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Number of 5-Matchings in Graphs

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Abstract

In this paper a formula for the number of 5-matchings in triangular–free and 4–cycle–free graph based on the number of vertices, edges, the degrees of vertices and the number of 5-cycles was obtained.

1. Introduction

A graph *G=(V,E)* is set containing vertices and edges that these edges are two elements sets of vertices that they are denoted by *V(G)* and *E(G)*, respectively. Graphs in this paper are finite, loopless and contains no multiple edges. For such a graph *G, n* and *m* are assumed the number of its vertices and edges respectively. We define a matching in *G* to be a spanning subgraph of *G*, whose components are vertices and edges. A *k*-matching is a matching with *k* edges. A perfect matching is a matching with edges only. We use the *p(G,k)* to denote the number of *k*-matching in *G* and it's assumed that *p(G,0)=1.*

The matching polynomial of graph *G* is denoted by *μ (G, x)* that defined by

$$\mu\,(G,x)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^k\;p(G,k)x^{n-2k}$$

The graphs that have the same matching polynomials are called co-matching. It is obvious that two isomorphic graphs are co-matching. But the reverse is not true [2]. However, some graphs that have this feature that co-matching is equal to isomorphism. These graphs can be characterized by their matching polynomials. For example, Petersen graph is one of these

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graphs [2]. The graphs that are characterized by their matching polynomial are called matching unique. The above-mentioned feature has vital role in graphs categorization.

The number of 3-matchings can be found in Farrel and Guo ([2]) to do this they used degrees of vertices and the number of vertices, edges and triangles also Behmaram ([1]) has calculated the number of 4-matchings in triangular-free graphs.

2. Preliminaries

It is obvious that the number of 1–matching is equal to the number of edges of *G, i.e.* $p(G\,,1) = m$. In this section we derived *p(G, k)* for k = 2,3,4.

Lemma 2.1. If the degrees of vertices of *G* are $d_1, d_2, \dots, d_n$، then the number of 2–matching is:

$$p\,(G,2) = \binom{m}{2} - \sum_{i=1}^{n} \binom{d_i}{2}$$

Lemma 2.2. Any graph that is co–matching with a regular graph is also regular of the same valency

Lemma 2.3.

$$p\,(G,3) = \binom{m}{3} - (m-2)\sum_{i}\binom{d_i}{2} + 2\sum_{i}\binom{d_i}{3} + \sum_{ij}(d_i - 1)(d_j - 1) - N_T$$

where N_T is the number of triangles in *G*

Corollary 2.4. Let *G* be a regular graph of degree *d* with n vertices. Then,

$$p\,(G,2) = \frac{(n-4)d+2}{8}\,(nd)$$

$$p\,(G,3) = \frac{(n^2-12n+40)d^2 + (6n-48)d + 16}{48}.(nd) - N_T$$

Proof. By $m = \frac{nd}{2}$, Lemma 2.1. and Lemma 2.3.، the result is obvious.

Corollary 2.5. suppose that *G* and *H* are two regular graphs which are co–matching, then the number of triangles in *G* is equal to the number of triangles in *H*.

Proof. By Corollary 2.4. and Lemma 2.2., it is obvious.

Lemma 2.6.[2] Let *G* be a triangular–free graph, with *V (G) = {1, 2, ..., n}* and let the degree of vertex *i* is d_i. Also, let *N(i)* be the set of neighbors of *i* in *G*. Hence, the number of 4–matching is:

$$p(G,4) = \binom{m}{4} + (m-2)\sum_{ij}(d_i - 1)(d_j - 1) - \sum_i \binom{d_i}{4} - \sum_{\{i,j\}\subset V} \binom{d_i}{2}\binom{d_j}{2}$$
$$-\sum_i \sum_{\{k,t\}\subset N(i)} (d_k - 1)(d_t - 1) - \sum_i \binom{d_i}{2} p(G-i,2) - \sum_i \binom{d_i}{3}(m - d_i)$$
$$-\sum_i \sum_{\{k,s.t\}\subset N(i)} (d_k + d_s + d_t - 3) + N_q$$

where N_q is the number of 4–cycles in *G*.

