

# On the Generalized Combinatorial Enumeration of Stereo and Position Isomers of Homopolysubstituted Derivatives of [m.n] Paracyclophanes (Part I: m=n)

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**Abstract:** A combinatorial method based on the determination of the averaged weight of permutations controlling the chirality/achirality fittingness of  $2j+8$  substitution sites of the parent [m.n] paracyclophane ([m.n] PCP) having  $m=n$ , allows to derive generalized functional equations for direct enumeration of enantiomers pairs and achiral skeletons of any homopolysubstituted [m.n]paracyclophane (Ho[m.n] PCP) with the empirical formula  $\varphi_2 C_j H_{2j+8-q} X_q$  where  $X$  is a non isomerisable ligand,  $q$  is the degree of substitution and  $j=2n$  is the number of methylene units of the two equivalent carbon bridges. The integer sequences of enantiomer pairs  $A_e(j, q_z)$  and achiral skeletons  $A_{ac}(j, q_z)$  are given for substituted derivatives of [2.2]-, [3.3]-, [4.4]-paracyclophanes having variable degrees of substitution  $1 \leq q \leq 2j+8$ .

## 1 Introduction

The [m.n]paracyclophanes ([m.n]PCP) are a series of polycyclic hydrocarbons symbolized by the empirical formula  $(C_6H_4)_2(CH_2)_{m+n}$  consisting of two eclipsing benzene rings attached to two carbon bridges having  $m$  and  $n$  methylene groups located at the para position. The first member of this series discovered in 1949 by Brown and Farthing<sup>1</sup> as a by product of the pyrolytic polymerization of xylene into poly-para-xylene is the [2.2]-paracyclophane which has open new research topics for the cyclophanes chemistry<sup>2</sup>. If one considers the variation of the numbers  $m$  and  $n$  these chemical compounds may be classified into two categories: the [m.n]paracyclophanes ([m.n]PCP) having the two carbon chains with

unequal lengths  $m \neq n$  and the  $[m.n]$  paracyclophanes having the two carbon chains with equal lengths  $m=n$  ( $[n.n]$ PCP). The substituted derivatives of these two categories of paracyclophanes largely reported in the chemical literature<sup>3-11</sup> result from distinct placements of ligands/substituents among 8 substitution sites located on the two phenyl rings and  $2(m+n)$  substitution sites located on the two carbon bridges. Such distributions of ligands give rise to a large number of chiral stereo and position isomers which exhibit a wide range of optical activities observed among numerous molecules of the cyclophanes series<sup>12,13</sup>. The enumeration of such molecular structures is of great interest both to mathematicians and chemists who need to count and classify isomers of chemical compounds. The focus of this first part of the study is to propose a generalized and direct combinatorial method for counting enantiomer pairs and achiral skeletons of any homopolysubstituted  $[m.n]$ paracyclophane derivative having  $m=n$  (Ho $[n.n]$ PCP).

## 2 Mathematical formulation

Let us symbolize a Ho $[n.n]$ PCP by the empirical formula  $\varphi_2 C_j H_{2j+8-q} X_q$  where the subscripts  $j=2n$ ,  $q$  and  $2j+8-q$  represent respectively the total number of methylene units or polymethylene index (PMI)<sup>14</sup> of the two carbon bridges, the number of non isomerisable substituents of the same kind X and the number of unsubstituted hydrogen atoms while the greek symbol  $\varphi$  represents the hydrogen depleted benzene ring. The parity of the integer number  $n$  suggests to classify  $[n.n]$ PCP into 2 sub categories as follows: (a)- the  $[n_+, n_+]$ PCP and b) the  $[n_-, n_-]$ PCP. Throughout this paper the subscripts (+) and (-) are indicating even and odd positive integer numbers respectively.

We refer to the structural description of Jones et al<sup>5</sup> who report that in small  $[m.n]$ PCP systems with bridges of equal length ( $m=n$ ) the rings are slightly parallel and rotated with respect to each other, to represent such molecular systems by tridimensional molecular graphs or stereographs  $G_{n,n_+}$  and  $G_{n,n_-}$  in  $D_{2h}$  symmetry which are shown in figure 1.

The stereograph  $G_{n,n_+}$  of the parent  $[n_+, n_+]$ PCP depicted in figure 1 contains in each bridge an even number  $n_+$  of unlabelled black vertices representing carbon atoms.

