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A Note on the Randić Spectral Radius*

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Abstract

Let G be a simple connected graph on n vertices and d_i be the degree of its vertex v_i , , $i = 1, 2, \ldots, n$. The (i, j)-entry of the Randić matrix of G is equal to $1/\sqrt{d_i d_j}$ if v_i and v_j are adjacent, and zero otherwise. In this paper, we obtain that the Randić spectral radius of G is $\rho_1(G) = 1$, which improves the results of Bozkurt et al. [*MATCH Commum. Math. Comput. Chem.* **64** (2010) 321–334].

1 Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let d_i denote the degree of vertex v_i, where $i = 1, 2, \ldots, n$. Use the notation $v_i \sim v_j$, if two vertices v_i and v_j of G are adjacent. The Randić matrix of G and the Randić degree of vertex v_i [1, 2] is defined as

$$\mathbf{R}(G) = (r_{ij})_{n\times n} \qquad \text{and} \qquad R_i = \sum_{j=1}^{n} r_{ij}$$

respectively, where

$$r_{ij} = \begin{cases} (d_i d_j)^{-\frac{1}{2}} & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

The Randić matrix is real symmetric, so that its eigenvalues can be arranged as follows:

$$\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_{n-1}(G) \geq \rho_n(G).$$

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The Randić energy [1] of G is defined to be

$$RE(G) = \sum_{i=1}^{n} |\rho_i|.$$

On the other hand, the famous Randić index [3] was put forward by Milan Randić already in 1975, given by

$$R(G) = \sum_{v_i \sim v_j} \left(d_i d_j\right)^{-\frac{1}{2}}$$

where the sum run over all pairs of adjacent vertices v_i and v_j. Countless applications and the mathematical properties of the Randić index were reported (see e.g. [4–6]). Besides, it's clear to see the connection between Randić matrix and Randić index, that is,

$$R(G) = \frac{1}{2} \sum_{i=1}^{n} R_i .$$

However, Randić matrix which is also called "weighted adjacency matrix" or "degree-adjacency matrix" is not so popular. Only a few papers (e.g. [7–10]) referred to it for the study of Randić index. Recently, in [1,2], the role of Randić matrix in the Laplacian theory was clarified. Moreover, some lower bounds for the Randić spectral radius and some upper bounds for the Randić energy were obtained (see [1,2]). Especially, Bozkurt et al. [1] pointed out that the Randić matrix plays an outstanding role in the theory of Laplacian spectra.

The Laplacian matrix [11] of a graph G is defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{A}$ is the adjacency matrix of G and $\mathbf{D}$ is the diagonal matrix whose ith diagonal entry is the degree of vertex v_i. The normalized Laplacian matrix [12] is defined as $\tilde{\mathbf{L}} = \mathbf{D}^{-\frac{1}{2}} \mathbf{L} \mathbf{D}^{-\frac{1}{2}}$. It can be seen that the normalized Laplacian matrix and the Randić matrix of a connected graph G are related [2] as

$$\tilde{\mathbf{L}} = \mathbf{I} - \mathbf{R} \tag{1}$$

where $\mathbf{I}$ is the unit matrix of order n and $\mathbf{R}$ is the Randić matrix. Let the eigenvalues of $\tilde{\mathbf{L}}$ be

$$\tilde{\mu}_1(G) \geq \tilde{\mu}_2(G) \geq \cdots \geq \tilde{\mu}_{n-1}(G) \geq \tilde{\mu}_n(G) = 0 .$$

In this paper, we obtain that the Randić spectral radius of G is $\rho_1(G) = 1$, which improves the results of [2].

2 Main results

Theorem 2.1 *The Randić spectral radius* $\rho_1(G) = 1$.

Proof. By Equation (1), we have

$$\tilde{\mu}_i(G) = 1 - \rho_{n-i+1}(G)$$

for $i = 1, 2, \ldots, n$. Hence $\rho_1(G) = 1 - \tilde{\mu}_n(G) = 1 - 0 = 1$. □

Remark 1 *The second Randić degree of vertex v_i is defined as*

$$S_i = \sum_{j=1}^{n} r_{ij} R_j \, .$$

Some lower bounds for ρ_1 were given in [2] as follows:

$$\rho_1 \geq \sqrt{\frac{\sum_{i=1}^{n} \left(L_i^{(p+1)}\right)^2}{\sum_{i=1}^{n} \left(L_i^{(p)}\right)^2}} \geq \sqrt{\frac{\sum_{i=1}^{n} S_i^2}{\sum_{i=1}^{n} R_i^2}} \geq \sqrt{\frac{2}{n} \sum_{v_i \sim v_j} \frac{1}{d_i d_j}}$$

where $L_i^{(1)} = R_i^{\alpha}$ (α is a real number) and for $p \geq 2$,

$$L_i^{(p)} = \sum_{v_i \sim v_j} \frac{1}{\sqrt{d_i d_j}} L_j^{(p-1)} \, .$$

However, we show that $\rho_1(G) = 1$, so it is no need to give the bounds for ρ_1.

Lemma 2.2 ([2]) *If G is a simple connected graph on n vertices with Randić spectral radius ρ_1 and Randić energy $RE(G)$, then*

$$RE(G) \leq \rho_1 + \sqrt{(n-1)\left[2 \sum_{v_i \sim v_j} \frac{1}{d_i d_j} - \rho_1^2\right]} \, . \qquad (2)$$

By Theorem 2.1 and Inequality (2), we obtain that

Corollary 2.3 *If G is a simple connected graph on n vertices with Randić energy $RE(G)$, then*

$$RE(G) \leq 1 + \sqrt{(n-1)\left[2 \sum_{v_i \sim v_j} \frac{1}{d_i d_j} - 1\right]} \, .$$

Remark 2 *Let*

$$f(x) = x + \sqrt{(n-1)\left[2 \sum_{v_i \sim v_j} \frac{1}{d_i d_j} - x^2\right]} \, .$$

By directly calculation, it can be seen that $f(x)$ is monotonically decreasing. From Theorem 2.1 and Remark 1,

$$\rho_1 = 1 \geq \sqrt{\frac{\sum_{i=1}^{n} \left(L_i^{(p+1)}\right)^2}{\sum_{i=1}^{n} \left(L_i^{(p)}\right)^2}} \geq \sqrt{\frac{\sum_{i=1}^{n} S_i^2}{\sum_{i=1}^{n} R_i^2}} \geq \sqrt{\frac{2}{n} \sum_{v_i \sim v_j} \frac{1}{d_i d_j}}$$

then we get

$$f(1) \leq f\left(\sqrt{\frac{\sum_{i=1}^{n}\left(L_i^{(p+1)}\right)^2}{\sum_{i=1}^{n}\left(L_i^{(p)}\right)^2}}\right) \leq f\left(\sqrt{\frac{\sum_{i=1}^{n} S_i^2}{\sum_{i=1}^{n} R_i^2}}\right).$$

Hence the upper bound of $RE(G)$ in Corollary 2.3 is better than the bounds given in [2].

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