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Estimating the Incidence Energy

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Abstract

Bounds for the incidence energy of connected bipartite graphs were recently reported. We now extend these results to connected non-bipartite graphs. In addition, these bounds are generalized so as to apply to the sum of α-th powers of signless Laplacian eigenvalues, for any real α.

1 Introduction

The *energy* of a graph is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix [14]. After the great success of this concept [15, 20, 27], the natural idea was to extend it to other matrices. The first such attempt was to conceive the "energy" of the Laplacian matrix [22, 44], followed by a plethora of other "graph energies" [16, 27]. A significant step forward in the theory of these novel "graph energies" was made by Nikiforov [33]. According to Nikiforov, the energy of any matrix is equal to the sum of singular values of this matrix.

Recall that the singular values of a (real) matrix $\mathbf{M}$ are equal to the positive square roots of the eigenvalues of $\mathbf{M}\,\mathbf{M}^T$. In particular, the ordinary graph energy coincides with the energy of the adjacency matrix, whereas the Laplacian energy of an (n,m)-graph G is the energy of the matrix $\mathbf{L}(G) - (2m/n)\,\mathbf{I}_n$, where $\mathbf{L}(G)$ and $\mathbf{I}_n$ are the Laplacian matrix of G and the unit matrix of order n.

The first non-trivial matrix energy defined via singular values was the *incidence energy* [24], namely the energy of the vertex–edge incidence matrix.

Let $\mathbf{I}(G)$ be the vertex–edge incidence matrix of the graph G. For a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, the (i,j)-entry of $\mathbf{I}(G)$ is equal to 1 if the vertex v_i is incident to the edge e_j, and is zero otherwise. Let $\xi_1, \xi_2, \ldots, \xi_n$ be the singular values of $\mathbf{I}(G)$. Then the incidence energy of the graph G is defined as [24]

$$IE = IE(G) = \sum_{i=1}^{n} \xi_i \,.$$

Some basic properties of the incidence energy were established in [18, 19, 24, 37, 38, 42].

It was soon recognized [18] that the incidence energy is intimately related with the eigenvalues of the Laplacian and signless Laplacian matrices. Therefore, for the considerations that follow, we need to define these matrices and recall their main spectral properties [8–11, 31, 32].

Let G be a simple connected graph with n vertices and m edges and vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $v_i \in V(G)$, the degree of the vertex v_i, denoted by d_i, is the number of vertices adjacent to v_i.

The *Laplacian matrix* of the graph G is defined as $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$, where $\mathbf{A}(G)$ is the (0,1)-adjacency matrix of G, and $\mathbf{D}(G)$ the diagonal matrix of the vertex degrees. The eigenvalues of $\mathbf{L}(G)$ are said to be the *Laplacian eigenvalues* of G and will be denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. It is well known that $\mu_n = 0$ and that the multiplicity of zero is equal to the number of connected components of G [13]. The sum of the Laplacian eigenvalues is equal to $2m$. For more details on Laplacian eigenvalues see [31, 32].

The *signless Laplacian matrix* of the graph G is defined as $\mathbf{Q}(G) = \mathbf{D}(G) + \mathbf{A}(G)$ and its eigenvalues are denoted by $q_1 \geq q_2 \geq \cdots \geq q_n$. The signless Laplacian eigenvalues are real non-negative numbers. Their sum is also equal to $2m$. For more details on signless Laplacian eigenvalues see [8–11].

As well known in graph theory, $\mathbf{I}(G)\,\mathbf{I}(G)^T = \mathbf{Q}(G)$. From this identity it immediately follows [18]

$$IE(G) = \sum_{i=1}^{n} \sqrt{q_i} \,. \tag{1}$$

Independently of, and somewhat earlier than, the introduction of the incidence energy concept, Liu and Liu [29] considered the quantity LEL defined as

$$LEL = LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i}$$

and named it *Laplacian–energy like invariant* of the graph G. The authors of [29] believed that LEL has properties analogous to the Laplacian energy, but it was later found [23] that it is much more similar to the ordinary graph energy. For survey and more information on LEL see [28].