Corollary 2.7. Let *G* be a triangular–free graph with *n* vertices which is regular of valency *d*. Then,

$$p(G,4) = \frac{(n^3 - 24n^2 + 208n - 672)d^3 + (12n^2 - 240n - 1344)d^2 + (76n - 960)d + 240}{384}(nd) + N_q$$

Proof: By Lemma 2.6.and the relations $m = \frac{nd}{2}$ and

$$\mathrm{p(G\text{–}i,2)} = \binom{\mathrm{m - d_i}}{2} - \sum_{\mathrm{j\neq i}} \binom{\mathrm{d_j'}}{2}$$

where $d_j' = \begin{cases} d_j - 1, & j \epsilon N(i) \\ d_j & , j \notin N(i) \end{cases}$ ؛ the result is obviously obtained.

Corollary 2.8. suppose that *G* and *H* are two triangular – free regular graphs which are co–matching, then the number of 4–cycles in *G* is equal to the number of 4–cycles in *H*.

Proof: By Corollary 2.7. and Lemma 2.2. the result is clearly verified.

3. The number of 5–matchings

In the following theorem we will obtain a formula for the sixth coefficient, i.e, *p (G, 5),* of the matching polynomial in triangular–free and 4–cycle–free graphs.

Theorem 3.1. Let *G* be a triangular–free and 4–cycle–free graph, with *V(G) = {1, 2, …, n}* and let the degree of vertex *i* is d_i. Also, let *N (i)* be the set of neighbors of *i* in *G*. Hence, the number of 5–matchings is:

$$p(G,5) = \binom{m}{5} - \sum_{i=1}^{n}\binom{d_i}{5} - \sum_{i=1}^{n}\binom{d_i}{4}(m - d_i) - \sum_{i=1}^{n}\binom{d_i}{3}\binom{m - d_i}{2}$$
$$-\sum_{i=1}^{n}\binom{d_i}{2} p(G-i,3) + \sum_{ij}(d_i - 1)(d_j - 1)\binom{m - d_i - d_j + 2}{2}$$

$$-3\sum_{ij}\binom{d_i-1}{2}\binom{d_j-1}{2}+\sum_{ij}\sum_{\substack{k\epsilon N(i)-\{j\}\\ t\epsilon N(j)-\{i\}}}(d_k-1)(d_t-1)$$

$$+\sum_{i}\sum_{\{k,t\}\subset N(i)}\left[\binom{d_t-1}{2}(d_k-1)+\binom{d_k-1}{2}(d_t-1)\right]$$

$$-(m-4)\sum_{i}\sum_{\{k,t\}\subset N(i)}(d_k-1)(d_t-1)$$

$$+2\sum_{i}\sum_{\{k,s,t\}\subset N(i)}[(d_k-1)(d_s-1)+(d_k-1)(d_t-1)+(d_s-1)(d_t-1)]$$

$$-\sum_{\{i,j\}\subset V}\binom{d_i}{2}\binom{d_j}{2}p(G-i-j,1)+(m-4)\sum_{i}\sum_{\{k,s,t\}\subset N(i)}(d_k+d_s+d_t-3)$$

$$-3\sum_{i}\sum_{\{k,s,t,r\}\subset N(i)}(d_k+d_s+d_t+d_r-4)-N_P$$

where N_P is the number of 5–cycles in *G*.

Proof. To find *p (G, 5)*, first we find the number of subsets of edges in *G* that have 5 edges, i.e., $\binom{m}{5}$. Then subtract the number of graphs in which they do not form a 5–matching.
The possible subgraphs which do not form a 5–matching are shown in Figure 1.

Let N_P، N_q، N_S، N_T، N_X، N_Y، N_Z، N_V، N_H، N_E، N_F، N_L، N_K، N_M، N_I and N_J denote the number of subgraphs of *G* that are isomorphic to *P*، *q*، *S*، *T*، *X*، *Y*، *Z*، *V*، *H*، *E*، *F*، *L*، *K*، *M*، *I* and *J* respectively. Now, we calculate each of these numbers, as follows:

N_s: For counting the number of graphs that are isomorphic to *S*, we choose one vertex and then five edges adjacent to this vertex. Therefore, we have:

$$N_S=\sum_{i=1}^{n}\binom{d_i}{5}$$

$\mathrm{N_X}$: For counting $\mathrm{N_X}$، first select an edge *ij* from *E(G)* , then choose two edges from each: *i* and *j*, except *ij* ,therefore, $\mathrm{N_X}$ is:

$$N_X=\sum_{ij}\binom{d_i-1}{2}\binom{d_j-1}{2}$$

$\mathrm{N_F}$: For counting $\mathrm{N_F}$, we choose a vertex *i* from *V (G)* and then select a subset *{k,s,t,r}* from *N(i)*.

