On Connective Eccentricity Index of Graphs[1]

Guihai Yu[a], Lihua Feng[b]

[a] *School of Mathematics, Shandong Institute of Business and Technology,
191 Binhaizhong Road, Yantai, Shandong, 264005, P. R. China*

[b] *Department of Mathematics, Central South University,
Railway Campus, Changsha, Hunan, P. R. China, 410075*
e-mail: `yuguihai@126.com, fenglh@163.com`

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Abstract

The connective eccentricity index of a graph G is defined as $\xi^{ce}(G) = \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)}$, where $\varepsilon(v)$ and $d(v)$ denote the eccentricity and the degree of the vertex v, respectively. In this paper we derive upper or lower bounds for the connective eccentricity index in terms of some graph invariants such as the radius, independence number, vertex connectivity, minimum degree, maximum degree etc. Moreover, we investigate the maximal and the minimal values of connective eccentricity index among all n-vertex graphs with fixed number of pendent vertices and characterize the extremal graphs. In addition, we study the cactus on n vertices with k cycles having the maximal connective eccentricity index.

1 Introduction

Throughout this paper, all graphs we considered are simple and connected. Let $G = (V(G), E(G))$ be a simple connected graph with n vertices and m edges. For a vertex $v \in V(G)$, $d_G(v)$ (or just $d(v)$ briefly) denotes the degree of v. $\delta(G)$, $\Delta(G)$ represent the minimum and maximum degree of G, respectively. For vertices $u, v \in V(G)$, the *distance* $d(u, v)$ is defined as the length of the shortest path between u and v in G. The *eccentricity* $\varepsilon(v)$ of a vertex v is the maximum distance from v to any other vertex. The *radius* $r(G)$

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of a graph is the minimum eccentricity of any vertex, while the *diameter* $D(G)$ of a graph is the maximum eccentricity of any vertex in the graph. Let K_n , S_n and P_n be a complete graph, a star and a path on n vertices, respectively.

In organic chemistry, topological indices have been found to be useful in chemical documentation, isomer discrimination, structure-property relationships, structure-activity (SAR) relationships and pharmaceutical drug design. These indices include Wiener index [32–34], Balaban's index [2–5], Hosoya index [17,18], Randić index [25] and so on. In recent years, some indices have been derived related to eccentricity such as eccentric connectivity index [14, 23, 26], eccentric distance sum [15], augmented and super augmented eccentric connectivity indices [1, 12, 29], adjacent eccentric distance sum index [27, 28].

The Wiener index is one of the most used topological indices with high correlation with many physical and chemical indices of molecular compounds (for a recent survey on Wiener index see [8]). The *Wiener index* of a graph G, denoted by $W(G)$, is defined as the sum of the distances between all pairs of vertices in graph G, that is

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v) = \frac{1}{2} \sum_{v \in V(G)} D(v) .$$

The parameter $DD(G)$ is called the *degree distance* of G and it was introduced by Dobrynin and Kochetova [9] and Gutman [16] as a graph-theoretical descriptor for characterizing alkanes; it can be considered as a weighted version of the Wiener index

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} (d(u) + d(v)) d(u,v) = \sum_{v \in V(G)} d(v) \cdot D(v) .$$

When G is a tree on n vertices, it has been demonstrated that Wiener index and degree distance are closely related by (see [19, 21]) $DD(G) = 4W(G) - n(n-1)$.

The *total eccentricity* of the graph G [6, 31], denoted by $\zeta(G)$, is defined as the sum of eccentricities of all vertices of graph G, i. e.,

$$\zeta(G) = \sum_{v \in V(G)} \varepsilon(v).$$

The *eccentric connectivity index* of G, denoted by $\xi^c(G)$, is defined as [26]

$$\xi^c(G) = \sum_{v \in V(G)} \varepsilon(v) d(v) .$$

The *eccentric distance sum* (EDS) of G is defined as [15]

$$\xi^d(G) = \sum_{v \in V(G)} \varepsilon(v) D(v) .$$

More recently, the mathematical properties of eccentric distance sum have been investigated. In [22,35], the authors studied the eccentric distance sum of trees, unicyclic graph with given girth and established some lower and upper bounds for the eccentric distance sum in terms of some graph invariants.

In [27], Sardana and Madan introduced a novel topological descriptor–*adjacent eccentric distance sum index*, which is defined to be

$$\xi^{sv}(G) = \sum_{v \in V(G)} \frac{\varepsilon(v) \, D(v)}{d(v)} .$$

In 2000, Gupta, Singh and Madan [13] introduced a novel, adjacency-cum-path length based, topological descriptor termed the *connective eccentricity index*. In order to explore the potential of connective eccentricity index in predicting biological activity, authors used nonpeptide N-benzylimidazole derivatives to investigate the predictability of the connective eccentricity index with respect to antihypertensive activity. They showed that results obtained using connective eccentricity index were better than the corresponding values obtained using Balaban's mean square distance index and the accuracy of prediction was found to be about 80% in the active range [13].

The *connective eccentricity index* (CEI) of a graph G was defined as

$$\xi^{ce}(G) = \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} .$$

As a newly introduced distance-based molecular descriptor, there are three groups of closed related problems which have attracted the attention of researchers naturally:

1. How $\xi^{ce}(G)$ depends on the structure of the graph G?

2. Given a set of molecular graphs $\mathcal{G}$, find upper and lower bounds for $\xi^{ce}(G)$ of graphs in $\mathcal{G}$ and characterize the extremal graphs which attain the maximal and minimal connective eccentricity index.