For the present considerations, it is important that because for bipartite graphs the Laplacian and signless Laplacian eigenvalues coincide [8,31,32], for bipartite graphs LEL is equal to the incidence energy [18].

In the present work we obtain some new estimates of the incidence energy. However, we will be able to establish more general results, of which the incidence–energy results are straightforward special cases. The details of this generalized approach are outlined in the subsequent section.

2 Sum of powers of Laplacian and signless Laplacian eigenvalues

Let α be a real number. In order to avoid trivialities, we may require that $\alpha \neq 0$ and $\alpha \neq 1$. For a connected graph G, several authors [1,5,30,36,40,41,45] considered the sum of the α-th powers of the non-zero Laplacian eigenvalues,

$$\sigma_\alpha = \sigma_\alpha(G) = \sum_{i=1}^{n-1} \mu_i^\alpha \ . \tag{2}$$

The cases $\alpha = 0$ and $\alpha = 1$ are trivial as $\sigma_0(G) = n - 1$ and $\sigma_1(G) = 2m$. Note that $\sigma_{1/2} \equiv LEL$.

Some properties of σ_2 were established in [25]. The author of [25], fully unjustified, referred to $\sigma_2(G)$ as to the Laplacian energy of the graph G. In fact, σ_2 is just the second Laplacian spectral moment, and its dependence on the structure of the graph G is trivially simple.

It is worth noting that for a connected graph G with n vertices, $n\,\sigma_{-1}(G)$ is equal to the Kirchhoff index, an invariant having extensive applications in the theory of electric circuits, probabilistic theory, and chemistry [2–4,12,21,35].

The graph invariant σ_α and especially its estimates were much studied in the literature. For instance, Zhou established bounds of σ_α and also discussed further properties for σ_2 and $\sigma_{1/2}$ [40]. The results obtained in [40] were eventually improved [30,36]. More bounds on σ_α are found in [1, 41, 45]. In [45] lower and upper bounds for incidence energy and lower bounds for Kirchhoff index and Laplacian Estrada index of bipartite graphs were also deduced.

Our main concern in this paper is the sum

$$s_\alpha = s_\alpha(G) = \sum_{i=1}^{n} q_i^\alpha$$

which is just the signless–Laplacian variant of Eq. (2). Also the sum of the α-th powers of the signless Laplacian eigenvalues was studied in the literature [1, 26], but to a somewhat lesser extent than σ_α .

The author of the paper [26] named s_α "α-incidence energy", because its special case for $\alpha = 1/2$ coincides with IE, cf. Eq. (1). This name for s_α is certainly not adequate, and "*sum of powers of the signless Laplacian eigenvalues*" should be preferred [1]. For the case when α is a positive integer, the name "*signless Laplacian spectral moment*" would be justified.

Formally speaking, s_α would be the "energy" of the signless Laplacian matrix raised to the power of α. However, Nikiforov's matrix–energy concept [33] is purposeful only for square and symmetric matrices whose trace (sum of diagonal entries) is zero.[1]

Returning to the sum of powers of signless Laplacian eigenvalues, we first note that the cases $\alpha = 0$ and $\alpha = 1$ are trivial as $s_0(G) = n$ and $s_1(G) = 2m$, respectively. Evidently, $s_{1/2}(G)$ is equal to the incidence energy of G. Note further that $\sigma_\alpha(G)$ and $s_\alpha(G)$ coincide in the case of bipartite graphs. This is an immediate consequence of the well known fact that the Laplacian and signless Laplacian eigenvalues of bipartite graphs coincide [8, 31, 32]. Some bounds for $s_\alpha(G)$ were established in [26].