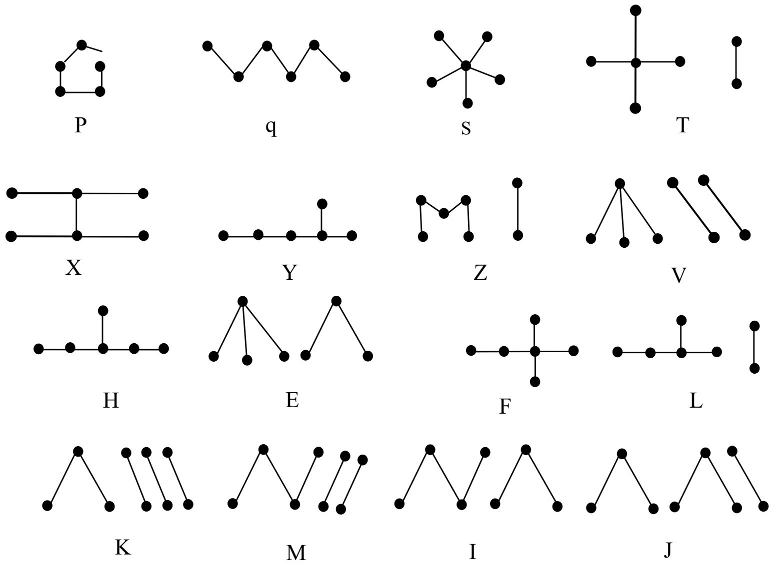

Figure 1. The possible subgraphs which do not form a 5-matching

Then we select an edge which is adjacent to *k, s, t* or *r*, other than edges connecting *k,s,t* and *r* to *i*. Therefore, N_F is:

$$N_F = \sum_{i} \sum_{\{k,s,t,r\}\subset N(i)} (d_k + d_s + d_t + d_r - 4)$$

N_T: For counting N_T choose a vertex *i* from *V(G)* and then four edges that are adjacent to *i*. Then we select another edge that is not adjacent to *i*.

But this single edge may be connected to four edges that are adjacent to *i* and so it makes a graph that is isomorphic to *F*. (see figure 2)

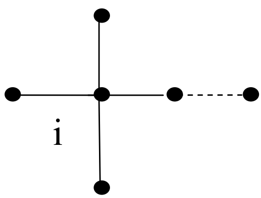

Figure 2. The case T in fig.1

Therefore, we subtract N_F. So we have

$$N_T = \sum_{i=1}^{n} \binom{d_i}{4} (m - d_i) - N_F$$

N_Y: For counting N_Y , we choose a vertex *i* from *V(G)* and then select a subset *{k,t}* from *N(i)*. We select an edge adjacent to *k* and two edges adjacent to *t*, or conversely, except edges connecting *k,t* to *i*. There fore N_Y is:

$$N_Y = \sum_{i} \sum_{\{k,t\} \subset N(i)} [\binom{d_t - 1}{2} (d_k - 1) + \binom{d_k - 1}{2} (d_t - 1)].$$

N_H: For counting N_H , we choose a vertex *i* from *V(G)* and then select a subset *{k,s,t}* from *N(i)*.after this we select an edge from <u>*k,t*</u> *,* <u>*k,s*</u> or <u>*s,t*</u> other than edges connecting *k,s,t* to *i*. Therefore N_H is:

$$N_H = \sum_{i} \sum_{\{k,s,t\} \subset N(i)} [(d_x - 1)(d_s - 1) + (d_k - 1)(d_t - 1) + (d_s - 1)(d_t - 1)].$$

N_q: For counting N_q, first select an edge *ij* from *E(G)*, then choose a vertex t from *N(j) – {i}* and choose a vertex *k* from *N(i) – {j}*. Then select an edge from each *t* and *k* *tj and ki.*

It is possible that the edge from *k* and the edge from *t* be adjacent to each other which this makes a graph isomorphic to P. (See figure 3) Hence we subtract the number of cases in which they do not form graphs isomorphic to p.