The  $2j$  numerical labels ( $1, 2, \dots, k, \dots, n; 1', 2', \dots, 3, \dots, k', \dots, n'; 1'', 2'', \dots, k'', \dots, n''; 1''', 2''', \dots, k''', \dots, n'''$ ) attached to  $j$  black vertices and the 8 alphabetical labels ( $a, b, c, d, a', b', c', d'$ ) located on the two phenyl groups represent the  $2j+8$  hydrogen atoms or

substitution sites of this molecular system. Similarly the stereograph  $G_{n,n_-}$  of the parent  $[n_-.n_-]$ PCP depicted in figure 1 contains in each bridge an odd number  $n$  of unlabelled vertices representing carbon atoms while its numerical and alphabetical labels correspond to the  $2j+8$  hydrogen atoms (or substitution sites).

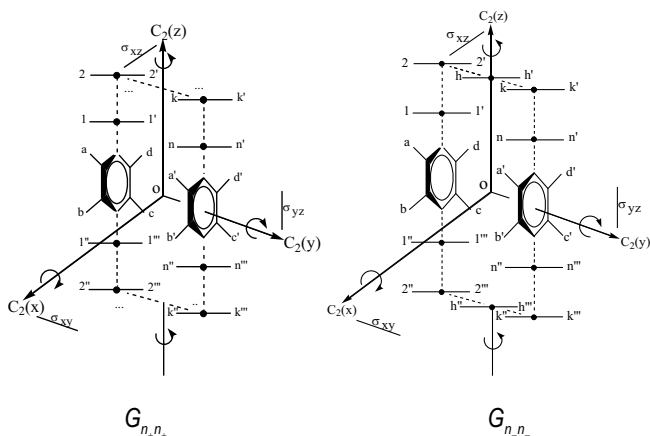


Figure 1: Molecular stereographs of  $[n_+.n_+]$ - and  $[n_-.n_-]$ -paracyclophanes.

## 2.1 Average weight of permutations controlling the chirality/achirality fittingness

Let  $D_{2h} = \{E, C_2(x), C_2(y), C_2(z), i, \sigma(xy), \sigma(xz), \sigma(yz)\}$  and  $|D_{2h}| = 8$ . Each symmetry operation of  $D_{2h}$  is a permutation operator which permutes equivalent atoms of  $G_{n_+, n_+}$  and  $G_{n_-, n_-}$ . Therefore the permutations of the  $2j+8$  substitution sites of a  $[n_+.n_+]$ PCP induced by the eight symmetry operations of  $D_{2h}$  as reported in table 1 column 3 are obtained as follows : E fixes each hydrogen atom and gives rise to  $2j+8$  unitary permutation cycles of length  $\ell_1 = 1$  which are symbolized in partition notation by the term  $1^{2j+8}$ . On the other hand, each one of the following symmetry operations  $C_2(x)$ ,  $C_2(y)$ ,  $C_2(z)$ ,  $i$ ,  $\sigma(xy)$ ,  $\sigma(xz)$ ,  $\sigma(yz)$  induces  $j+4$  permutation cycles of length  $\ell_2 = 2$  (transpositions) represented by the term  $2^{j+4}$ . The permutations of the substitution sites of  $G_{n_+, n_+}$  are reported in column 3 of table 1. In the case of a  $[n_-.n_-]$ PCP depicted in figure 1 by the stereograph  $G_{n_-, n_-}$  each one of these symmetry

operations E,  $C_2(x)$ ,  $C_2(y)$ ,  $C_2(z)$ ,  $I$ ,  $\sigma(xy)$ ,  $\sigma(yz)$  induces permutation cycles as previously indicated in table 1 column 4 except the symmetry plane  $\sigma(xz)$  which fixes four hydrogen atoms and generates simultaneously 4 unitary cycles and  $j+2$  transpositions.

**Table 1:** Permutations of the  $2j+8$  substitution sites of  $G_{n,n}$  and  $G_{n,n}$  induced by  $D_{2h}$  symmetry operations.