The rest of the paper is organized in the following way. In Section 3, we give some useful lemmas that will be used later. In Section 4, we present some lower and upper bound on $s_\alpha(G)$ for a connected graph G and show that some of our results improve the

[1] Recall that the "Laplacian energy" of an (n, m)-graph G is not the energy of the Laplacian matrix $\mathbf{L}(G)$, but of the modified matrix $\mathbf{L}(G) - (2m/n)\,\mathbf{I}_n$. The trace of the signless Laplacian matrix $\mathbf{Q}(G)$ is $2m$. Therefore, the consistent definition of "α-incidence energy" would be the energy of the matrix $\mathbf{Q}(G)^\alpha - (2m/n)^\alpha\,\mathbf{I}_n$. The present authors hope that nobody ever would endeavor to examine this awkward and ill-conceived "graph energy".

results obtained in [26]. These results yield, as immediate special cases, estimates for the incidence energy.

3 Lemmas

Let $Zg(G)$ be the first Zagreb index of a graph G, defined as the sum of squares of the vertex degrees of G [17,34]. Let $\mathcal{L}(G)$ denote the line graph of the graph G and let $G_1 \times G_2$ be the Cartesian product of the graphs G_1 and G_2 [7]. We now introduce an auxiliary quantity for a graph G as

$$T = T(G) = \frac{1}{2}\left[\Delta + \delta + \sqrt{(\Delta - \delta)^2 + 4\Delta}\right] \tag{3}$$

where Δ and δ are the maximum and the minimum vertex degree of G, respectively.

Lemma 3.1. ([10]) *Let G be a connected non-bipartite graph with n vertices and signless Laplacian eigenvalues $q_1 \geq q_2 \geq \cdots \geq q_n$. Then*

$$\prod_{i=1}^{n} q_i = \frac{2t(G \times K_2)}{t(G)}$$

where $t(G)$ and $t(G \times K_2)$ are the number of spanning trees of G and $G \times K_2$, respectively.

Lemma 3.2. ([42, 43]) *Let G be a graph with at least two edges. Then*

$$q_1 \geq \frac{Zg(G)}{m}$$

with equality if and only if $\mathcal{L}(G)$ is regular.

Lemma 3.3. ([6, 39]) *Let G be a connected graph with $n \geq 3$ vertices and let Δ be the maximum vertex degree of G. Then*

$$q_1 \geq T \geq \Delta + 1$$

with either equalities if and only if G is a star graph $K_{1,n-1}$.

Lemma 3.4. ([8]) *Let G be a connected graph with diameter d. If G has exactly k distinct signless Laplacian eigenvalues, then $d + 1 \leq k$.*

4 Main Results

Theorem 4.1. *Let G be a connected graph with n vertices, m edges, Zagreb index $Zg(G)$, and $t(G)$ spanning trees. If G is bipartite, then*

$$IE(G) \geq \sqrt{\frac{Zg(G)}{m}} + (n-2)\left(\frac{nm\,t(G)}{Zg(G)}\right)^{1/(2(n-2))} \tag{4}$$

with equality if and only if $G \cong K_{1,n-1}$ or (provided n is even) $G \cong K_{n/2,n/2}$.

If G is non-bipartite, then

$$IE(G) \geq \sqrt{\frac{Zg(G)}{m}} + (n-1)\left(\frac{2m\,t(G \times K_2)}{Zg(G)t(G)}\right)^{1/(2(n-1))} \tag{5}$$

with equality if and only if $G \cong K_n$.

The inequality (4) was earlier established by Zhou and Ilić [45], whereas (5) is stated here for the first time.

Instead of proving Theorem 4.1, we demonstrate the validity of a more general result, from which Theorem 4.1 follows as an immediate special case for $\alpha = 1/2$.