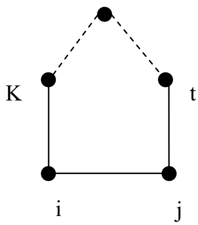

Figure3. subgraph p

In the above figure, we count N_P five times.

Thus:

$$N_q = \sum_{ij} \sum_{\substack{k \epsilon N(i) - \{j\} \\ t \epsilon N(j) - \{i\}}} (d_k - 1)(d_t - 1) - 5N_p$$

N_E : For counting N_E, we choose a vertex *i* from *V(G)* and then select three edges adjacent to *i*. After this we select two edges adjacent to each other except edges from *i*. But these two edges may be connected to edges of *i*. Now we subtract the number of cases in which these two edges are connected to edges of *i*. (See figure 4)

Figure 4. The case of E in figure 1

In the above counting we count N_X twice (fig.4–a) and N_Y once (fig.4–b). Therefore, N_E is:

$$N_E = \sum_{i=1}^{n} \binom{d_i}{3} [\binom{m - d_i}{2} - p\,(G - i, 2)] - 2N_X - N_Y.$$

N_Z: For counting N_Z, we choose a vertex i from *V(G)* and then select a subset *{k,t}* from *N(i).* Then select an edge from each *t* and *k*, except edges connecting *k, t* to *i*. Now we have a path of length four (*P5*). The number of subgraphs in *G* that are isomorphic to *P5* is

$$\sum_{i} \sum_{\{k,t\}\subset N(i)} (d_k - 1)(d_t - 1).$$

After selecting *P5*, we choose one edge of graph that does not belong to *P5*. This edge must not be connected to the edges of *P5*. Therefore, we subtract the number of graphs in which this single edge is connected to *P5*. (see figure 5).

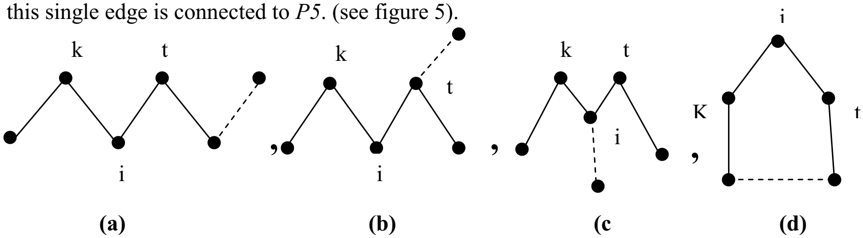

Figure 5. The case of Z in figure 1

Considering the above figure, we count N_q and N_Y two times (fig.5–a,b) and N_H one time (fig.5–c) and N_P five times (fig.5–d).

Thus:

$$N_Z = (m-4)\sum_i \sum_{\{k,t\}\subset N(i)} (d_k - 1)(d_t - 1) - 2N_q - 2N_Y - N_H - 5N_P.$$

N_L: For counting N_L, we choose a vertex *i* from *V(G)* and then select a subset *{k,s,t}* from *N(i)*. Then we select an edge which adjacent to *k, s* or *t*, exept edges connecting *k,s,t* to *i*. Finally, we choose another edge, except mentioned edges. The last edge must not be connected to edges of *i, t, s* and *k*. (see figure 6). Hence, subtract the number of cases in which they do not form graphs isomorphic to *L*.

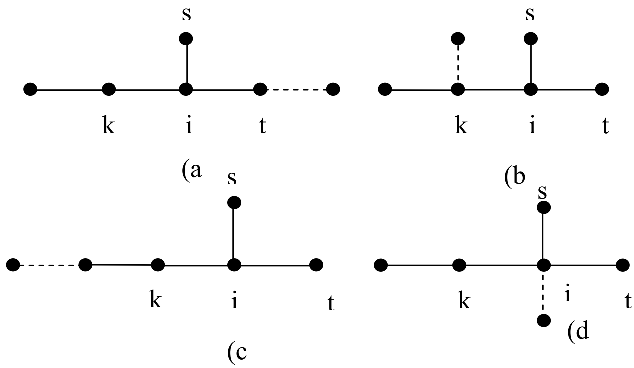

Figure 6. The case of L in figure 1

the above counting we count N_H twice (fig.6–a) and N_Y once (fig.6–b) and N_X four times (fig.6–c) and N_F three times (fig.6–d). Therefore, N_L is:

$$N_L = (m-4)\sum_i \sum_{\{k,s,t\}\subset N(i)} (d_k + d_s + d_t - 3) - 2N_H - N_Y - 4N_X - 3N_F.$$

