Terminal Wiener Index of Line Graphs

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Abstract

The terminal Wiener index of a graph is defined as the sum of the distances between the pendant vertices of a graph. In this paper we obtain results for the terminal Wiener index of line graphs.

1. Introduction

Let $G$ be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The degree of a vertex $v$ in $G$ is the number of edges incident to it and is denoted by $\text{deg}_G(v)$. If $\text{deg}_G(v) = 1$ then $v$ is called a pendant vertex. An edge $e = uv$ of a graph $G$ is called a pendant edge if $\text{deg}_G(u) = 1$ or $\text{deg}_G(v) = 1$. Two edges are said to be independent if they are not adjacent to each other. An edge $e$ is called a bridge if removal of $e$ from $G$ increases the number of components. The distance between the vertices $v_i$ and $v_j$ in $G$ is equal to the length of a shortest path joining them and is denoted by $d(v_i, v_j)(G)$. 
The Wiener index $W = W(G)$ of a graph $G$ is defined as the sum of the distances between all pairs of vertices of $G$, that is,

$$W = W(G) = \sum_{1 \leq i < j \leq n} d(v_i, v_j | G).$$

This molecular structure descriptor was conceived by Harold Wiener [21] in 1947. For details on its chemical applications and mathematical properties one may refer to [4,5,8,13,14,18,20] and the references cited therein.

If $G$ has $k$ pendent vertices labeled by $v_1, v_2, \ldots, v_k$, then its terminal distance matrix is the square matrix of order $k$ whose $(i, j)$-th entry is $d(v_i, v_j | G)$. Terminal distance matrices were used for modeling amino acid sequences of proteins and of the genetic code [10,16,17].

The terminal Wiener index $TW(G)$ of a connected graph $G$ is defined as the sum of the distances between all pairs of its pendent vertices.

Thus, if $V_T(G) = \{v_1, v_2, \ldots, v_k\}$ is the set of all pendent vertices of $G$, then

$$TW(G) = \sum_{\{v_i, v_j\} \subseteq V_T(G)} d(v_i, v_j | G) = \sum_{1 \leq i < j \leq k} d(v_i, v_j | G).$$

This distance–based molecular structure descriptor was recently put forward by Gutman, Furtula and Petrović [7]. The same idea was, independently, developed by Székely, Wang, and Wu [19].

If the graph $G$ has no pendent vertex or has only one pendent vertex, then $TW(G) = 0$. If $G$ has at least two pendent vertices then $TW(G) \geq 1$. More details on the terminal Wiener index are found in the review [6] and in the recent papers [3,9].

Of the numerous results on the Wiener index of line graphs are we mention here [1,2,11,12,15,22]. In this paper we offer a few results on the terminal Wiener index of line graphs.

2. Terminal Wiener index of line graphs

The line graph of $G$, denoted by $L(G)$ is the graph whose vertices are the edges of $G$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in $G$. 
Observations

1. Let $e = uv$ be an edge of $G$ such that $\deg_G(u) = 1$ and $\deg_G(v) = 2$. Then $\deg_{L(G)}(e) = \deg_G(u) + \deg_G(v) - 2 = 1$. Therefore $e$ is a pendent vertex of $L(G)$.

2. Let $e = uv$ be a pendent edge of $G$ such that $\deg_G(u) = 1$ and $\deg_G(v) = 2$. Then $v$ is a pendent vertex of $G'$, where $G'$ is the graph obtained by removing pendent vertices of $G$.

We define the set $D_2(G)$ as

$$D_2(G) = \{v \mid \deg_G(v) = 2 \text{ and one neighbour of } v \text{ is pendent} \}.$$

**Theorem 2.1.** Let $G$ be a connected graph with $n \geq 4$ vertices and let $D_2(G) = \{v_1, v_2, \ldots, v_q\}$. Then

$$TW(L(G)) = \sum_{1 \leq i < j \leq q} d(v_i, v_j|G) + \frac{q(q-1)}{2}. \quad (1)$$

**Proof.** Let $E_k = \{e_1, e_2, \ldots, e_k\}$ be the set of pendent edges of $G$ and $E_q = \{e_1, e_2, \ldots, e_q\}$ be the subset of $E_k$ where for each $e_i \in E_q$, the edge $e_i$ is incident to $v_i \in D_2(G)$, $i = 1, 2, \ldots, q$. Thus, if $e_i = uv \in E_q$ then $\deg_G(u) = 1$ and $\deg_G(v) = 2$ (or vice versa), $i = 1, 2, \ldots, q$.

Consider two edges $e_i = uv_i$ and $e_j = v_jw$ of $E_q$ where $\deg_G(u) = \deg_G(w) = 1$ and $\deg_G(v_i) = \deg_G(v_j) = 2$, $i = 1, 2, \ldots, q$.