Theorem 4.2. *Let G be a connected graph with $n \geq 3$ vertices, m edges, Zagreb index $Zg(G)$, and $t(G)$ spanning trees. Let α be a real number. If G is bipartite, then*

$$s_\alpha(G) \geq \left(\frac{Zg(G)}{m}\right)^{\alpha} + (n-2)\left(\frac{nm\,t(G)}{Zg(G)}\right)^{\alpha/(n-2)} \tag{6}$$

with equality if and only if $G \cong K_{1,n-1}$ or (provided n is even) $G \cong K_{n/2,n/2}$.

If G is non-bipartite, then

$$s_\alpha(G) \geq \left(\frac{Zg(G)}{m}\right)^{\alpha} + (n-1)\left(\frac{2m\,t\,(G \times K_2)}{Zg(G)\,t(G)}\right)^{\alpha/(n-1)} \tag{7}$$

with equality if and only if $G \cong K_n$.

Proof. Inequality (6) has been obtained by Zhou and Ilić [45]. Therefore, its proof will be omitted.

Using Lemma 3.1 and the arithmetic–geometric mean inequality, we obtain

$$\begin{aligned} s_\alpha(G) &= q_1^\alpha + \sum_{i=2}^{n} q_i^\alpha \geq q_1^\alpha + (n-1)\left(\prod_{i=2}^{n} q_i^\alpha\right)^{1/(n-1)} \\ &= q_1^\alpha + (n-1)\left(\frac{2t(G \times K_2)}{t(G)\,q_1}\right)^{\alpha/(n-1)} \end{aligned}$$

with equality if and only if $q_2 = \cdots = q_n$.

Consider now the following function:

$$f(x) = x^\alpha + (n-1)\left(\frac{2t(G\times K_2)}{t(G)\,x}\right)^{\alpha/(n-1)} .$$

It is easy to see that $f(x)$ is increasing for $x > \left(\frac{2t(G\times K_2)}{t(G)}\right)^{1/n}$ for both $\alpha > 0$ and $\alpha < 0$. By Lemma 3.2 and the Cauchy–Schwarz inequality, we have [42]

$$q_1 \geq \frac{Zg(G)}{m} \geq 2\sqrt{\frac{Zg(G)}{n}} > \frac{2m}{n} .$$

Using the arithmetic–geometric inequality and Lemma 3.1, we get

$$\frac{2m}{n} = \frac{1}{n}\sum_{i=1}^{n} q_i \geq \left(\prod_{i=1}^{n} q_i\right)^{1/n} = \left(\frac{2t(G\times K_2)}{t(G)}\right)^{1/n} .$$

Therefore

$$s_\alpha(G) \geq f\left(\frac{Zg(G)}{m}\right) = \left(\frac{Zg(G)}{m}\right)^\alpha + (n-1)\left(\frac{2m\,t(G\times K_2)}{Zg(G)\,t(G)}\right)^{\alpha/(n-1)} .$$

Hence (7) follows. Equality holds in (7) if and only if $q_1 = Zg(G)/m$ and $q_2 = \cdots = q_n$.

Suppose that the equality holds in (7). Then by Lemma 3.2, we get that $\mathcal{L}(G)$ is regular. Note further that $q_1 > q_2 = \cdots = q_n$, otherwise we would have that $q_1 = 2m/n < 2\sqrt{Zg(G)/n} \leq Zg(G)/m$, which is a contradiction. Thus, G has exactly two distinct signless Laplacian eigenvalues. Then, by Lemma 3.4, we conclude that $G \cong K_n$.

Conversely, it can be easily verified that the equality holds in (7) for the complete graph K_n. □

Setting $\alpha = 1/2$ into the inequalities (6) and (7), we readily arrive at the inequalities (4) and (5) for the incidence energy.

The following results were given by Zhou and Ilić in [45].

