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On the Determinant of the Adjacency Matrix of a Type of Plane Bipartite Graphs

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Abstract

Let G be a simple graph and $A(G)$ its adjacency matrix. Based on some results of Rara (H. M. Rara, *Discr. Math.* **151** (1996) 213–219), we show that the determinant of $A(G)$ of a plane graph G which has the property that every face-boundary is a cycle of size divisible by 4, equals $-1, 0$ or 1, provided the inner dual graph of G is a tree. As applications, we compute the algebraic structure count of some polygonal chains.

1. INTRODUCTION

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix of graph G is an $n \times n$ $(0,1)$-matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if and only if (v_i, v_j) is an edge of G and $a_{ij} = 0$ otherwise. Let $d_G(u)$ be the degree of vertex u of G. If $V_1 \subset V(G)$ and $E_1 \subset E(G)$, we use $G - V_1$ and $G - E_1$ to denote the subgraphs of G induced by $V(G)\backslash V_1$ and $E(G)\backslash E_1$, respectively. Particulary, if $V_1 = \{u\}$ and $E_1 = \{e\}$, we use $G - u$ and $G - e$ to denote $G - \{u\}$ and $G - \{e\}$. Let $G^{\perp}$ be the dual graph of a plane graph G and f the vertex of $D^{\perp}$ corresponding to the unbounded face of G. Call $G^{\perp} - f$ to be the inner dual graph of G, denoted by G^* (see Figure 1).

Deift and Tomei [5] proved an interesting result: The determinant of the adjacency matrix of a finite subgraph G of $\mathbb{Z} \times \mathbb{Z}$ equals $-1, 0$ or 1, provided G has no "hole". This

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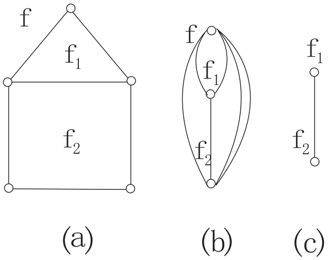


Figure 1: (a). A plane graph G with three faces f, f_1 and f_2. (b). The dual graph $G^{\perp}$ of G. (c). The inner dual graph G^* of G.

implies that the algebraic structure count of a finite subgraph G of $\mathbb{Z} \times \mathbb{Z}$ equals 0 or 1, provided G has no "hole" (The algebraic structure count of a bipartite graph G is defined as the square root of the absolute value of $\det(A(G))$, see [6,13,14]). For some results on the determinant of the adjacency matrix of graphs (resp. the algebraic structure count of bipartite graphs) see for example [1,3,7,11,12]. (resp. [3,8–10]).

A polyomino system is a finite 2-connected plane graph such that each interior face is surrounded by a regular square of length one. A special case of the above result by Deift and Tomei is that if G is a polyomino system whose inner dual is a tree, then the determinant of $A(G)$ equals $-1, 0$ or 1. It is natural to ask whether there exists a similar result for the determinant of $A(G)$ of a plane graph G which has the property that every face-boundary is a cycle of size divisible by 4, provided the inner dual graph of G is a tree. The main result of this short note, Theorem 2.6, answers this question in the affirmative. Finally, as applications we compute the algebraic structure count of some polygonal chains.

2. Main results

We use P_n and C_n to denote the path and cycle with n vertices. Now, we introduce some known lemmas.

Lemma 2.1. [12] Let $P_6 = [1, 2, 3, 4, 5, 6]$ be an induced subgraph of G with $d_G(2) = d_G(3) = d_G(4) = d_G(5) = 2$. If H is the graph formed from $G - \{2, 3, 4, 5\}$ by joining vertices 1 and 6 with an edge, then

$$\det(A(G)) = \det(A(H)) .$$

Lemma 2.2. [12] Let $C_4 = [v_1, v_2, v_3, v_4, v_1]$ be a subgraph of G with $d_G(v_1) = 2$. If G' is the graph obtained from G by removing the edges v_2v_3 and v_3v_4, then

$$\det(A(G')) = \det(A(G)) \ .$$

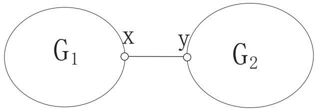


Figure 2: The graph G obtained from G_1 and G_2.