$D_{2h}$ symmetry operations	Permutations of $2j+8$ substitution sites	Partition notation	
		$G_{n,n}$	$G_{n,n}$
E	$(1)(2)...(k)...(n)(1')(2')...(k')...(n')(1'')(2'')...(k'')...(n'')(1''')(2''')...(k''')...(n''')(1'''')(2'''')...(k'''')(n'''')$ $...(k''')...(n'''')(a)(b)(c)(d)(a')(b')(c')(d')$	$1^{2j+8}$	$1^{2j+8}$
$C_{2(x)}$	$(1,n)(2,k)(k,2'')(n,1'')(1',n'')(2',k'')(k',2'')(n',1'')$ $(a,b')(b,a')(c,d')(d,c')...$	$2^{j+4}$	$2^{j+4}$
$C_{2(y)}$	$(1,1'')(2,2'')(1',1'')(2',2'')(k,k'')(k',k'')(n,n'')(n',n'')$ ... $(a,c)(b,d)(a',c')(b',d')$	$2^{j+4}$	$2^{j+4}$
$C_{2(z)}$	$(1,n')(1',n')(2,k')(2',k')(1'',n''')(1''',n'''')(2'',k'')(2''',k''')$ ... $(a,d')(d,a')(b,c')(c,b')$	$2^{j+4}$	$2^{j+4}$
$I = \sigma_{(yz)}C_{2(x)}$	$(2,k'')(2',k'')(1,n'')(1',n'')(k,2'')(k',2'')(n,1'')(n',1'')$ $(a,c')(b,d')(d,b')(c,a')$	$2^{j+4}$	$2^{j+4}$
$\sigma_{(xy)}$	$(1,1'')(2,2'')(1',1'')(2',2'')(k,k'')(n,n'')(n',n'')(k',k'')$ ... $(a,b)(c,d)(a',b')(c',d')$	$2^{j+4}$	$2^{j+4}$
$\sigma_{(yz)}$	$(1,1')(2,2')(1'',1''')(2'',2'')(k,k')(n,n')(k',k')(n',n'')$ ... $(a,d)(a',d')(b,c)(b',c')$	$2^{j+4}$	$2^{j+4}$
$\sigma_{(xz)}$	$(1,n)(1',n'')(1'',n'')(2'',k'')(2',k')(2'',k')(1''',n''')(2''',k'')$ ... $(a,a')(b,b')(c,c')(d,d')$	$2^{j+4}$	--
	$(1,n)(1',n'')(1'',n'')(2'',k'')(2',k')(2'',k')(1''',n''')(2''',k'')$ $(a,a')(b,b')(c,c')(d,d')(h)(h')(h'')(h''')$	--	$1^4 2^{j+2}$

Such combined permutations are symbolized by the term  $1^4 2^{j+2}$ . The permutations of the substitution sites for  $G_{n,n}$  are reported in column 4 of table 1. Now let P denote in eqs 1-2 the set which contains all the permutations generated by the eight symmetry operations of  $D_{2h}$  acting on the  $2j+8$  substitution sites.

$$P = \left\{ [1^{2j+8}], [7[2^{j+4}]] \right\} \text{ for } [n_+ . n_+] PCP \tag{1}$$

or

$$P = \{ [1^{2j+8}], 6 [2^{j+4}], [1^4 2^{j+2}] \} \text{ for } [n_- n_-] \text{PCP} \quad (2)$$

By eliminating in  $P$  the permutations induced by reflections and improper rotations one may obtain the set  $P'$  of cardinality 4 defined in eq.3.

$$P' = \{ [1^{2j+8}], 3 [2^{j+4}] \} \text{ for } [n_+ n_+] \text{PCP or } [n_- n_-] \text{PCP} \quad (3)$$

Let us consider  $\bar{P}$  and  $\bar{P}'$  in eqs.4-5, as the averaged sum of permutations collected in  $P$  and  $P'$ :

$$\bar{P} = \frac{1}{8} \begin{cases} ([1^{2j+8}] + 7 [2^{j+4}]) & \text{for } n_+ n_+ \\ ([1^{2j+8}] + 6 [2^{j+4}] + [1^4 2^{j+2}]) & \text{for } n_- n_- \end{cases} \quad (4)$$

$$\bar{P}' = \frac{1}{4} ([1^{2j+8}] + 3 [2^{j+4}]) \quad \text{for } n_+ n_+ \text{ or } n_- n_- \quad (5)$$

and derive the differences  $\Delta_c = \bar{P}' - \bar{P}$  and  $\Delta_a = 2\bar{P} - \bar{P}'$  given in eqs.6-7 :

$$\Delta_c = \frac{1}{8} \begin{cases} ([1^{2j+8}] - [2^{j+4}]) & \text{for } n_+ n_+ \\ ([1^{2j+8}] - [1^4 2^{j+2}]) & \text{for } n_- n_- \end{cases} \quad (6)$$

and

$$\Delta_a = \frac{1}{4} \begin{cases} (4 [2^{j+4}]) & \text{for } n_+ n_+ \\ (3 [2^{j+4}] + [1^4 2^{j+2}]) & \text{for } n_- n_- \end{cases} \quad (7)$$