3. Compare the values of $\xi^{ce}(G)$ with other molecular topological indices such as Wiener index and its generalizations.

In view of these natural problems, in this paper, we present some mathematical results for this new molecular descriptor. This paper is organized as follows. In Section 2, we give some bounds for the CEI in terms of some graph invariants such as the radius, independence number, vertex connectivity, minimum degree, and the number of vertices with eccentricity 1. In Section 3, we investigate the CEI among all connected graphs on n vertices with fixed number of pendent vertices and characterize the graphs with the maximal, minimal CEI, respectively. In Section 4, we determine the maximal CEI among all n-vertex graphs with fixed number of cut edges. In Section 5, we consider the maximal CEI among the cacti on n vertices with k cycles and obtain the extremal graph.

2   Bounds for connective eccentricity index

Recall that the first and second Zagreb indices are defined as [11]

$$M_1(G) = \sum_{u \in V(G)} [d(u)]^2 \quad , \quad M_2(G) = \sum_{uv \in E(G)} d(u)\, d(v)$$

whereas the first and second Zagreb coindices are [7, 10, 20]

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d(u) + d(v)] \quad , \quad \overline{M}_2(G) = \sum_{uv \notin E(G)} d(u)\, d(v) \ .$$

For more details on vertex–degree–based topological indices see [11] and the references cited therein.

Theorem 2.1. *Let G be a connected simple graph. Then we have*

1. *$\xi^{ce}(G) \leq 2m \leq n(n-1)$, with equality if and only if $G \cong K_n$.*

2. *$\xi^{ce}(G) \geq \frac{n^2\delta}{\zeta(G)}$, with equality if and only if $G \cong K_n$, or $G \cong K_n - \frac{n}{2}e$ for even n.*

3. *$\xi^{ce}(G) \leq \xi^c(G)$, with equality if and only if $G \cong K_n$.*

4. *$\xi^{ce}(G) \geq \frac{DD(G)+2M_2+2\overline{M}_2}{\xi^d(G)}$, with equality if and only if $G \cong K_n$.*

5. *$\xi^{ce}(G) \geq \frac{2W(G)+n(n-1)^2}{\xi^{sv}(G)}$, with equality if and only if $G \cong K_n$.*

6. *$\xi^{ce}(G) \geq \frac{4m^2}{\xi^c(G)}$, with equality if and only if $\varepsilon(v)$ is a constant for all v.*

7. *$\xi^{ce}(G) \geq \frac{M_1}{\zeta(G)}$.*

Proof. (1). Since $\varepsilon(v) \geq 1$ for any $v \in V(G)$, then

$$\xi^{ce}(G) = \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} \leq \sum_{v \in V(G)} d(v) = 2m$$

the equality holds in the above inequality if and only if $\varepsilon(v) = 1$ for any $v \in V(G)$, i. e, G is a complete graph.

(2). From the harmonic-arithmetic inequality, we have

$$\xi^{ce}(G) = \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} \geq \delta \sum_{v \in V(G)} \frac{1}{\varepsilon(v)} \geq \frac{n^2 \delta}{\sum_{v \in V(G)} \varepsilon(v)} = \frac{n^2 \delta}{\zeta(G)} .$$

The first equality holds if and only if $d(v) = \delta$ for all v and the second equality holds if and only if $\varepsilon(v)$ is a constant.

Let $n_i(v)$ be the number of vertices at distance i from the vertex v. It is evident that $d(v) \leq n - \varepsilon(v)$ with equality if and only if $\varepsilon(v) = 1$ and $deg(v) = n - 1$, or $\varepsilon(v) \geq 2$ and $n_2(v) = n_3(v) = \cdots = n_{\varepsilon(v)}(v) = 1$.

If $\varepsilon(v) = 1$ and $deg(v) = n - 1$ for any $v \in V(G)$, then $G \cong K_n$.

If $\varepsilon(v) = 2$ and $deg(v) = n - 2$ for any $v \in V(G)$, then $G \cong K_n - \frac{n}{2}e$ for even n.

If $\varepsilon(v) = 2$ and $deg(v) \leq n-3$ for some vertex $v \in V(G)$, then there is no such regular graphs.

If there exists some vertex $u \in V(G)$ such that $\varepsilon(u) \geq 3$. Then the diameter of G is 3. In fact, assume to the contrary that there exists an induced path P with length $D(G) > 3$ in G. Then there exists some vertex $u_i \in V(P)$ such that $\varepsilon(u_i) \geq 2$ and $n_2(u_i) \geq 2$. This contradicts that $n_2(u_i) = 1$. Since $n_2(u) = n_3(u) = \cdots = n_{\varepsilon(u)}(u) = 1$, then $G \cong P_4$. This contradicts that G is regular.

The converse is easy to check.