Lemma 2.3. [12] Let G be the graph obtained by joining the vertex x of the graph G_1 to the vertex y of the graph G_2 by an edge (see Fig. 2). Then

$$\det(A(G)) = \det(A(G_1)) \det(A(G_2)) - \det(A(G_1 - x)) \det(A(G_2 - y)) \ .$$

The following lemma is immediate from the lemma above.

Lemma 2.4. Let G be a graph and v be any vertex of G. If G^* is the graph obtained from G by joining v to a new vertex u, then

$$\det(A(G^*)) = -\det(A(G - v)) \ .$$

By induction, the following result follows from Lemma 2.4.

Corollary 2.5. If T is a tree with n vertices, then the determinant of $A(T)$ equals $(-1)^{n/2}$ if T has a perfect matching and zero otherwise.

Lemma 2.6. Let G be a plane graph each bounded face of which is a cycle with length equal to 0 (mod 4). If the inner dual G^* is a tree, then the determinant of the adjacency matrix of G equals $-1, 0$ or 1, i.e.,

$$\det(A(G)) = 0 \, , \, \pm 1 \ .$$

Proof. Since each bounded face of G is a cycle with even number of edges, G is a bipartite graph. First, we prove the following claim.

Claim. Let G be the graph obtained by joining the vertex x of the bipartite graph G_1 to the vertex y of the bipartite graph G_2 by an edge. If $\det(A(G_i)) = 0\,,\,\pm 1$, $\det(A(G_1 - x)) = 0\,,\,\pm 1$ and $\det(A(G_2 - y)) = 0\,,\,\pm 1$, then $\det(A(G)) = 0\,,\,\pm 1$. By Lemma 2.3,

$$\det(A(G)) = \det(A(G_1))\,\det(A(G_2)) - \det(A(G_1 - x))\,\det(A(G_2 - y))\ . \tag{1}$$

If $\det(A(G_1))\,\det(A(G_2)) = 0$, then by (1) the claim holds. If $\det(A(G_1)) = \pm 1$ and $\det(A(G_2)) = \pm 1$, then both $|V(G_1 - x)|$ and $|V(G_2 - y)|$ are odd, implying $\det(A(G_1 - x)) = \det(A(G_2 - y)) = 0$. So the $\det(A(G)) = \pm 1$. Hence the claim follows.

Now we prove the theorem by induction on $|V(G)|$, the number of vertices of G. If G contains no cycle, that is, $|V(G^*)| = 0$, then, by Corollary 2.5, $\det(A(G)) = 0\,,\,\pm 1$. If G has a cut edge $e = (u, v)$, then by induction and the claim above, the theorem follows. Hence we may assume that G is 2-edge connected. Note that the inner dual G^* is a tree. If $|V(G^*)| = 1$, then G is a cycle with $4s$ vertices for some integer s. Obviously, $\det(A(C_{4s})) = 0$. Hence we suppose that $|V(G^*)| \geq 2$. Let f be a vertex of degree one of G^*. Then f can be regarded as a bounded face of G whose boundary is a cycle with $4k$ vertices for some integer k. Hence G has the form of the graphs illustrated in Figure 3, where G_0 is plane graph each bounded face of which is a cycle with length equal to 0 (mod 4) and $G_0^* = G^* - f$ is a tree. We distinguish the following two cases.

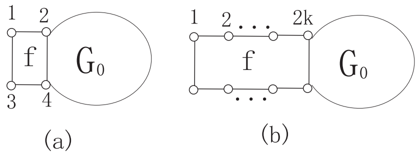


Figure 3: (a). The graph G_1 with a face f whose boundary is a cycle with four vertices. (b). The graph G_2 with a face f whose boundary is a cycle with $4k$ vertices ($k \geq 2$).

Case 1. $k = 1$.