Therefore $e_i$ and $e_j$ are the pendent vertices of $L(G)$ and $d(e_i, e_j|L(G)) = d(v_i, v_j|G) + 1$. Therefore

$$TW(L(G)) = \sum_{1 \leq i < j \leq q} d(e_i, e_j|L(G)) = \sum_{1 \leq i < j \leq q} [d(v_i, v_j|G) + 1]$$

$$= \sum_{1 \leq i < j \leq q} d(v_i, v_j|G) + \frac{q(q-1)}{2}. \quad \square$$

**Corollary 2.2.** $TW(L(G)) = 0$ if and only if the graph $G$ satisfies one of the following conditions. (i) $G$ has no edge $e = uv$ where $\deg_G(u) = 1$ and $\deg_G(v) = 2$. (ii) $G$ has only one edge $e = uv$ where $\deg_G(u) = 1$ and $\deg_G(v) = 2$. (iii) $G$ has no pendent vertices. (iv) $G$ has only one pendent vertex. (v) $G$ has no vertex of degree 2.
Theorem 2.3. Let \( G \) be a connected graph with \( n \geq 4 \) vertices and \( G' \) be the graph obtained from \( G \) by removing pendent vertices of \( G \). If \( p \) is the number of pendent vertices of \( G' \), then

\[
TW(L(G)) \leq TW(G') + \frac{p(p-1)}{2}.
\]

Equality holds if and only if (i) \( G = K_{1,n-1} \) or (ii) \( G \) has no bridge \( e \) such that one of the components of \( G - e \) is \( K_{1,s}, s \geq 2 \) and \( G \neq K_{1,n-1} \).

Proof. Let \( D_2(G) = \{v_1, v_2, \ldots, v_q\} \). Then the number of pendent vertices of \( G' \) is at least \( q \). Let \( p \) be the number of pendent vertices of \( G' \). Then \( p \geq q \). From Theorem 2.1,

\[
TW(L(G)) = \sum_{1 \leq i < j \leq q} d(v_i, v_j|G) + \frac{q(q-1)}{2} \leq \sum_{\{u,v\} \subseteq V_T(G')} d(u, v|G') + \frac{p(p-1)}{2}
\]

\[
= TW(G') + \frac{p(p-1)}{2}.
\]

For equality we consider the following cases:

Case 1. It is obvious that equality holds for \( G = K_{1,n-1} \).

Case 2. If \( G \neq K_{1,n-1} \) and if there is no edge \( e \) in \( G \) such that one of the components of \( G - e \) is \( K_{1,s}, s \geq 2 \), then \( q = p \). That is, the vertices of the set \( D_2(G) \) become pendent in \( G' \). Therefore

\[
\sum_{1 \leq i < j \leq p} d(v_i, v_j|G) = \sum_{\{v_i, v_j\} \subseteq V_T(G')} d(v_i, v_j|G').
\]

Substituting this in Eq. (1) we get

\[
TW(L(G)) = \sum_{\{v_i, v_j\} \subseteq V_T(G')} d(v_i, v_j|G') + \frac{p(p-1)}{2} = TW(G') + \frac{p(p-1)}{2}.
\]

Conversely, let \( G \) contain a bridge \( e \) such that one of the component of \( G - e \) is \( K_{1,s}, s \geq 2 \). Then \( p > q \) implying

\[
\sum_{1 \leq i < j \leq q} d(v_i, v_j|G) < \sum_{\{v_i, v_j\} \subseteq V_T(G')} d(v_i, v_j|G').
\]

From Eq. (1),

\[
TW(L(G)) = \sum_{1 \leq i < j \leq q} d(v_i, v_j|G) + \frac{q(q-1)}{2} < \sum_{\{v_i, v_j\} \subseteq V_T(G')} d(v_i, v_j|G') + \frac{p(p-1)}{2} \hspace{1cm} \text{by Eq. (2) and } q < p
\]

\[
= TW(G') + \frac{p(p-1)}{2}.
\]
which is a contradiction. This completes the proof. \(\square\)

**Corollary 2.4.** Let \(G\) be a connected graph with \(n \geq 4\) vertices and \(G'\) be the graph obtained from \(G\) by removing pendent vertices. Let \(p\) be the number of pendent vertices of \(G'\). If all pendent edges of \(G\) are mutually independent, then

\[
TW(L(G)) = TW(G') + \frac{p(p - 1)}{2}.
\]

**Proof.** Follows from the equality part of Theorem 2.3. \(\square\)

3. Terminal Wiener index of line graphs of some graphs

Let the vertices of \(G\) be \(v_1, v_2, \ldots, v_n\) then \(G^+\) is the graph obtained from \(G\) by adding \(n\) new vertices \(v'_1, v'_2, \ldots, v'_n\) and joining \(v'_i\) to \(v_i\) by an edge, \(i = 1, 2, \ldots, n\).

**Theorem 3.1.** If \(G\) has \(k\) pendent vertices, then \(L(G^+)\) also has \(k\) pendent vertices.

**Proof.** If \(G\) has \(n\) vertices of which \(k\) are pendent vertices, then \(G^+\) has \(n\) pendent edges of which \(k\) pendent edges say \(e_1, e_2, \ldots, e_k\) are such that for each \(e_i = uv, i = 0, 1, 2, \ldots, k\), \(deg_{G^+}(u) = 1\) and \(deg_{G^+}(v) = 2\).