(3). Evidently, $\frac{1}{\varepsilon(v)} \leq \varepsilon(v)$, with equality if and only if $\varepsilon(v) = 1$. Therefore

$$\xi^{ce}(G) = \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} \leq \sum_{v \in V(G)} d(v) \varepsilon(v) = \xi^c(G) .$$

(4). Since $D(v) \geq n - 1 \geq d(v)$, with equality if and only if $G \cong K_n$, we have

$$
\begin{aligned}
\sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} \sum_{v \in V(G)} D(v)\varepsilon(v) \;\geq\; & \left(\sum_{v \in V(G)} \sqrt{d(v) D(v)} \right)^2 \quad \text{by Cauchy inequality} \\
=\; & \sum_{v \in V(G)} d(v) D(v) + 2 \sum_{u,v \in V(G)} \sqrt{d(v) D(v)\, d(u)\, D(u)} \\
\geq\; & DD(G) + 2 \sum_{u,v \in V(G)} d(u)\, d(v) \\
=\; & DD(G) + 2M_2 + 2\overline{M}_2 .
\end{aligned}
$$

The first equality holds if and only if $\frac{d(v)}{\varepsilon(v)} = c\,D(v)\,\varepsilon(v)$ for some constant c and any $v \in V(G)$. The second equality holds if and only if $G \cong K_n$. Therefore

$$\xi^{ce}(G) \geq \frac{DD(G) + 2M_2 + 2\overline{M}_2}{\xi^d(G)}$$

with equality if and only if $G \cong K_n$.

(5). Similar as above,

$$\sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} \sum_{v \in V(G)} \frac{\varepsilon(v)\,D(v)}{d(v)} \geq \left(\sum_{v \in V(G)} \sqrt{D(v)} \right)^2$$
$$= \sum_{v \in V(G)} D(v) + 2 \sum_{u,v \in V(G)} \sqrt{D(v)\,D(u)}$$
$$\geq 2W(G) + 2(n-1)\binom{n}{2}.$$

Therefore

$$\xi^{ce}(G) \geq \frac{2W(G) + n(n-1)^2}{\xi^{sv}(G)}$$

with equality if and only if $G \cong K_n$.

(6). From the Cauchy inequality, one has

$$\sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} \sum_{v \in V(G)} d(v)\,\varepsilon(v) \geq \left(\sum_{v \in V(G)} d(v) \right)^2.$$

Therefore

$$\xi^{ce}(G) \geq \frac{4m^2}{\xi^c(G)}$$

with equality if and only if $\varepsilon(v)$ is a constant for all v.

(7). It is proved in [30] that $\left(\sum_{v \in V(G)} [d(v)]^{1/p} \right)^p \geq \sum_{v \in V(G)} [d(v)]^p$. Therefore

$$\sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} \sum_{v \in V(G)} \varepsilon(v) \geq \left(\sum_{v \in V(G)} \sqrt{d(v)} \right)^2 \geq M_1$$

Therefore

$$\xi^{ce}(G) \geq \frac{M_1}{\zeta(G)}.$$

$\square$

Let $K_n - ke$ be the graph obtained from K_n by deleting k independent edges for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Theorem 2.2. *Let G be a connected graph on n vertices. Let n_0 be the number of vertices with eccentricity 1 in graph G. Then*

$$\xi^{ce}(G) \le \frac{n(n-2)}{2} + \frac{n}{2}n_0$$

with equality if and only if $G \cong K_{n_0} \vee (K_{n-n_0} - \frac{n-n_0}{2}e)$, where $n - n_0$ is even.

Proof. Let $S = \{v_1, v_2, \cdots, v_{n_0}\}$ be the set of vertices with eccentricity 1. It follows that $\varepsilon(u) \ge 2$, $deg(u) \le n - 2$ for any $u \in V(G) \setminus S$. By the definition of CEI, we have

$$
\begin{aligned}
\xi^{ce}(G) &= \sum_{i=1}^{n_0} \frac{d(v_i)}{\varepsilon(v_i)} + \sum_{u \in V(G) \setminus S} \frac{d(u)}{\varepsilon(u)} \\
&\le n_0(n-1) + \sum_{u \in V(G) \setminus S} \frac{n-2}{2} \\
&= \frac{n(n-2)}{2} + \frac{n}{2}n_0 \ .
\end{aligned}
$$

The above equality holds if and only if $\varepsilon(v) = 2$ and $deg(v) = n - 2$ for any vertices $v \in V(G) \setminus S$, i. e., $G \cong K_{n_0} \vee (K_{n-n_0} - \frac{n-n_0}{2}e)$, $n - n_0$ is even. □

Theorem 2.3. *Let G be a connected graph on n vertices with m edges. Let*

$$a = \left\lfloor \frac{2n - 1 - \sqrt{(2n-1)^2 - 8m}}{2} \right\rfloor$$

be the largest integer satisfying that $x^2 + (1 - 2n)x + 2m \ge 0$. Then

$$\xi^{ce}(G) \le \frac{n(n-2)}{2} + \frac{n}{2}a$$

with equality if and only if $G \cong K_a \vee (K_{n-a} - \frac{n-a}{2}e)$, where $n - a$ is even.