N_M: For counting N_M, we select an edge *ij*, then choose two edges that are adjacent to it. Then select a 2–matching from *G-{i,j}*. Now subtract the number of cases in which edges of the 2–matching are connected to edges that are adjacent to *ij*. (see figure 7).

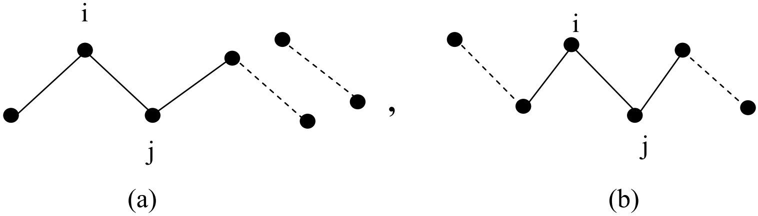

Figure 7. The case of M in figure 1

In the above figure , we count N_Z twice (fig.7–a) and N_q once (fig.7–b). Therefore, N_M is:

$$N_M = \sum_{ij} (d_i - 1)(d_j - 1)p(G - i - j, 2) - 2N_Z - N_q.$$

N_K: For counting N_K, choose a vertex *i* of *V(G)* and select two edges adjacent to *i.* Then select a 3–matching from *G-i*. Now subtract the number of cases in which edges of the 3–matching are connected to edges of *i.* (see figure 8)

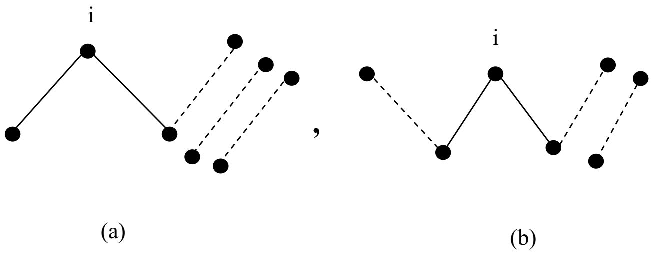

Figure 8. The case of K in figure 1

In the above we count N_M twice (fig.8–a) and N_Z once (fig.8–b). Therefore, N_K is:

$$N_K = \sum_{i=1}^{n} \binom{d_i}{2} P(G - i, 3) - 2N_M - N_Z.$$

N_V: For counting the number of graphs which are isomorphic to V, choose a vertex *i* of *V(G*) and select three edges adjacent to *i.* Then select a 2–matching from *G–i.* Now subtract the number of cases in which edges of the 2–matching are connected to edges of *i*. (see figure 9).

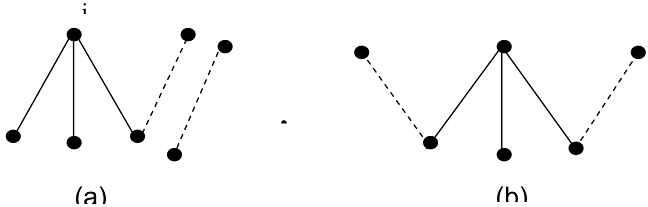

Figure 9.

Considering the above figure, we count N_L and N_H once (fig.9–a,b).

Thus:

$$N_V = \sum_{i=1}^{n} \binom{d_i}{3} p(G - i, 2) - N_L - N_H.$$

N_I: For counting N_I, first select an edge *ij* from *E(G),* then choose two edges that are adjacent to *ij*. Hence we have a path of length three *(P4).* After selecting *P4,* we choose two edges adjacent to each other of graph *G* that does not belong to *P4.* Now, subtract the number of cases in which two adjacent edges are connected to at least one edge of *P4* except *ij* (see figure 10).