Theorem 4.3. *Let G be a connected bipartite graph with $n \geq 3$ vertices and m edges.*
(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha(G) \geq \left(\frac{Zg(G)}{m}\right)^\alpha + \frac{\left(2m - \frac{Zg(G)}{m}\right)^\alpha}{(n-2)^{\alpha-1}} . \tag{8}$$

(ii) If $0 < \alpha < 1$, then

$$s_\alpha(G) \leq \left(\frac{Zg(G)}{m}\right)^\alpha + \frac{\left(2m - \frac{Zg(G)}{m}\right)^\alpha}{(n-2)^{\alpha-1}} . \tag{9}$$

Equality in both (8) and (9) occurs if and only if $G \cong K_{1,n-1}$ *or (provided* n *is even)* $G \cong K_{n/2,n/2}$ *.*

(iii) As a special case of (9) for $\alpha = 1/2$*,*

$$IE(G) \leq \sqrt{\frac{Zg(G)}{m}} + \sqrt{(n-2)\left(2m - \frac{Zg(G)}{m}\right)} \,. \tag{10}$$

Theorem 4.4. *Let* G *be a connected non-bipartite graph with* $n \geq 3$ *vertices and* m *edges.*
(i) If $\alpha < 0$ *or* $\alpha > 1$*, then*

$$s_\alpha(G) \geq \left(\frac{Zg(G)}{m}\right)^\alpha + \frac{\left(2m - \frac{Zg(G)}{m}\right)^\alpha}{(n-1)^{\alpha-1}} \,. \tag{11}$$

(ii) If $0 < \alpha < 1$*, then*

$$s_\alpha(G) \leq \left(\frac{Zg(G)}{m}\right)^\alpha + \frac{\left(2m - \frac{Zg(G)}{m}\right)^\alpha}{(n-1)^{\alpha-1}} \,. \tag{12}$$

Equality in both (11) and (12) occurs if and only if $G \cong K_n$ *.*

Proof. Note that x^α is convex when $x > 0$ and $\alpha < 0$ or $\alpha > 1$. Therefore we get

$$\left(\sum_{i=2}^{n} \frac{1}{n-1}\, q_i\right)^\alpha \leq \sum_{i=2}^{n} \frac{1}{n-1}\, q_i^\alpha$$

i. e.,

$$\sum_{i=2}^{n} q_i^\alpha \geq \frac{1}{(n-1)^{\alpha-1}} \left(\sum_{i=2}^{n} q_i\right)^\alpha$$

with equality if and only if $q_2 = \cdots = q_n$. Then

$$s_\alpha(G) \geq q_1^\alpha + \frac{1}{(n-1)^{\alpha-1}} \left(\sum_{i=2}^{n} q_i\right)^\alpha = q_1^\alpha + \frac{(2m-q_1)^\alpha}{(n-1)^{\alpha-1}} \,.$$

Consider the auxiliary function

$$g(x) = x^\alpha + \frac{(2m-x)^\alpha}{(n-1)^{\alpha-1}}$$

and note that it is increasing for $x > 2m/n$, see [26]. By Lemma 3.2, and bearing in mind that $q_1 \geq Zg(G)/m \geq 2\sqrt{Zg(G)/n} > 2m/n$, we get

$$s_\alpha(G) \geq g\left(\frac{Zg(G)}{m}\right) = \left(\frac{Zg(G)}{m}\right)^\alpha + \frac{\left(2m - \frac{Zg(G)}{m}\right)^\alpha}{(n-1)^{\alpha-1}} \,.$$

for $\alpha < 0$ or $\alpha > 1$. Hence (11) follows.

Now we consider the case of $0 < \alpha < 1$. Note that x^α is concave when $x > 0$ and $0 < \alpha < 1$. Thus

$$\left(\sum_{i=2}^{n} \frac{1}{n-1}\, q_i\right)^{\alpha} \geq \sum_{i=2}^{n} \frac{1}{n-1}\, q_i^{\alpha}$$

with equality if and only if $q_2 = \cdots = q_n$.

Note that $g(x)$ is decreasing for $x > 2m/n$, see [26]. Then, by a parallel argument as above, we can prove (12).