Proof. By Theorem 2.2, we have $\xi^{ce}(G) \le \frac{n(n-2)}{2} + \frac{n}{2}n_0$, where n_0 is the number of vertices with degree $n - 1$. The equality holds if and only if all vertices of degree less than $n - 1$ have eccentricity 2. Since $2m = \sum_{v \in V(G)} d(v) \ge n_0(n-1) + n_0(n-n_0)$, so this implies that $n_0 \le a$. Therefore

$$\xi^{ce}(G) \le \frac{n(n-2)}{2} + \frac{n}{2}n_0 \le \frac{n(n-2)}{2} + \frac{n}{2}a$$

with equality if and only if G has exactly a vertices with eccentricity one and all other vertices have eccentricity two. □

Theorem 2.4. *Let G be a connected graph on n vertices with radius r. Then*

$$\xi^{ce}(G) \leq \frac{n^2}{r} - n$$

with equality if and only if $G \cong K_n$, or $G \cong K_n - \frac{n}{2}e$ for even n.

Proof. It is evident that $d(v) \leq n - \varepsilon(v)$ for any $v \in V(G)$. Then

$$
\begin{aligned}
\xi^{ce}(G) &= \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} \\
&\leq \sum_{v \in V(G)} \frac{n - \varepsilon(v)}{\varepsilon(v)} \qquad (1) \\
&\leq n \sum_{v \in V(G)} \frac{1}{r} - n \qquad (2) \\
&= \frac{n^2}{r} - n .
\end{aligned}
$$

If the equality holds in the above proof, then both (1) and (2) must be equalities. The equality in (1) holds if and only if $\varepsilon(v) = n - d(v)$ for any vertex v. The equality in (2) holds if and only if $\varepsilon(v) = r$ for any vertex v. So if the equalities hold in the above inequalities, then G is regular with degree $n - r$ and $\varepsilon(v) = r$ for any vertex v. The remaining is similar to (1) of Theorem 2.1. $\qquad \square$

¿From the definition of CEI, one has

Lemma 2.5. *Let G be a non-complete graph. Then $\xi^{ce}(G) < \xi^{ce}(G + e)$ for $e \in E(\overline{G})$.*

We adopt the traditional symbols in graph theory. We use $\alpha(G), \gamma(G), \kappa(G), \kappa'(G)$ to denote the independence number, the vertex covering number, the vertex connectivity, the edge connectivity, respectively. It is well known that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ and $\alpha(G) + \gamma(G) = n$.

Let $G^\star = K_{n-\alpha} \vee \overline{K_\alpha}$. Let e_1 be an edge in $G^\star$ joining two vertices in $K_{n-\alpha}$ and $\overline{K_\alpha}$. Let e_2 be an edge incident two vertices in $\overline{K_\alpha}$.

Theorem 2.6. *Let G be a connected graph on $n \, (\geq 6)$ vertices with independence number α. Then*

1. $\xi^{ce}(G) \leq (n - \alpha)(n - 1 + \frac{\alpha}{2})$, *with equality if and only if $G \cong G^\star$;*

2. *if $G \in \mathcal{G}_{n,\alpha} \backslash \{G^\star\}$, then $\xi^{ce}(G) \leq \frac{1}{2}\big(2n^2 - (3 + \alpha)n - \alpha^2 + \alpha + 1\big)$, with equality if and only if $G \cong G^\star - e_1$.*

Proof. The result (1) follows from Lemma 2.5. We next consider result (2). By deleting an edge in $G^\star$, we get two graphs: $G^\star - e_1$, $G^\star - e_2$. From Lemma 2.5, we need only compare $\xi^{sv}(G^\star - e_1)$ with $\xi^{ce}(G^\star - e_2)$.

By direct calculation we get

$$\xi^{ce}(G^\star - e_1) = \frac{1}{2}\big(2n^2 - (3+\alpha)n - \alpha^2 + \alpha + 1\big)$$

$$\xi^{sv}(G^\star - e_2) = \frac{1}{2}\big(2n^2 - (4+\alpha)n - \alpha^2 + 2\alpha\big) \ .$$

It follows that

$$\xi^{sv}(G^\star - e_1) - \xi^{sv}(G^\star - e_2) = \frac{1}{2}(n - \alpha + 1) > 0$$

which implies the result. $\qquad\qquad\square$

Corollary 2.7. *Let G be a connected graph on n vertices with covering number γ. Then*

$$\xi^{ce}(G) \le \gamma\left(n - 1 + \frac{n-\gamma}{2}\right)$$

with equality holding if and only if $G \cong K_\gamma \vee \overline{K_{n-\gamma}}$.

Ilić et al. [22] investigated the EDS of graphs with given vertex connectivity. By modifying their method, we have

Theorem 2.8. *Let G be a connected graph on n vertices with vertex connectivity κ. Then*

$$\xi^{ce}(G) \le \frac{1}{2}\big(n^2 + (\kappa - 3)n + \kappa + 2\big)$$

with equality holding if and only if $G \cong K_\kappa \vee (K_1 \cup K_{n-\kappa-1})$.

Theorem 2.9. *Let G be a connected graph on n vertices with edge connectivity κ'. Then*

$$\xi^{ce}(G) \le \frac{1}{2}\big(n^2 + (\kappa' - 3)n + \kappa' + 2\big)$$

with equality holding if and only if $G \cong K_{\kappa'} \vee (K_1 \cup K_{n-\kappa'-1})$.