Therefore \(deg_{L(G^+)}(e_i) = deg_{G^+}(u) + deg_{G^+}(v) - 2 = 1, i = 0, 1, 2, \ldots, k\). Therefore \(L(G^+)\) has \(k\) pendent vertices. \(\square\)

**Theorem 3.2.** Let \(G\) be a connected graph with \(k\) pendent vertices, then

\[
TW(L(G^+)) = TW(G) + \frac{k(k - 1)}{2}.
\]

**Proof.** If \(n\) is the number of vertices of \(G\), then \(G^+\) has \(n\) pendent edges of which \(k\) pendent edges say \(e_1, e_2, \ldots, e_k\) are such that for each \(e_i = uv, i = 0, 1, 2, \ldots, k\), \(deg_{G^+}(u) = 1\) and \(deg_{G^+}(v) = 2\), since \(G\) has \(k\) pendent vertices. Removing pendent vertices of \(G^+\) we get the graph \(G\). Knowing that the pendent edges of \(G^+\) are mutually independent, from Corollary 2.4,

\[
TW(L(G^+)) = TW(G) + \frac{k(k - 1)}{2}.
\]

Let \(K_n\) and \(S_n\) be the complete graph and star, respectively, on \(n\) vertices.
Corollary 3.3. $TW(L(S_n^+)) = TW(K_{n-1}^+)$.

Proof. Star $S_n$ has $n - 1$ pendant vertices. Therefore from Theorem 3.2,

$$
TW(L(S_n^+)) = TW(S_n) + \frac{(n-1)(n-2)}{2} = \frac{2(n-1)(n-2)}{2} + \frac{(n-1)(n-2)}{2} = \frac{3(n-1)(n-2)}{2} = TW(K_{n-1}^+). \quad \square
$$

Theorem 3.4. Let $G$ be a connected graph with $k$ pendant vertices and $H_t = L(H_{t-1}^+)$, $t = 1, 2, \ldots$, where $H_0 = G$ and $H_1 = L(G^+)$, then

$$
TW(H_t) = TW(G) + \frac{tk(k-1)}{2}.
$$

Proof. As $G$ has $k$ pendant vertices, from Theorem 3.1, the graph $H_t$ also has $k$ pendant vertices, $t = 1, 2, \ldots$. From Theorem 3.2,

$$
TW(H_1) = TW(L(G^+)) = TW(G) + \frac{k(k-1)}{2}.
$$

By induction, let

$$
TW(H_{t-1}) = TW(G) + \frac{(t-1)k(k-1)}{2}.
$$

Therefore

$$
TW(H_t) = TW(L(H_{t-1}^+)) = TW(H_{t-1}) + \frac{k(k-1)}{2} = TW(G) + \frac{(t-1)k(k-1)}{2} + \frac{k(k-1)}{2} = TW(G) + \frac{tk(k-1)}{2}. \quad \square
$$

The subdivision graph $S(G)$ is obtained from $G$ by inserting a new vertex of degree 2 on each edge of $G$. The graph $S_l(G)$ is obtained from $G$ by inserting $l$ new vertices of degree 2 on each edge of $G$. Thus $S_1(G) = S(G)$.

Theorem 3.5. Let $G$ be a connected graph with $k$ pendant vertices. Then

$$
TW(L(S_l(G))) = (l + 1)TW(G) - \frac{k(k-1)}{2}, \quad l \geq 1.
$$
**Proof.** If $G$ has $k$ pendent vertices then $S_l(G)$ also has $k$ pendent vertices. That is $S_l(G)$ has $k$ pendent edges.

If $u$ and $v$ are pendent vertices of $G$, then $u$ and $v$ are pendent vertices of $S_l(G)$. Let $u'$ and $v'$ be the subdivision vertices where $u'$ is adjacent to $u$ and $v'$ is adjacent to $v$ in $S_l(G)$, where $u$ and $v$ are pendent vertices of $G$. Let $(S_l(G))'$ be the graph obtained by removing all pendent vertices of $S_l(G)$. Therefore $u'$ and $v'$ are pendent vertices of $(S_l(G))'$ and $d(u', v')((S_l(G))') = (l + 1)d(u, v|G) - 2$. Since the pendent edges of $S_l(G)$ are mutually independent, from Corollary 2.4,

$$TW(L(S_l(G))) = TW((S_l(G))') + \frac{k(k - 1)}{2}$$

$$= \sum_{\{u', v': (S_l(G))'} d(u', v')((S_l(G))') + \frac{k(k - 1)}{2}$$

$$= \sum_{\{u, v|G\} V_T(G)} ((l + 1)d(u, v|G) - 2) + \frac{k(k - 1)}{2}$$

$$= (l + 1) \sum_{\{u, v|G\} V_T(G)} d(u, v|G) - \frac{2k(k - 1)}{2} + \frac{k(k - 1)}{2}$$

$$= (l + 1)TW(G) - \frac{k(k - 1)}{2}.$$  \qquad \Box$$

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**References**