For connected non-bipartite connected graphs, either equality in (11) or (12) holds if and only if $q_1 = Zg(G)/m$ and $q_2 = \cdots = q_n$. Then, by using similar arguments as in the proof of Theorem 4.2, we conclude that $G \cong K_n$. □

For $\alpha = 1/2$, Theorem 4.4 yields the following corollary.

Corollary 4.5. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices and m edges. Then*

$$IE(G) \leq \sqrt{\frac{Zg(G)}{m}} + \sqrt{(n-1)\left(2m - \frac{Zg(G)}{m}\right)} \tag{13}$$

with equality if and only if $G \cong K_n$.

In view of (10), the inequality (13) holds also for bipartite graphs, which then coincides with Proposition 1 in [42].

Theorem 4.6. ([26]) *Let G be a connected graph with $n \geq 3$ vertices and m edges. Let $\alpha \neq 0, 1$. Then the following holds:*

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha(G) \geq \left(2\sqrt{\frac{Zg(G)}{n}}\right)^{\alpha} + \frac{\left(2m - 2\sqrt{\frac{Zg(G)}{n}}\right)^{\alpha}}{(n-1)^{\alpha-1}} \tag{14}$$

with equality if and only if $G \cong K_n$.

(ii) If $0 < \alpha < 1$, then

$$s_\alpha(G) \leq \left(2\sqrt{\frac{Zg(G)}{n}}\right)^{\alpha} + \frac{\left(2m - 2\sqrt{\frac{Zg(G)}{n}}\right)^{\alpha}}{(n-1)^{\alpha-1}} \tag{15}$$

with equality if and only if $G \cong K_n$.

Remark 4.7. *From the proof of Theorem 4.4, we conclude that if $\alpha < 0$ or $\alpha > 1$, then the lower bound (11) is better than the lower bound (14) and if $0 < \alpha < 1$, then the upper bound (12) is better than the upper bound (15). Thus our Theorem 4.4 provides an improvement of Rao Li's Theorem 4.6.*

Theorem 4.8. *Let G be a connected graph with $n \geq 3$ vertices. If G is bipartite, then*

$$IE(G) \geq \sqrt{T} + (n-2)\left(\frac{n\,t(G)}{T}\right)^{1/(2(n-2))}$$

with equality if and only if $G \cong K_{1,n-1}$.

If G is non-bipartite, then

$$IE(G) > \sqrt{T} + (n-1)\left(\frac{2t(G \times K_2)}{t(G)\,T}\right)^{1/(2(n-1))} .$$

Proof. Theorem 4.8 is an immediate special case of the below Theorem 4.9, obtained by setting $\alpha = 1/2$. □

Theorem 4.9. *Let G be a connected graph with $n \geq 3$ vertices and let the parameter T be given by Eq. (3). Let α be a real number. If G is bipartite, then*

$$s_\alpha(G) \geq T^\alpha + (n-2)\left(\frac{n\,t(G)}{T}\right)^{\alpha/(n-2)} \tag{16}$$

with equality if and only if $G \cong K_{1,n-1}$.

If G is non-bipartite, then

$$s_\alpha(G) > T^\alpha + (n-1)\left(\frac{2t(G \times K_2)}{t(G)\,T}\right)^{\alpha/(n-1)} . \tag{17}$$

Proof. Inequality (16) was earlier communicated in [5].

By Lemma 3.3, $q_1 \geq T \geq 1 + \Delta > \Delta \geq 2m/n \geq \left(\frac{2t(G \times K_2)}{t(G)}\right)^{1/n}$. Thus by similar arguments as in the proof of Theorem 4.2, we get that $s_\alpha(G) \geq f(T)$. Then (17) follows and the equality holds in (17) if and only if $q_1 = T$ and $q_2 = \cdots = q_n$.

The condition $q_2 = \cdots = q_n$ would imply that $G \cong K_n$. However, $q_1(K_n)$ is equal to $2n-2$, which differs from $T(K_n) = n - 1 + \sqrt{n-1}$. Thus we conclude that (17) cannot become an equality.