Proof. Let $f(x) = \frac{1}{2}\big(n^2 + (x-3)n + x + 2\big)$. It is easy to verify that $f(\kappa') \ge f(\kappa)$ due to the fact that $\kappa \le \kappa'$. By Theorem 2.8, we have $f(\kappa') \ge f(\kappa) \ge \xi^{ce}(G)$. If $\xi^{ce}(G) = f(\kappa')$, then $\kappa' = \kappa$. This implies the result. $\qquad\qquad\square$

Theorem 2.10. *Let G be a connected graph on n vertices with minimum degree δ. Then*

$$\frac{n\delta}{n-\delta} \le \xi^{ce}(G) \le \frac{1}{2}\big(n^2 + (\delta - 3)n + \delta + 2\big)$$

with the right–hand side equality holding if and only if $G \cong K_\delta \vee (K_1 \cup K_{n-\delta-1})$ and the left–hand side equality holding if and only if $G \cong K_n$, or $G \cong K_n - \frac{n}{2}e$ for even n.

Proof. Similarly to the proof of Theorem 2.9, we can obtain the right–hand side inequality. According to the discussion in Theorem 2.4

$$\xi^{ce}(G) \geq \sum_{v \in V(G)} \frac{d(v)}{n - d(v)} \geq \sum_{v \in V(G)} \frac{\delta}{n - \delta} = \frac{n\delta}{n - \delta} \ .$$

The left–hand side inequality follows. □

3 Connected graphs with k pendent vertices

Let K_n^k be the graph obtained from K_{n-k} by attaching k pendent edges to one vertex of K_{n-k}. For nonnegative integers p, q, let $\mathcal{T}_n^k$ be the set of trees obtained by identifying two end-vertices of P_{n-k} with the centers of stars S_{p+1}, S_{q+1} $(p + q = k)$, respectively. It is evident that any two trees in $\mathcal{T}_n^k$ have the same CEI. Let $H(n)$ be the n-th harmonic number, i. e, $H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Theorem 3.1. *Let G be a connected graph on n vertices with k pendent vertices. Then*

$$C \leq \xi^{ce}(G) \leq \frac{1}{2}(n^2 - 2kn + k^2 + 3k - 1)$$

where

$$C = \begin{cases} \frac{(2k+2)n - 2k^2 - k + 2}{(n-k)(n-k+1)} + 4H(n - k - 1) - 4H(\frac{n-k}{2}) & \text{if } n - k \text{ is even} \\[2mm] \frac{(2k+6)n - 2k^2 - 5k + 2}{(n-k)(n-k+1)} + 4H(n - k - 1) - 4H(\frac{n-k+1}{2}) & \text{if } n - k \text{ is odd.} \end{cases}$$

The left–hand side equality holds if and only if $G \in \mathcal{T}_n^k$ and the right–hand side equality holds if and only if $G \cong K_n^k$.

Proof. Let G_{max} be the graph with the maximal CEI among all connected graphs on n vertices with k pendent vertices. Assume that $\{v_1, v_2, \cdots, v_k\}$ be the set of pendent vertices in G_{max}. By Lemma 2.5, the subgraph G' induced by $V(G_{max})\backslash\{v_1, \cdots, v_k\}$ is a complete graph. All k pendent edges are attached at some vertices of G'. Next we would prove that $G_{max} \cong K_n^k$.

Assume that there exist only two vertices u_i, u_j of G' such that $d(u_i) \geq d(u_j) > n - k - 1$. Let G_1 be the new graph obtained from G_{max} by removing all pendent edges attached to u_j and attaching them to u_i. It follows that $\varepsilon_{G_{max}}(v) = \varepsilon_{G_1}(v)$ for $v \in V(G')\backslash\{u_i\}$ and $\varepsilon_{G_{max}}(u_i) > \varepsilon_{G_1}(u_i)$. The degrees of vertices in $V(G_{max})\backslash\{u_i, u_j\}$ do not change, but the degree of u_i increases by $d_{G_{max}}(u_j) - n + k + 1$ and the one of u_j

decreases by $d_{G_{max}}(u_j) - n - k + 1$. The eccentricities of all pendent vertices decrease by 1. So we have

$$
\begin{aligned}
\xi^{ce}(G_{max}) - \xi^{ce}(G_1) \quad &< \quad \frac{d_{G_{max}}(u_i)}{\varepsilon_{G_{max}}(u_i)} + \frac{d_{G_{max}}(u_j)}{\varepsilon_{G_{max}}(u_j)} - \frac{d_{G_1}(u_i)}{\varepsilon_{G_1}(u_i)} - \frac{d_{G_1}(u_j)}{\varepsilon_{G_1}(u_j)} \\[2mm]
&< \quad \frac{d_{G_{max}}(u_i)}{\varepsilon_{G_1}(u_i)} - \frac{d_{G_1}(u_i)}{\varepsilon_{G_1}(u_i)} + \frac{d_{G_{max}}(u_j)}{\varepsilon_{G_{max}}(u_j)} - \frac{d_{G_1}(u_j)}{\varepsilon_{G_{max}}(u_j)} \\[2mm]
&= \quad \frac{-d_{G_{max}(u_j)} + n - k - 1}{\varepsilon_{G_1}(u_i)} + \frac{d_{G_{max}(u_j)} - n + k + 1}{\varepsilon_{G_{max}}(u_j)} \\[2mm]
&< \quad \frac{-d_{G_{max}(u_j)} + n - k - 1}{\varepsilon_{G_{max}}(u_j)} + \frac{d_{G_{max}(u_j)} - n + k + 1}{\varepsilon_{G_{max}}(u_j)} \\[2mm]
&= \quad 0 \, .
\end{aligned}
$$

This contradicts to the fact that G_{max} has the maximal CEI.

