that $|S| = 1$. Suppose, to the contrary, that $|S| \geq 2$. Since $N(x) = V - \{x\}$, G^c has at least three components. Since the eigenvalues of G are

$$(n, \lambda_2, \lambda_3, \ldots, \lambda_{n-1}, 0),$$

the eigenvalues of G^c are

$$((n - d_n), (n - d_n), \ldots, (n - d_n), (n - \lambda_2), 0, 0).$$

Since $n - \lambda_2 = \lambda_1 - \lambda_2 > 0$, G^c has two components, a contradiction. Hence $|S| = 1$.

Now we have that $\lambda_{n-1} = d_n = |S| = 1$. Therefore G^c has eigenvalues

$$((n - 1), (n - 1), \ldots, (n - 1), (n - \lambda_2), 0, 0),$$

and $G^c - \{x\}$ has eigenvalues

$$((n - 1), (n - 1), \ldots, (n - 1), (n - \lambda_2), 0).$$

By Theorem 8, we have that $G^c - \{x\}$ is K_{n-1} or $K_{n-1} - e$. Since $(n-1) = (n - d_n) > (n - \lambda_2)$, $G^c - \{x\}$ cannot be K_{n-1}. Therefore $G^c - \{x\} = K_{n-1} - e$. Hence G is $S_n + e$ with $n \geq 4$.

Case 8. $\lambda_1 > \lambda_2$, $\lambda_2 > \lambda_3$, and $\lambda_{n-2} > \lambda_{n-1}$.

Using arguments similar to those in Case 7, we have that $d(x) = (n-1)$, $N(x) = V - \{x\}$, G is the same as the graph $G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$ with $|S| = d_n = \lambda_{n-1} = 1$ and $m \geq 2$, and x is in S.

Since $d_n = 1$, v_n is only adjacent to x in G and v_n is adjacent to each vertex of $V - \{x\}$ in G^c. Since the eigenvalues of G are

$$(n, \lambda_2, \ldots, \lambda_{n-2}, \lambda_{n-1}, 0),$$

the eigenvalues of G^c are

$$((n-1), (n-\lambda_{n-2}), \ldots, (n-\lambda_3), (n-\lambda_2), 0, 0).$$

This implies that the eigenvalues of $G^c - \{x\}$ are

$$((n-1), (n-\lambda_{n-2}), \ldots, (n-\lambda_3), (n-\lambda_2), 0).$$

Therefore the eigenvalues of $(G^c - \{x\})^c$ are

$$((n-1)-(n-\lambda_2), (n-1)-(n-\lambda_3), \ldots, (n-1)-(n-\lambda_{n-2}), 0, 0),$$

which further implies that the eigenvalues of $(G^c - \{x\})^c - \{v_n\}$, i.e., $G - \{x, v_n\}$, are

$$((\lambda_2 - 1), (\lambda_3 - 1), \ldots, (\lambda_{n-2} - 1), 0).$$

Since $(\lambda_3 - 1) = \cdots = (\lambda_{n-2} - 1)$, we, by Theorem 7, have that $(G^c - \{x\})^c - \{v_n\} = G - \{x, v_n\}$ is K_{n-2}, S_{n-2}, or $K_{\frac{n-2}{2}, \frac{n-2}{2}}$. Since $\lambda_2 - 1 > \lambda_3 - 1$, $G - \{x, v_n\}$ cannot be K_{n-2} when $n \geq 5$. If $G - \{x, v_n\}$ is $K_{\frac{n-2}{2}, \frac{n-2}{2}}$, then $\lambda_2 = d_2^* = (n-1) \neq \frac{n}{2} + 1 = a$ when $n \geq 5$. Thus $G - \{x, v_n\}$ cannot be $K_{\frac{n-2}{2}, \frac{n-2}{2}}$ when $n \geq 5$. Notice that K_{n-2} and $K_{\frac{n-2}{2}, \frac{n-2}{2}}$ are K_2 when $n = 4$. Therefore $G - \{x, v_n\} = K_2$ or S_{n-2}. Notice again that $K_1 \vee (K_1 \cup K_2) = K_1 \vee (K_1 \cup S_{n-2})$ when $n = 4$. Hence G is $K_1 \vee (K_1 \cup S_{n-2})$ with $n \geq 4$.

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