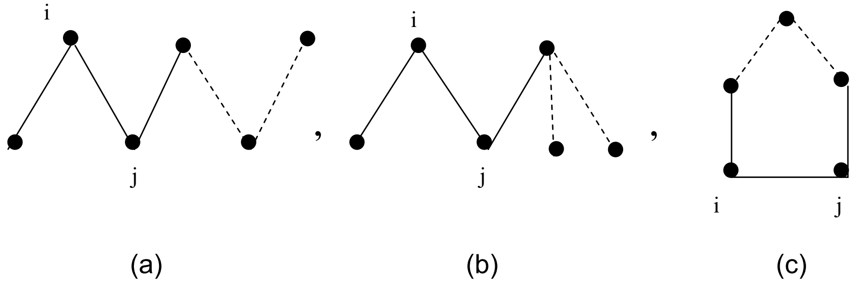

Figure 10. The case of Z in figure 1

In mentioned process, we count N_q twice (fig.10–a) and N_Y once (fig.10–b) and N_P five times (fig.10–c). Thus:

$$N_I = \sum_{ij} (d_i - 1)(d_j - 1)[\binom{m - d_i - d_j + 1}{2} - P(G - I - j, 2)] - 2N_q - N_Y - 5N_P$$

N_J: In this case, first select a subset *{i,j}* from *V(G),* then choose two edges from *i* and *j*, too Then choose another edge that is not adjacent to *i* and *j.*

Now subtract the number of cases in which they do not from graphs isomorphic to J. These cases are as follows:

1. One of the edges of *i* is connected to one of *j*.
2. One edge of *i* is connected to *j*.
3. The last single edge is connected to edges of *i* and *j.*

Cases of 1,2 and 3 are shown in figure 11.

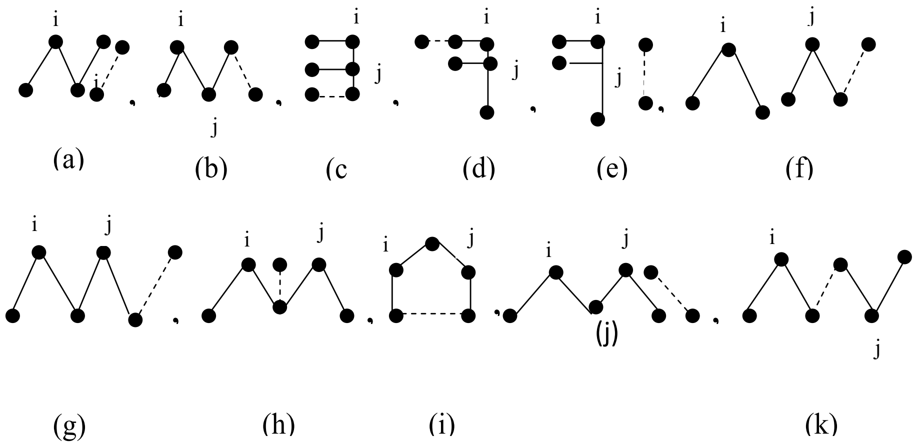

Figure 11.

In the above we count *N(a)* once (fig.11–a) and *N(b)* twice (fig.11–b). so we have:

$$N(a) + 2N(b) = \sum_{ij} (d_i - 1)(d_j - 1)(m - d_i - d_j + 1)$$

Where *N(a)* and *N(b)* denote the number of subgraphs of *G* that are isomorphic to (a) and (b) respectively. Also we count N_H twice (fig.11–c), N_Y once (fig.11–d), N_L once (fig.11–e), N_I twice (fig.11–f), N_q twice (fig.11–g), N_H once (fig.11–h), N_P five times (fig.11–i), N_Z once (fig.11–j) and N_q once (fig.11–k).

Thus:

$$N_J = \sum_{\{i,j\}\subset V} \binom{d_i}{2}\binom{d_j}{2} P(G-i-j,1) - \sum_{ij}(d_i - 1)(d_j - 1)(m - d_i - d_j + 1)$$
$$- 3N_H - N_Y - N_L - 2N_I - 3N_q - 5N_P - N_Z.$$

Now, the number of 5–matching is:

$$p(G,5) = \binom{m}{5} - N_P - N_q - N_s - N_T - N_X - N_Y - N_Z - N_V - N_H - N_E - N_F - N_L$$
$$- N_K - N_M - N_I - N_J$$

The result is obtained by direct substitution into above formula.