$\Delta_c$  and  $\Delta_a$  are respectively the averaged weight of permutations controlling the chirality and the achirality fittingness of the  $2j+8$  substitution sites of the parent  $[n_- n_-]$  PCP. It is to be noticed that the concept of chirality/achirality fittingness of orbits that consist of a set of atoms, bonds, substitution sites, substituents or ligands in organic molecules has been widely presented by Fujita<sup>15-18</sup>.

Let  $\ell_i$  denote the length of a permutation cycle induced by the symmetry operation  $i \in D_{2n}$ . To obtain distinct homopolysubstitutions of degree  $q$  we have to place in distinct ways  $q$  substituents of the same kind  $X$  among  $2j+8$  substitution sites permuted as given in eqs.1-2. This condition is satisfied if the positive integer number  $\ell_i$  is a common divisor of the two positive integer numbers  $2j+8$  and  $q$ . Consequently each  $\ell_i$  – cycle of permutations generates  $\frac{2j+8}{\ell_i}$  subsets of positions (or boxes) where  $\frac{q}{\ell_i}$  subsets of substituents are to be put. We recall from combinatorics<sup>19</sup> that the number of combinations with repetition (distinct homopolysubstitutions of degree  $q$ ) resulting from the placements in distinct ways of  $\frac{q}{\ell_i}$  objects among  $\frac{2j+8}{\ell_i}$  undistinguishable boxes is derived from the binomial coefficient given in eq.8:

$$T\left(\frac{2j+8}{\ell_i}, \frac{q}{\ell_i}\right) = \binom{\frac{2j+8}{\ell_i}}{\frac{q}{\ell_i}} \tag{8}$$

The numbers of distinct placements of  $q$  substituents among  $2j+8$  positions induced by the permutations of types  $1^{2j+8}$  and  $2^{j+4}$  are obtained from eqs. 9-11 given in column 2 of table 2.

**Table 2:** The numbers of distinct placements of  $q$  substituents among  $2j+8$  positions induced by the permutations of types  $1^{2j+8}$ ,  $2^{j+4}$ ,  $1^4 2^{j+2}$ .

Types of Permutations	Number of distinct placements of $q$ substituents among $2j+8$ substitution sites	
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$[1^{2j+8}]$	$T(2j+8, q)$	(9)
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$[2^{j+4}]$	$T\left(j+4, \frac{q}{2}\right)$	(10)
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$[1^4 2^{j+2}]$	$\sum_{\alpha_1=0}^4 T(4, \alpha_1) T\left(j+2, \frac{q-\alpha_1}{2}\right)$	(11)
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It is to be noticed that for the term  $1^4 2^{j+2}$  which is the product of 4 unitary cycles of permutations and  $j+2$  transpositions, we have to solve the Young's partition<sup>20</sup> equation  $\alpha_1 + \alpha_2 = 4$ . The couples of solutions  $(\alpha_1, \alpha_2) = (4, 0), (3, 1), (2, 2), (1, 3), (0, 4)$  determine the numbers of substituents of type X and of hydrogen atoms to distribute in 4 invariants positions. The number of distinct ways of putting  $\alpha_1$  substituents of type X and  $\alpha_2$  hydrogen atoms in 4 invariants positions is  $T(4, \alpha_1)$ . By fixing the couple of integer numbers  $(\alpha_1, \alpha_2)$  aforementioned, it remains  $\frac{q - \alpha_1}{2}$  couples of substituents of the same kind X to be distributed among  $j+2$  positions. Then the number of placements of  $\frac{q - \alpha_1}{2}$  objects X among  $j+2$  boxes is derived from the binomial coefficient  $T\left(j+2, \frac{q - \alpha_1}{2}\right)$ . Finally the number of distinct placements resulting from the term  $1^4 2^{j+2}$  including the four different choices  $(\alpha_1, \alpha_2)$  previously mentioned is derived from eq.11 where  $\frac{q - \alpha_1}{2} \geq 0$ .