If $k = 1$, G is of the form of graph G_1 shown in Figure 3(a). Since $d_{G_1}(1) = 2$ and $1 - 2 - 4 - 3 - 1$ is a cycle of $G = G_1$, by Lemma 2.2, $\det(A(G)) = \det(A(G_1)) = \det(A(G_1 - e_1 - e_2))$, where $e_1 = (2, 4)$ and $e_2 = (3, 4)$. By Lemma 2.4,

$$\det(A(G_1 - e_1 - e_2)) = -\det(A(G_0 - e_1))\ .$$

Note that $G_0 - e_1$ is a plane graph each bounded face of which is a cycle with length equal to 0 (mod 4). Moreover, the inner dual graph of $G_0 - e_1$ is a forest. By induction, $\det(A(G_0 - e_1)) = 0\,,\,\pm 1$. Hence $\det(A(G)) = \det(A(G_1)) = 0\,,\,\pm 1$.

Case 2. $k \geq 2$.

If $k \geq 2$, then by a repeated application of Lemma 2.1, $\det(A(G)) = \det(A(G_2)) = \det(A(G_1))$. By Case 1, $\det(A(G)) = 0\,, \pm 1$.

Hence we have completed the proof of the theorem. ■

The following result is immediate from the theorem above.

Corollary 2.7. Let G be a plane graph each bounded face of which is a cycle with length equal to 0 (mod 4). If the inner dual G^* is a tree, then the algebraic structure count of G equals 0 or 1.

3. Applications

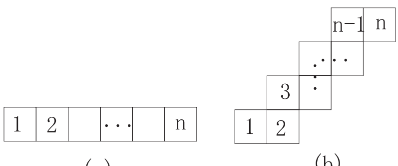
As applications of Theorem 2.6, in this section, we compute the determinant of adjacency matrices of some polygonal chains.

The following lemma is well-known.

Lemma 3.1. [4] Let G be a simple graph and $A(G)$ the adjacency matrix. Then $\det(A(G)) \equiv 0$ (mod 2) if and only if G has an even number of perfect matchings.

Lemma 3.2. [2,4] The spectrum of a bipartite graph is symmetric with respect to zero.

Remark 3.3. By Lemmas 3.1 and 3.2, if a bipartite graph G with n vertices satisfies the property $\det(A(G)) = 0$ or ± 1, then $\det(A(G)) = 0$ if G has an even number of perfect matchings and $\det(A(G)) = (-1)^{n/2}$ otherwise.



(a) (b)

Figure 4: (a). The linear polyomino chain L_n. (b). The zigzag polyomino chain Z_n.

Let L_n and Z_n denote the linear polyomino chain and zigzag polyomino chain with n squares, which are illustrated in Figure 4. Let L_n^{4k} and Z_n^{4k} be the linear polygonal chain and zigzag polygonal chain with n polygons of size $4k$, which are illustrated in Figure 5. Obviously, $L_n = L_n^4$ and $Z_n = Z_n^4$.

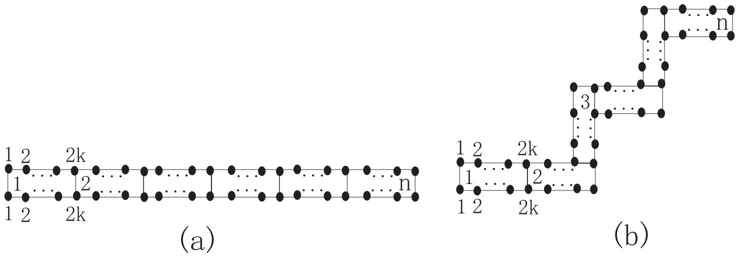


Figure 5: (a). The linear polygonal chain L_n^{4k} with n polygons of size $4k$. (b). The zigzag polygonal chain Z_n^{4k} with n polygons of size $4k$.