Corollary 3.2. Let G be a triangular–free and 4–cycle–free graph with n vertices which is regular of valency d.

Then,

$$P(G,5) = \binom{m}{5} - n\binom{d}{5} - n\,(m + 11\,d - 12)\binom{d}{4}$$
$$+ n\left[6\,(d-1)^2 + 3\,(m-4)(d-1) - \binom{m-d}{2}\right]\binom{d}{3}$$
$$+ n\,[dp\,(G,2) - P(G,3) - d^2\,(m-2d+1) - d(d-1)(m-3d+2)$$
$$+ (d-1)^2\,(d-2) - (m-4)(d-1)^2]\binom{d}{2} - \left[(m - 2d)\binom{n}{2} + m\right]\binom{d}{2}^2$$
$$+ m\,(d-1)^2\binom{m-2d+2}{2} - 3m\binom{d-1}{2}^2 + m\,(d-1)^4 - N_P$$

Proof. In this case we have

$$\sum_{i=1}^{n}\binom{d_i}{5} = n\binom{d}{5}$$

$$\sum_{i=1}^{n}\binom{d_i}{4}(m - d_i) = n\,(m-d)\binom{d}{4}$$

$$\sum_{i=1}^{n}\binom{d_i}{3}\binom{m - d_i}{2} = n\binom{m-d}{2}\binom{d}{3}$$

$$\sum_{i=1}^{n}\binom{d_i}{2} p(G-i,3)$$
$$= [p(G,3) - dp(G,2) + d^2\,(m-2d+1)$$
$$+ d(d-1)(m-3d+2)].\,n.\binom{d}{2}$$

$$\sum_{ij}(d_i - 1)(d_j - 1)\binom{m - d_i - d_j + 2}{2} = m\,(d-1)^2\binom{m-2d+2}{2}$$

$$\sum_{ij}\binom{d_i - 1}{2}\binom{d_j - 1}{2} = m\binom{d-1}{2}^2$$

$$\sum_{ij}\sum_{\substack{k \in N(i)-\{j\} \\ t\in N(j)-\{i\}}}(d_k - 1)\,(d_t - 1) = m\,(d-1)^4$$

$$\sum_{i}\sum_{\{k,t\}\subset N(i)}[\binom{d_t - 1}{2}(d_k - 1) + \binom{d_k - 1}{2}(d_t - 1)] = 2n\,(d-1)\binom{d-1}{2}\binom{d}{2}$$

$$\sum_{i}\sum_{\{k,t\}\subset N(i)}(d_k - 1)(d_t - 1) = n\,(d-1)^2\binom{d}{2}$$

$$\sum_{i}\sum_{\{k,s,t\}\subset N(i)}[(d_k - 1)\,(d_s - 1) + (d_k - 1)(d_t - 1) + (d_s - 1)(d_t - 1)]$$

$$= 3n\,(d-1)^2\binom{d}{3}$$

$$\sum_{\{i,j\}\subset V}\binom{d_i}{2}\binom{d_j}{2}\,p(G-i-j,1) = \left[\binom{n}{2}(m-2d) + m\right]\binom{d}{2}^2$$

$$\sum_{i}\sum_{\{k,s,t\}\subset N(i)}(d_k + d_s + d_t - 3) = 3n\,(d-1)\binom{d}{3}$$

$$\sum_{i}\sum_{\{k,s,t,r\}\subset N(i)}(d_k + d_s + d_t + d_r - 4) = 4n\,(d-1)\binom{d}{4}$$

The result is obtained by direct substitution into the formula for *p(G,5)*, given in the theorem.

Corollary 3.3. Let *G* and *H* be triangular–free and 4–cycle–free regular graphs and suppose that *G* and *H* are co–matching graphs. Then the number of 5–cycles in *G* and *H* are equal.

Proof: From $\mu(G,x) = \mu(H,x)$, we deduce that

p(G,5) = p(H,5) and it follows from corollary 3.2. and lemma 2.2. that $\mathrm{N_P(G)} = \mathrm{N_P(H)}$.

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