## 2.2 Generalized equations for counting enantiomer pairs and achiral skeletons of Ho[n.n]PCP

By replacing the right hand side terms of eqs.6-7 with their equivalent algebraic expressions given in eqs.9-11, we convert  $\Delta_c$  and  $\Delta_a$  into generalized equations used to derive the number of enantiomer pairs  $A_c(j, q_{\pm})$  and the number of achiral skeletons  $A_{ac}(j, q_{\pm})$  for molecules of the series Ho[n+.n+]PCP and Ho[n.n.]PCP symbolized by the empirical formula  $\varphi_2 C_j H_{2j+8-q} X_q$ . These generalized counting formulas are declined in accordance with the parity of the variable  $q_-$  or  $q_+$  as follows:

### 2.2a Determination of the numbers of enantiomer pairs and achiral skeletons of a Ho[n+.n+]PCP

In the series Ho[n+.n+]PCP consisting of  $j = 2n_+$  methylene units and a degree of substitution  $q_{\pm}$  the number of enantiomer pairs  $A_c(j, q_{\pm})$  is obtained from the generalized counting formula given in eq.12:

$$A_c(j, q) = \frac{1}{8} \left[ \binom{2j+8}{q} - \binom{j+4}{\frac{q}{2}} \right] \quad (12)$$

which splits into eqs.13 -14 for  $q_-$  and  $q_+$  respectively.

$$A_c(j, q_-) = \frac{1}{8} \left[ \binom{2j+8}{q} \right] \quad (13)$$

$$A_c(j, q_+) = \frac{1}{8} \left[ \binom{2j+8}{q} - \binom{j+4}{\frac{q}{2}} \right] \quad (14)$$

The associate number  $A_{ac}(j, q_{\pm})$  of achiral skeletons of  $\text{Ho}[n_+, n_+]$ PCP with a degree of substitution  $q_{\pm}$  is obtained from eq-15 :

$$A_{ac}(j, q_{\pm}) = \binom{j+4}{\frac{q}{2}} \quad (15)$$

which becomes eqs.16-17 in accordance with the variables  $q_+$  and  $q_-$  :

$$A_{ac}(j, q_-) = 0 \quad (16)$$

$$A_{ac}(j, q_+) = \binom{j+4}{\frac{q}{2}} \quad (17)$$

Eqs.16 -17 suggest according to the parity of the positive integers  $j$  and  $q$  the following remark:

*Any  $\text{Ho}[n_+, n_+]$ PCP consisting of  $j_+ = 2n_+$  methylene units and a degree of substitution  $q_-$  is exclusively a chiral structure and consequently, no achiral structure is observed in such a molecular system.*

## 2.2b Determination of the numbers of enantiomer pairs and achiral skeletons of a $\text{Ho}[n_-, n_-]$ PCP

In the series  $\text{Ho}[n_-, n_-]$  PCP comprising  $j = 2n_-$  methylene units and a degree of substitution  $q_{\pm}$  the number of enantiomer pairs  $A_c(j, q_{\pm})$  is obtained from the generalized counting formula given in eq.18:



$$A_c(j, q) = \frac{1}{8} \left[ \binom{2j+8}{q} - \sum_{\alpha_1=0}^4 \binom{4}{\alpha_1} \binom{j+2}{\frac{q-\alpha_1}{2}} \right] \quad (18)$$

which becomes eqs.19-20 in accordance with the variables  $q$  and  $q_{\pm}$ :

$$A_c(j, q_{-}) = \frac{1}{8} \left[ \binom{2j+8}{q} - 4 \binom{j+2}{\frac{q-1}{2}} - 4 \binom{j+2}{\frac{q-3}{2}} \right]$$

$$(19) \quad A_c(j, q_{+}) = \frac{1}{8} \left[ \binom{2j+8}{q} - \binom{j+2}{\frac{q}{2}} - 6 \binom{j+2}{\frac{q-2}{2}} + \binom{j+2}{\frac{q-4}{2}} \right]$$

(20)

The associate number  $A_{ac}(j, q_{\pm})$  of achiral skeletons of Ho[ $n.n.$ ] PCP with a degree of substitution  $q$  is obtained from eq. 21 :

$$A_{ac}(j, q) = \frac{1}{4} \left[ 3 \binom{j+4}{\frac{q}{2}} + \sum_{p_1=0}^4 \binom{4}{p_1} \binom{j+2}{\frac{q-p_1}{2}} \right] \quad (21)$$

which becomes eqs.22-23 in accordance with the variables  $q_{+}$  and  $q_{-}$  :

$$A_{ac}(j, q_{-}) = \left[ \binom{j+2}{\frac{q-1}{2}} + \binom{j+2}{\frac{q-3}{2}} \right] \quad (22)$$

$$A_{ac}(j, q_{+}) = \frac{1}{4} \left[ 3 \binom{j+4}{\frac{q}{2}} + \binom{j+2}{\frac{q}{2}} + 6 \binom{j+2}{\frac{q-2}{2}} + \binom{j+2}{\frac{q-4}{2}} \right] \quad (23)$$