Theorem 3.4. The determinants of adjacency matrices of L_n^{4k} and Z_n^{4k} $(k \geq 1)$ are

$$\det(A(L_n^{4k})) = \begin{cases} 0 & \text{if } n \equiv 1 \ (mod\ 3) \\ (-1)^{n+1} & \text{if } n \equiv 0, 2 \ (mod\ 3) \end{cases}$$

$$\det(A(Z_n^{4k})) = \begin{cases} -1 & \text{if } n \equiv 0 \ (mod\ 2) \\ 0 & \text{if } n \equiv 1 \ (mod\ 2)\ . \end{cases}$$

Proof. By Theorem 2.6, we know that $\det(A(L_n^{4k})) = 0\,, \pm 1$ and $\det(A(Z_n^{4k})) = 0\,, \pm 1$. Hence by Lemma 3.1 we only need to enumerate perfect matchings of L_n^{4k} and Z_n^{4k}. Let a_n and b_n be the number of perfect matchings of L_n^{4k} and Z_n^{4k}. It is easy to obtain the following recurrences:

$$\begin{cases} a_n = a_{n-1} + a_{n-2} & \text{if } n \geq 3 \\ a_1 = 2\,,\ a_2 = 3 \end{cases} \qquad \begin{cases} b_n = b_{n-1} + 1 & \text{if } n \geq 2 \\ b_1 = 2\ . \end{cases}$$

Hence $b_n = n + 1$ and $\{a_n\}$ is the Fibonacci sequence. We know easily that a_n is even if $n \equiv 1$ $(mod\ 3)$ and a_n is odd otherwise, and b_n is even if $n \equiv 1$ $(mod\ 2)$ and b_n is odd otherwise. Note that both L_n^{4k} and Z_n^{4k} have $(4k-2)n+2$ vertices. By Remark 3.2,

$$\det(A(L_n^{4k})) = \begin{cases} 0 & \text{if } n \equiv 1 \ (mod\ 3) \\ (-1)^{\frac{(4k-2)n+2}{2}} & \text{if } n \equiv 0, 2 \ (mod\ 3) \end{cases}$$

$$\det(A(Z_n^{4k})) = \begin{cases} (-1)^{\frac{(4k-2)n+2}{2}} & \text{if } n \equiv 0 \ (mod\ 2) \\ 0 & \text{if } n \equiv 1 \ (mod\ 2) \end{cases}$$

implying the theorem holds. ■

Corollary 3.5. The algebraic structure count of L_n^{4k} and Z_n^{4k} $(k \geq 1)$ are

$$\det(A(L_n^{4k})) = \begin{cases} 0 & \text{if } n \equiv 1 \ (mod\ 3) \\ 1 & \text{otherwise.} \end{cases}$$

$$\det(A(Z_n^{4k})) = \begin{cases} 1 & \text{if } n \equiv 0 \ (mod\ 2) \\ 0 & \text{otherwise.} \end{cases}$$

4. Some remarks

Deift and Tomei [5] proved that the determinant of the adjacency matrix of a subgraph G of $\mathbb{Z} \times \mathbb{Z}$ equals $-1, 0$ or 1, provided G contains no "hole". Note that if G is a subgraph G of $\mathbb{Z} \times \mathbb{Z}$ and has no "hole", then each bounded face of G is a cycle C_4. So it is nature to ask whether there exists a similar result for the determinant of the adjacency matrix of plane graph each bounded face of which is a cycle with length $4k$. In Section 2, we have obtained a more special result by showing that the determinant of $A(G)$ of a plane graph G which has the property that every face-boundary is a cycle of length of the form $4k$ $(k = 1, 2, \cdots)$, equals $-1, 0$ or 1, provided the inner dual graph of G is a tree. But The following Example 4.1 gives a negative answer for the above question.

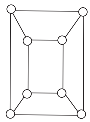


Figure 6: The graph $G = C_4 \times P_2$.

Example 4.1. Let G be the graph shown in Figure 6, that is, G is the Cartesian product of C_4 and P_2. Although each bounded face of G is a cycle C_4, G is not a subgraph of $\mathbb{Z} \times \mathbb{Z}$. Obviously, $\det(A(G)) = 9$.

The following classifying theorem follows directly form Theorem 2.6 and Remark 3.3.

Theorem 4.2. Let G be a plane graph with n vertices each bounded face of which is a cycle with length equal to 0 $(mod\ 4)$. If the inner dual G^* is a tree, then

(1). The determinant of the adjacency matrix of G equals 0 if G has an even number of perfect matchings and $(-1)^{n/2}$ otherwise.

(2). The algebraic structure count of G equals 0 if G has an even number of perfect matchings and 1 otherwise.

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