Assume that there exist at least three vertices in G' such that their degrees are at least $n - k$. We can repeatedly apply the above procedure and get a new graph with larger CEI, this yields a contraction.

Therefore, $G_{max} \cong K_n^k$.

Let G_{min} be the graph with the minimal CEI among all connected graph on n vertices with k vertices. By Lemma 2.5, G_{min} is a tree. Next we intend to prove that $G_{min} \in \mathcal{T}_n^k$.

In the following we deal with two claims.

Claim 1. G_{min} must be a caterpillar.

Proof of Claim 1. Let $P_{t+1} = v_0 v_1 \cdots v_t$ be the longest path in G_{min}. If $t = 3$, then G_{min} must be a caterpillar. so we only consider the case $t \geq 4$.

Assume that $i \in \{2, 3, \cdots, \lceil \frac{k+1}{2} \rceil\}$ is the smallest integer such that there exists a vertex u different to v_{i-1} and v_{i+1}, which is adjacent to v_i and $|N_{G_{min}}(u)| \geq 2$. Let $N_{G_{min}}(u) = \{v_i, w_1, w_2, \cdots, w_s\}$. Let T_1 be the subtree of $G_{min} - v_i v_{i+1} - v_i u$ containing v_i. Let T_2 (resp. T_3) be the subtree of $G_{min} - v_i v_{i+1} - v_i u$ containing v_{i+1} (resp. u).

Let $T' = G_{min} - \{uw_1, uw_2, \cdots, uw_s\} + \{v_t w_1, v_t w_2, \cdots, v_t w_s\}$. It is evident that T' has k pendent vertices. It can be checked that

- For any $v \in V(T_1)$, $\varepsilon_{G_{min}}(v) < \varepsilon_{T'}(v)$ and $d_{G_{min}}(v) = d_{T'}(v)$;

- For any $v \in V(T_3 - u)$, $\varepsilon_{G_{min}}(v) < \varepsilon_{T'}(v)$ and $d_{G_{min}}(v) = d_{T'}(v)$;

- $\varepsilon_{G_{min}}(u) < \varepsilon_{T'}(u)$, $d_{G_{min}}(u) = s + 1$ and $d_{T'}(u) = 1$;

- For any $v \in V(T_2 - v_t)$, $\varepsilon_{G_{min}}(v) \leq \varepsilon_{T'}(v)$ and $d_{G_{min}}(v) = d_{T'}(v)$;

- $\varepsilon_{G_{min}}(v_t) = \varepsilon_{T'}(v_t)$, $d_{G_{min}}(v_t) = 1$ and $d_{T'}(v) = s + 1$.

Bearing in mind that $\varepsilon_{T'}(u) \leq \varepsilon_{T'}(v_t)$, it follows that

$$
\begin{aligned}
\xi^{ce}(G_{min}) - \xi^{ce}(T') \quad &> \quad \frac{d_{G_{min}}(u)}{\varepsilon_{G_{min}}(u)} - \frac{d_{T'}(u)}{\varepsilon_{T'}(u)} + \frac{d_{G_{min}}(v_t)}{\varepsilon_{G_{min}}(v_t)} - \frac{d_{T'}(v_t)}{\varepsilon_{T'}(v_t)} \\
&= \quad \frac{s+1}{\varepsilon_{G_{min}}(u)} - \frac{1}{\varepsilon_{T'}(u)} + \frac{1}{\varepsilon_{G_{min}}(v_t)} - \frac{s+1}{\varepsilon_{T'}(v_t)} \\
&> \quad s\left(\frac{1}{\varepsilon_{T'}(u)} - \frac{1}{\varepsilon_{T'}(v_t)}\right) \geq 0 \ .
\end{aligned}
$$

This leads to a contradiction.

Assume that $j \in \{\lceil \frac{k+1}{2} \rceil, \cdots, t-2\}$ is the largest integer such that there exists a vertex v different from v_{j-1} and v_{j+1}, which is adjacent to v_j and $|N_{G_{min}}(v)| \geq 2$. Let $N_{G_{min}}(v) = \{v_j, w_1', w_2', \cdots, w_s'\}$. Similarly to the above discussion, we construct a new graph

$$
T'' = G_{min} - \{vw_1', vw_2', \cdots, vw_s'\} + \{v_0w_1', v_0w_2', \cdots, v_0w_s'\}
$$

and get $\xi^{ce}(G_{min}) > \xi^{ce}(T'')$, which is a contradiction.

Therefore, G_{min} must be a caterpillar.

Claim 2. $G_{min} \in \mathcal{T}_n^k$.

By Claim 1, G_{min} is a caterpillar with k pendent vertices, so G_{min} has the diameter $n - k + 1$. Let $P_{n-k+2} = v_0v_1 \cdots v_{n-k+1}$ be the longest path in G_{min}.