This completes the proof of the theorem. □

Theorem 4.10. *Let G be a connected bipartite graph with $n \geq 3$ vertices and m edges.*
(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha(G) \geq T^\alpha + \frac{(2m-T)^\alpha}{(n-2)^{\alpha-1}} . \tag{18}$$

(ii) If $0 < \alpha < 1$*, then*

$$s_\alpha(G) \le T^\alpha + \frac{(2m-T)^\alpha}{(n-2)^{\alpha-1}} \ . \tag{19}$$

Equality in both (18) and (19) occurs if and only if $G \cong K_{1,n-1}$ *.*

Let G *be a connected non-bipartite graph with* $n \ge 3$ *vertices and* m *edges.*
(iii) If $\alpha < 0$ *or* $\alpha > 1$*, then*

$$s_\alpha(G) > T^\alpha + \frac{(2m-T)^\alpha}{(n-1)^{\alpha-1}} \ . \tag{20}$$

(iv) If $0 < \alpha < 1$*, then*

$$s_\alpha(G) < T^\alpha + \frac{(2m-T)^\alpha}{(n-1)^{\alpha-1}} \ . \tag{21}$$

Proof. Inequalities (18) and (19) was previously proven in [5].

By Lemma 3.3, we have $q_1 \ge T \ge 1 + \Delta > \Delta \ge 2m/n$. Thus by similar arguments as in the proof of Theorem 4.4, we get that $s_\alpha(G) \ge g(T)$ for $\alpha < 0$ or $\alpha > 1$. Then (20) follows.

From the proof of Theorem 4.4, we also obtain $s_\alpha(G) \le g(T)$ for $0 < \alpha < 1$ and then (21) follows.

Either equality in (20) or (21) holds if and only if $q_1 = T$ and $q_2 = \cdots = q_n$. By similar arguments as in the proof of Theorem 4.9, we conclude that the above inequalities cannot become equalities. □

From Theorem 4.10, we also have the following sharp upper bound for the incidence energy of connected graphs.

Corollary 4.11. *Let* G *be a connected graph with* $n \ge 3$ *vertices and* m *edges. If* G *is bipartite, then*

$$IE(G) \le \sqrt{T} + \sqrt{(n-2)(2m-T)}$$

with equality if and only if $G \cong K_{1,n-1}$ *. If* G *is non-bipartite, then*

$$IE(G) < \sqrt{T} + \sqrt{(n-1)(2m-T)} \ .$$

Rao Li [26] obtained also the following result:
Theorem 4.12. ([26]) *Let* G *be a connected graph with* $n \ge 3$ *vertices,* m *edges, and maximum vertex degree* Δ*. Let* $\alpha \ne 0, 1$*. Then the following holds:*
(i) If $\alpha < 0$ *or* $\alpha > 1$*, then*

$$s_\alpha(G) > (1+\Delta)^\alpha + \frac{(2m-1-\Delta)^\alpha}{(n-1)^{\alpha-1}} \ . \tag{22}$$

(ii) If $0 < \alpha < 1$*, then*

$$s_\alpha(G) < (1+\Delta)^\alpha + \frac{(2m-1-\Delta)^\alpha}{(n-1)^{\alpha-1}} \ . \tag{23}$$

Remark 4.13. *From the proof of Theorem 4.10, we conclude that if* $\alpha < 0$ *or* $\alpha > 1$*, then the lower bound (20) is better than the lower bound (22) and if* $0 < \alpha < 1$*, then the upper bound (21) is better than the upper bound (23). Thus our Theorem 4.10 provides an improvement of Rao Li's Theorem 4.12 for the case of non-bipartite graphs.*

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Note added in proof: After the completion of this paper, the article M. Liu, B. Liu, On sum of powers of the signless Laplacian eigenvalues of graphs, *Hacettepe J. Math. Stat.* **41** (2012) 527–536 came to our attention. In it some results identical to ours are communicated.

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