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Estrada Index of Random Graphs*

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Abstract: The Estrada index of a graph G of order n is defined as $\mathbb{EE}(G) = \sum_{i=1}^{n} e^{\lambda_i}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the graph G. By the limiting behavior of the spectrum of random symmetric matrices, we formulate an exact estimate to $\mathbb{EE}(G)$ for almost all graphs G, and establish a lower bound and an upper bound to $\mathbb{EE}(G)$ for almost all multipartite graphs G.

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of G is a $(0,1)$-matrix $\mathbf{A}(G) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if there is an edge between the vertices v_i and v_j, and 0 otherwise. Evidently, the adjacency matrix $\mathbf{A}(G)$ is a real symmetric matrix and its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real numbers. In what follows we assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The basic properties of graph eigenvalues can be referred to Cvetković et al. [7]. A graph-spectrum-based invariant, recently put forward by Estrada [9], is defined as

$$\mathbb{EE}(G) = \sum_{i=1}^{n} e^{\lambda_i},$$

which was introduced in 2000 as a molecular structure-descriptor. Since then, there were various applications of the Estrada index. Initially, it was used to quantify the degree

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of folding of long chain polymeric molecules, especially those of proteins [10, 11]. And later, a connection between $\mathbb{EE}(G)$ and the concept of extended atomic branching was established [14], which was an attempt to apply $\mathbb{EE}(G)$ in quantum chemistry. In addition, Estrada and Rodríguez-Velázquez showed that $\mathbb{EE}(G)$ provides a measure of the centrality of complex networks [12, 13]. Recently, a information-theoretical application of $\mathbb{EE}(G)$ was put forward. Carbó-Dorca endeavored to find connections between $\mathbb{EE}(G)$ and the Shannon entropy [6].

In spite of the fact that $\mathbb{EE}(G)$ has numerous practical applications, investigations of its basic properties started only short time ago [16, 18]. It is rather hard, as well-known, to compute the eigenvalues for a large matrix even for $\mathbf{A}(G)$. So, in order to estimate the invariant, researchers established some lower and upper bounds [4, 17, 24] by algebraic approaches in the last few years. However, there are, as examples given below, only a few classes of graphs attaining the equalities of those bounds. Consequently, one can hardly see the major behavior of $\mathbb{EE}(G)$ for most graphs. In this paper, however, we shall formulate an exact estimate to $\mathbb{EE}(G)$ for almost all graphs by investigating the random graphs constructed from classical Erdös-Rényi model [5]. Furthermore, we explore random multipartite graphs. As a matter of fact, the adjacency matrix of Erdös-Rényi model is a random matrix, which has been widely researched and generalized [2, 3, 8]. We shall use those generalizations to study random multipartite graphs [1]. We finally note that in [19] the authors have investigated the asymptotic behavior of the Estrada index for trees.

In Section 2, we consider the Erdös-Rényi model $\mathscr{G}_n(p)$ and formulate an exact estimate to $\mathbb{EE}(G_n(p))$ for almost all $G_n(p) \in \mathscr{G}_n(p)$ where p is a constant with $0 < p < 1$. To be precise, we show that

$$\mathbb{EE}(G_n(p)) = e^{np}\left(e^{O(\sqrt{n})} + o(1)\right) \quad a.s. \tag{1.1}$$

In Section 3, we investigate random m-partite graph model $\mathscr{G}_{n;\nu_1,\ldots,\nu_m}(p)$ and establish the estimate below to $\mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(p))$ for almost all $G_{n;\nu_1,\ldots,\nu_m}(p) \in \mathscr{G}_{n;\nu_1,\ldots,\nu_m}(p)$.

$$e^{np(1-\max\{\nu_1,\nu_2,\ldots,\nu_m\})}\left(e^{O(\sqrt{n})} + o(1)\right) \leq \mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(p)) \leq e^{np}\left(e^{O(\sqrt{n})} + o(1)\right) \quad a.s. \tag{1.2}$$

According to the following three examples, one can readily see that Eq. (1.1) is better than those established by algebraic approaches.

Example 1. Let G be a simple graph with n vertices and m edges. Peña et al. [22] proved

$$\sqrt{n^2+4m} \le \mathbb{EE}(G) \le n-1+e^{\sqrt{2m}}, \tag{1.3}$$

with equalities on both sides hold if and only if $G \cong K_n^c$ (the complement of the complete graph K_n). If $G \in \mathbb{EE}(G_n(p))$, then the number of edges of G is $m = \frac{pn(n-1)}{2}$ *a.s.* by the strong law of large numbers (see [23]). Hence the bounds in (1.3) become

$$\sqrt{n^2+4m} = \sqrt{(2p+1)n^2-2pn} \;\; a.s., \qquad n-1+e^{\sqrt{2m}} = n-1+e^{\sqrt{pn(n-1)}} \;\; a.s.$$

Consequently

$$\frac{\sqrt{n^2+4m}}{\mathbb{EE}(G_n(p))} = \frac{\sqrt{(2p+1)n^2-2pn}}{e^{np}\left(e^{O(\sqrt{n})}+o(1)\right)} = o(1) \;\; a.s.$$

$$\frac{\mathbb{EE}(G_n(p))}{e^{\sqrt{2m}}+n-1} = \frac{e^{np}\left(e^{O(\sqrt{n})}+o(1)\right)}{e^{\sqrt{pn(n-1)}}+n-1} = \frac{e^{np}\left(e^{O(\sqrt{n})}+o(1)\right)}{e^{np}\left(e^{(\sqrt{p}-p)n}+o(1)\right)} = o(1) \;\; a.s.$$