Assume that $i \in \{2, 3, \cdots, \lceil \frac{k+1}{2} \rceil\}$ is the smallest integer such that $d(v_i) > 2$. Let $N_{G_{min}}(v_i) = \{v_{i-1}, v_{i+1}, w_1, \cdots, w_s\}$. We construct a new tree

$$
T_1 = G_{min} - \{v_iw_1, v_iw_2, \cdots, v_iw_s\} + \{v_1w_1, v_1w_2, \cdots, v_1w_s\} \ .
$$

The eccentricities of these pendent vertices are increased, while the eccentricities of other vertices remain the same. Moreover, the degree of v_i decreases by s, while the degree of v_1 increases by s. The degrees of other vertices remain the same. Therefore it follows that

$$
\begin{aligned}
\xi^{ce}(G_{min}) - \xi^{ce}(T_1) \quad &> \quad \frac{d_{G_{min}}(v_1)}{\varepsilon_{G_{min}}(v_1)} + \frac{d_{G_{min}}(v_i)}{\varepsilon_{G_{min}}(v_i)} - \frac{d_{T_1}(v_1)}{\varepsilon_{T_1}(v_1)} - \frac{d_{T_1}(v_i)}{\varepsilon_{T_1}(v_i)} \\
&= \quad -\frac{s}{\varepsilon_{G_{min}}(v_1)} + \frac{s}{\varepsilon_{G_{min}}(v_i)} \\
&> \quad 0 \quad \left(\text{since } \varepsilon_{G_{min}}(v_i) < \varepsilon_{G_{min}}(v_1)\right) \ .
\end{aligned}
$$

This contradicts to the fact that G_{min} has the minimal CEI.

Assume that $j \in \{\lceil \frac{k+1}{2} \rceil, \cdots, n-k-1\}$ is the largest integer such that $d(v_j) > 2$. Let $N_{G_{min}}(v_j) = \{v_{j-1}, v_{j+1}, w_1', \cdots, w_s'\}$. Let

$$T_2 = G_{min} - v_j w_1' - v_j w_2' \cdots - v_j w_s' + v_{n-k} w_1' + v_{n-k} w_2' + \cdots + v_{n-k} w_s' .$$

As above, we get $\xi^{ce}(G_{min}) > \xi^{ce}(T_2)$, which is a contradiction. Therefore, $G_{min} \in \mathcal{T}_n^k$. □

4 Connected graphs with k cut edges

Lemma 4.1. *Let H_1 and H_2 be two disjoint connected graphs of order at least 2 with $u \in V(H_1)$, $v \in V(H_2)$. Let G_1 be the graph obtained from $H_1 \cup H_2$ by adding an edge uv. Let G_2 be the graph obtained from $H_1 \cup H_2$ by identifying u and v (to a new vertex, say, u) and adding a pendent edge, say uv without confusion. Then $\xi^{ce}(G_1) < \xi^{ce}(G_2)$.*

Proof. It is evident that $\varepsilon_{G_1}(x) \geq \varepsilon_{G_2}(x)$ and $d_{G_1}(x) = d_{G_2}(x)$ for any $x \in \big(V(H_1)\backslash\{u\}\big) \cup \big(V(H_2)\backslash\{v\}\big)$.

For u and v, we have

- $\varepsilon_{G_1}(u) = \max\{\varepsilon_{H_1}(u), \varepsilon_{H_2}(v) + 1\}$, $d_{G_1}(u) = 1 + d_{H_1}(u)$;

- $\varepsilon_{G_2}(u) = \max\{\varepsilon_{H_1}(u), \varepsilon_{H_2}(v)\}$, $d_{G_2}(u) = 1 + d_{H_1}(u) + d_{H_2}(v)$;

- $\varepsilon_{G_1}(v) = \max\{\varepsilon_{H_2}(v), \varepsilon_{H_1}(u) + 1\}$, $d_{G_1}(v) = 1 + d_{H_2}(v)$;

- $\varepsilon_{G_2}(v) = \max\{\varepsilon_{H_1}(u) + 1, \varepsilon_{H_2}(v) + 1\}$, $d_{G_2}(v) = 1$.

Case 1. $\varepsilon_{H_1}(u) \geq 1 + \varepsilon_{H_2}(v)$.

In this case we have $\varepsilon_{G_1}(u) = \varepsilon_{H_1}(u)$, $\varepsilon_{G_2}(u) = \varepsilon_{H_1}(u)$, $\varepsilon_{G_1}(v) = 1 + \varepsilon_{H_1}(u)$, $\varepsilon_{G_2}(v) = 1 + \varepsilon_{H_1}(u)$.

Therefore, it follows that

$$
\begin{aligned}
\xi^{ce}(G_1) - \xi^{ce}(G_2) &\leq \frac{d_{G_1}(u)}{\varepsilon_{H_1}(u)} - \frac{d_{G_2}(u)}{\varepsilon_{H_1}(u)} + \frac{d_{G_1}(v)}{\varepsilon_{H_1}(u) + 1} - \frac{d_{G_2}(v)}{\varepsilon_{H_1}(u) + 1} \\
&= -\frac{d_{H_2}(v)}{\varepsilon_{H_1}(u)} + \frac{d_{H_2}(v)}{\varepsilon_{H_1}(u) + 1} < 0 .
\end{aligned}
$$

Case 2. $\varepsilon_{H_1}(u) \leq \varepsilon_{H_2}(v)$.

Subcase 2.1 $\varepsilon_{H_1}(u) = \varepsilon_{H_2}(v)$.