It is to be noticed that  $A_c(j, q_{\pm})$  and  $A_{ac}(j, q_{\pm})$  satisfy the following conditions

$$A_c(j, q_{\pm}) = A_c(j, 2j+8-q_{\pm}) \quad (24)$$

$$A_{ac}(j, q_{\pm}) = A_{ac}(j, 2j+8-q_{\pm}) \quad (25)$$

due to the complementarity of the degrees of substitution  $q$  and  $2j+8-q$  in the systems  $\varphi_2 C_j H_{2j+8-q} X_q$  and  $\varphi_2 C_j H_q X_{2j+8-q}$ .

### 3 Application to direct combinatorial enumeration

To exemplify the application of the present pattern inventory, let us attribute numerical values to the positive integers numbers  $j$  and  $q_{\pm}$  in the molecular formula  $\varphi_2 C_j H_{2j+8-q} X_q$  of a

given homopolysubstituted derivative of a [n.n] PCP in order to compute its numbers  $A_c(j, q_{\pm})$  of enantiomer pairs and  $A_{ac}(j, q_{\pm})$  of achiral skeletons.

**Example 1.** Let  $n_+ = 2$  and  $j = 2n_+ = 4$  and consider the molecular formula  $\varphi_2 C_4 H_{16-q} X_q$  for the series of Ho - [2.2] PCP where  $1 \leq q_{\pm} \leq 16$ . From eqs 1 and 3 we derive the set of permutations  $P = ([1^{16}], 7 [2^8])$  and  $P' = ([1^{16}], 3 [2^8])$  and from eq.4-5 we determine the average sum of permutations  $\bar{P} = \frac{1}{8} ([1^{16}] + 7 [2^8])$  and  $\bar{P}' = \frac{1}{4} ([1^{16}] + 3 [2^8])$ . The averaged weight of permutations controlling the chirality fittingness of 16 substitution sites of the [2.2]PCP determined from eqs.6 is  $\Delta_c = \frac{1}{8} ([1^{16}] - [2^8])$ . The numbers  $A_c(4; q_{\pm})$  of enantiomer pairs are calculated by means of eqs 13 and 14. For instance if  $q_- = 1$  and  $q_+ = 2$ , hence:

$$A_c(4;1) = \frac{1}{8} \left[ \binom{16}{1} \right] = 2 \quad \text{and} \quad A_c(4;2) = \frac{1}{8} \left[ \binom{16}{2} - \binom{8}{1} \right] = 14$$

Conversely, from eq.7, we deduce the averaged weight of permutations controlling the achirality fittingness of 16 substitution sites of [2.2]PCP by the difference  $\Delta_a = [2^8]$ . From eq.16 and 17 we compute the numbers of achiral skeletons for  $q_- = 1$  and  $q_+ = 2$  as follows:

$$A_{ac}(4; 1) = 0 \quad \text{and} \quad A_{ac}(4; 2) = \binom{8}{1} = 8$$

**Example 2.** Let  $n_- = 3$  and  $j = 2n_- = 6$  and consider the molecular formula  $\varphi_2 C_6 H_{20-q} X_q$  for the series Ho-[3.3]PCP where  $1 \leq q_{\pm} \leq 20$ . From eqs 2 and 3 we derive the sets of permutations  $P = \{[1^{20}], 6 [2^{10}], [1^4 2^8]\}$ ,  $P' = \{[1^{20}], 3 [2^{10}]\}$  and from eq.4-5 we determine the average sum of permutations  $\bar{P} = \frac{1}{8} ([1^{20}] + 6 [2^{10}] + [1^4 2^8])$  and  $\bar{P}' = \frac{1}{4} ([1^{20}] + 3 [2^{10}])$ . The averaged weight of permutations controlling the chirality fittingness of 20 substitution sites of the [3.3]PCP determined from eqs.6 is  $\Delta_c = \frac{1}{8} ([1^{20}] - [1^4 2^8])$ . The numbers  $A_c(6; q_{\pm})$  of enantiomer pairs with the degrees of substitution  $q_+ = 2$  and  $q_- = 3$  are calculated by means of eqs 19 and 20 as follows:

$$A_c(6; 2) = \frac{1}{8} \left[ \binom{20}{2} - \binom{8}{1} - 6 \binom{8}{0} \right] = 22$$

$$A_c(6; 3) = \frac{1}{8} \left[ \binom{20}{3} - 4 \binom{8}{1} - 4 \binom{8}{0} \right] = 138$$