In this subcase, $\varepsilon_{G_1}(u) = \varepsilon_{H_1}(u) + 1$, $\varepsilon_{G_2}(u) = \varepsilon_{H_1}(u)$, $\varepsilon_{G_1}(v) = 1 + \varepsilon_{H_1}(u)$, $\varepsilon_{G_2}(v) = 1 + \varepsilon_{H_1}(u)$.

It follows that

$$
\begin{aligned}
\xi^{ce}(G_1) - \xi^{ce}(G_2) &\leq \frac{1 + d_{H_1}(u)}{1 + \varepsilon_{H_1}(u)} - \frac{1 + d_{H_1}(u) + d_{H_2}(v)}{\varepsilon_{H_1}(u)} + \frac{1 + d_{H_2}(v)}{1 + \varepsilon_{H_1}(u)} - \frac{1}{1 + \varepsilon_{H_1}(u)} \\
&= -\frac{1 + d_{H_1}(u) + d_{H_2}(v)}{\varepsilon_{H_1}(u)} + \frac{1 + d_{H_1}(u) + d_{H_2}(v)}{\varepsilon_{H_1}(u) + 1} < 0 .
\end{aligned}
$$

Subcase 2.2 $\varepsilon_{H_1}(u) < \varepsilon_{H_2}(v)$.

In this subcase, $\varepsilon_{G_1}(u) = \varepsilon_{H_2}(v) + 1$, $\varepsilon_{G_2}(u) = \varepsilon_{H_2}(v)$, $\varepsilon_{G_1}(v) = \varepsilon_{H_2}(v)$, $\varepsilon_{G_2}(v) = 1 + \varepsilon_{H_2}(v)$.

It follows that

$$
\begin{aligned}
\xi^{ce}(G_1) - \xi^{ce}(G_2) &\leq \frac{1 + d_{H_1}(u)}{1 + \varepsilon_{H_2}(v)} - \frac{1 + d_{H_1}(u) + d_{H_2}(v)}{\varepsilon_{H_2}(v)} + \frac{1 + d_{H_2}(v)}{\varepsilon_{H_2}(v)} - \frac{1}{1 + \varepsilon_{H_2}(v)} \\
&= -\frac{d_{H_1}(u)}{\varepsilon_{H_2}(v)} + \frac{d_{H_1}(u)}{\varepsilon_{H_2}(v) + 1} < 0 .
\end{aligned}
$$

This completes the proof. $\qquad\square$

Theorem 4.2. *Let G be a connected graph on n vertices with $k \geq 1$ cut edges. Then*

$$
\xi^{ce}(G) \leq \frac{1}{2}(n^2 - 2kn + k^2 + 3k - 1)
$$

with equality if and only if $G \cong K_n^k$.

Proof. By Lemma 4.1, it follows that all cut edges in G_0 must be pendent edges. By the first half of the proof of Theorem 3.1, we obtain the result. $\qquad\square$

5   Cacti with k cycles

Let C_n^k be a cactus obtained by adding k independent edges among pendent vertices of S_n.

Theorem 5.1. *Let G be a cactus on $n \geq 5$ vertices with k cycles. Then $\xi^{ce}(G) \leq \frac{3}{2}n + k - \frac{3}{2}$, with equality if and only if $G \cong C_n^k$.*

Proof. Let $V_1 = \{v \in V(G) | \varepsilon(v) = 1\}$ and $V_2 = \{v \in V(G) | \varepsilon(v) \geq 2\}$. Then $|V_1| + |V_2| = n$. Assume that $|V_1| \geq 2$. Let u, v be two vertices in G such that $\varepsilon(u) = \varepsilon(v) = 1$, then $d(u) = d(v) = n - 1$. It follows that G is not a cactus since there exists cycles sharing common edges in G and hence $|V_1| \leq 1$.

Case 1. $|V_1| = 1$.

Let v be the unique vertex in G such that $\varepsilon(v) = 1$. Then $d(v) = n - 1$. So each vertex in $V(G) \backslash \{v\}$ is adjacent to v. Hence the cactus G is obtained by introducing k independent edges among pendent vertices of n-vertex star S_n, i. e., $G \cong C_n^k$. By direct calculation we get $\xi^{ce}(C_n^k) = \frac{3}{2}n + k - \frac{3}{2}$.

Case 2. $|V_1| = 0$.

In this case, $\varepsilon(v) \geq 2$ for any $v \in V(G)$. Note that there are exactly $n + k - 1$ edges in G, it follows that

$$\xi^{ce}(G) \leq \frac{1}{2} \sum_{v \in V(G)} d(v) = n + k - 1$$

and hence

$$\xi^{ce}(G) - \xi^{ce}(C_n^k) \leq n + k - 1 - \left(\frac{3}{2}n + k - \frac{3}{2}\right) = -\frac{1}{2}(n - 1) < 0 .$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

¿From the above result, we have

Corollary 5.2. *Let T be a tree on n vertex. Then $\xi^{ce}(T) \leq \frac{3}{2}n - \frac{3}{2}$, with equality if and only if $T \cong C_n^0 = S_n$.*

Corollary 5.3. *Let G be a unicyclic graph on n vertices. Then $\xi^{ce}(G) \leq \frac{3}{2}n - \frac{1}{2}$, with equality if and only if $G \cong C_n^1$.*

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