From eq.7 we deduce the averaged weight of permutations controlling the achirality fittingness of 20 substitution sites of [3.3]PCP by the difference

$\Delta_{ac} = \frac{1}{4} \left( 3 \left[ 2^{j+4} \right] + \left[ 4^j 2^{j+2} \right] \right)$ . We compute from eqs. 22-23 numbers  $A_{ac}(6; q_{\pm})$  of achiral skeletons for  $q_{+}=2$  and  $q_{+}=3$  as follows:

$$A_{ac}(6; 2) = \frac{1}{4} \left[ 3 \binom{10}{1} + \binom{8}{1} + 6 \binom{8}{0} \right] = 11$$

$$A_{ac}(6; 3) = \left[ \binom{8}{1} + \binom{8}{0} \right] = 9$$

The integer sequences  $A_c(j, q_{\pm})$  and  $A_{ac}(j, q_{\pm})$  have been calculated in the series Ho[n.n]PCP where  $n=2,3,4$ ;  $j=4, 6, 8$  and  $1 \leq q \leq 2j+8$ . The results are reported in table 3.

**Table 3: The numbers  $A_c(j, q_{\pm})$  of enantiomer pairs and  $A_{ac}(j, q_{\pm})$  of achiral skeletons for [2-2]-, [3.3]-, [4.4]-homopolysubstituted paracyclophanes derivatives.**

q	Ho[2-2]-PCP			Ho[3-3]-PCP			Ho[4-4]-PCP		
	$A_c$	$A_{ac}$	$C_{\varepsilon}(j, q_{\pm})$	$A_c$	$A_{ac}$	$C_{\varepsilon}(j, q_{\pm})$	$A_c$	$A_{ac}$	$C_{\varepsilon}(j, q_{\pm})$
1	2	0	4	2	1	5	3	0	6
2	14	8	36	22	11	55	33	12	78
3	70	0	140	138	9	285	253	0	506
4	224	28	476	596	53	1245	1320	66	2706
5	546	0	1092	1920	36	3876	5313	0	10626
6	994	56	2044	4816	148	9780	16797	220	33814
7	1430	0	2860	9648	84	19380	43263	0	86526
8	1600	70	3270	15692	266	31650	91872	495	184239
9	1430	0	2860	20932	126	41990	163438	0	326876
10	994	56	2044	23028	322	46378	245058	792	490908
11	546	0	1092	20932	126	41990	312018	0	624036
12	224	28	476	15692	266	31650	337904	924	676732
13	70	0	140	9648	84	19380	312018	0	624036
14	14	8	36	4816	148	9780	245058	792	490908
15	2	0	4	1920	36	3876	163438	0	326876
16	0	1	1	596	53	1245	91872	495	184239
17				138	9	285	43263	0	86526
18				22	11	55	16797	220	33814
19				2	1	5	5313	0	10626
20				0	1	1	1320	66	2706
21							253	0	506
22							33	12	78
23							3	0	6
24							0	1	1

The accuracy of our theoretical results is testified by the method of drawing and counting graphs of systems with smaller size. An illustration is given in figure 2 which depicts 22 molecular graphs of  $\varphi_2 C_4 H_{14} X_2$  systems. For the sake of comparison it is to be noticed that the results of our combinatorial method verify the following relations:

$$A_c(j, q_{\pm}) + A_{ac}(j, q_{\pm}) = C_T(j, q_{\pm}) \quad (26)$$

and

$$A_c(j, q_{\pm}) + A_{ac}(j, q_{\pm}) = C_E(j, q_{\pm}) \quad (27)$$

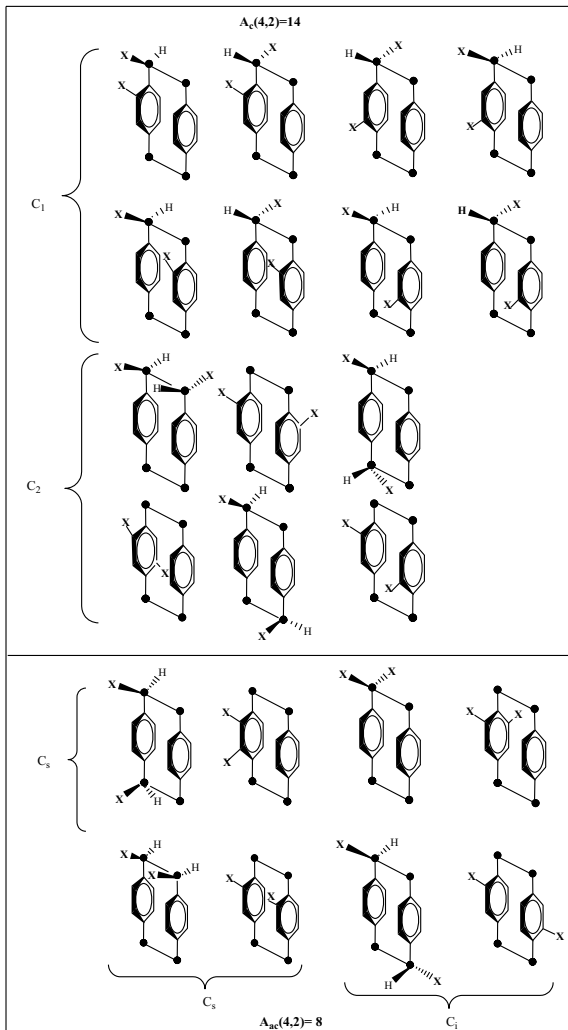


Figure 2: Molecular graphs of  $\varphi_2 C_4 H_{14} X_2$  systems.

where  $C_T(j, q_{\pm})$  and  $C_E(j, q_{\pm})$  are topological and enantiomeric integer numbers obtained through Pólya's enumerations method<sup>21</sup>. For instance the number of enantiomer pairs  $A_c(4, 2) = 14$  and the number of achiral skeletons  $A_{ac}(4, 2) = 8$  predicted by our calculations for the series  $\varphi_2C_4H_{14}X_2$  satisfy eqs 24-25 and match up with the Pólya's coefficients  $C_E(j, 2) = (14 \times 2) + 8 = 36$  (see in table 3 row 4 column 2) and  $C_T(j, 2) = 14 + 8 = 22$ . The symmetry characterization of the 22 graphs reveals that the class of chemical compounds  $\varphi_2C_4H_{14}X_2$  depicted in figure 2 comprises 8 chiral isomers with  $C_1$ -symmetry, 6 chiral isomers with  $C_2$ -symmetry, 6 achiral isomers with  $C_s$  symmetry and 2 achiral isomers with  $C_i$ -symmetry.

## 4 Conclusion

This combinatorial method for enumerating enantiomer pairs and achiral skeletons of Ho[n.n] PCP symbolized by the empirical formulae  $\varphi_2C_jH_{2j+8-q}X_q$  is applicable to any molecular system of these series where  $j=2n$  and  $1 \leq q_{\pm} \leq 2j+8$ . It includes from the determination of permutations of substitution sites induced by the symmetry point group of the parent hydrocarbon: (1) - the derivation of the averaged weight of permutations controlling the chirality and the achirality fittingness of the  $2j+8$  substitution sites of the parent [n.n]PCP. (2) - the transformation of these averaged contributions into generalized counting equations which allow the direct computation of the numbers  $A_c(j, q_{\pm})$  of enantiomer pairs and  $A_{ac}(j, q_{\pm})$  of achiral skeletons. This procedure has already been applied with success by Nemba and Balaban<sup>22</sup> and its advantage is to circumvent the Pólya's counting method<sup>21</sup> which is largely presented by Pólya, Tarjan and Woods<sup>23</sup>, Harary, Palmer, Robinson and Read<sup>24</sup>, Tucker<sup>25</sup> and Rouvray<sup>26</sup> and requires first to derive the cycle index in accordance with the symmetry point group of the molecular system and second the transformation of the cycle index into a generating function of order  $2j+8$  (in the case of [m.n] PCP) the coefficients of which are solution of the enumeration problem. In this work we have used basic concepts of permutation group theory and algebraic combinatorics as tools which allow to express mathematically the chirality and the achirality fittingness induced by substitution sites of [n.n]PCP molecular systems./.

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