Example 2. Liu et al. [20] investigated the relations between the Estrada index $\mathbb{EE}(G)$ and the graph energy $\mathbb{E}(G)$ of a graph G as follows.

$$\mathbb{EE}(G) \le n-1+e^{\frac{\mathbb{E}(G)}{2}}, \tag{1.4}$$

with equality if and only if $\mathbb{E}(G)=0$ or equivalently $G = K_n^c$. By means of Nikiforov's result that $\mathbb{E}(G) = n^{\frac{3}{2}}\left(\frac{4}{3\pi}+o(1)\right)$ *a.s.* in [21], we have

$$\mathbb{EE}(G) \le n-1+e^{n^{\frac{3}{2}}\left(\frac{2}{3\pi}+o(1)\right)} \;\; a.s.$$

Therefore

$$\frac{\mathbb{EE}(G_n(p))}{e^{\frac{\mathbb{E}(G)}{2}}+n-1} = \frac{e^{np}\left(e^{O(\sqrt{n})}+o(1)\right)}{e^{n^{\frac{3}{2}}\left(\frac{2}{3\pi}+o(1)\right)}+n-1} = \frac{e^{np}\left(e^{O(\sqrt{n})}+o(1)\right)}{e^{np}\left(e^{n^{\frac{3}{2}}\left(\frac{2}{3\pi}+o(1)\right)-np}+o(1)\right)} = o(1) \;\; a.s.$$

Example 3. Let G be a simple graph with n vertices and m edges. Gutman [15] proved if $\frac{2m}{n} \ge 1$, then

$$\mathbb{EE}(G) \ge n\frac{e^{\sqrt{\frac{2m}{n}}}+e^{-\sqrt{\frac{2m}{n}}}}{2}, \tag{1.5}$$

with equality if and only if G is a regular graph of degree 1. If $G \in \mathbb{EE}(G_n(p))$, then the number of edges of G is $m = \frac{pn(n-1)}{2}$ *a.s.*, and the lower bound in (1.5) becomes $n\frac{e^{\sqrt{p(n-1)}}+e^{-\sqrt{p(n-1)}}}{2}$ *a.s.* Thus

$$\frac{\frac{n}{2}\left(e^{\sqrt{2m/n}}+e^{-\sqrt{2m/n}}\right)}{\mathbb{EE}(G_n(p))} = \frac{\frac{n}{2}\left(e^{\sqrt{p(n-1)}}+e^{-\sqrt{p(n-1)}}\right)}{e^{np}\left(e^{O(\sqrt{n})}+o(1)\right)} = \frac{e^{\sqrt{np}}\left(\frac{n}{2}+o(1)\right)}{e^{np}\left(e^{O(\sqrt{n})}+o(1)\right)} = o(1) \;\; a.s.$$

2 Estrada index of $G_n(p)$

As well known, the theory of random graphs was founded by Erdös and Rényi in 1950s. In this section, we shall formulate an exact estimate to the Estrada index for the classical Erdös-Rényi model. We first recall the definition of the Erdös-Rényi model: $\mathscr{G}_n(p)$ consists of all graphs on n vertices in which the edges are chosen independently with probability $p \in (0,1)$. Throughout this paper, we use $\mathbf{A}_n$ to denote the adjacency matrix $\mathbf{A}(G_n(p))$ of the random graph $G_n(p) \in \mathscr{G}_n(p)$. Apparently, $\mathbf{A}_n$ is a symmetric random matrix in which the diagonal entries are zeros while $a_{ij}\,(i<j)$ is 1 or 0, with probability p or $1-p$. Therefore, one can readily evaluate $\mathbb{EE}(G_n(p))$ once the spectral distribution of the random matrix $\mathbf{A}_n$ is known. By virtue of the following two lemmas, we shall describe the spectral distribution of the random matrix $\mathbf{A}_n$ based on the centralizing random matrix $\overline{\mathbf{A}}_n = \mathbf{A}_n - p(\mathbf{J}_n - \mathbf{I}_n)$, where $\mathbf{J}_n$ is $n\times n$ matrix in which all entries equal 1 and $\mathbf{I}_n$ is the unit $n\times n$ matrix. The limiting behavior of the spectrum of $\overline{\mathbf{A}}_n$ has been characterized and extensively researched [2, 3].

Lemma 2.1 (**Bai [2]**) *Let* $\overline{\mathbf{A}}_n = \mathbf{A}_n - p(\mathbf{J}_n - \mathbf{I}_n)$. *Then*

$$\lim_{n\to\infty} \| n^{-\frac{1}{2}}\overline{\mathbf{A}}_n \| = 2\sqrt{p(1-p)} \quad a.s.$$

where $\| n^{-\frac{1}{2}}\overline{\mathbf{A}}_n \|$ *is the spectral radius of the matrix* $n^{-\frac{1}{2}}\overline{\mathbf{A}}_n$.

Lemma 2.2 (**Weyl inequality [7]**) *Let* $\lambda_1(\mathbf{M}),\ldots,\lambda_n(\mathbf{M})$ *be the eigenvalues of a real symmetric matrix* $\mathbf{M}$ *of order* n*, where* $\lambda_1(\mathbf{M}) \geq \cdots \geq \lambda_n(\mathbf{M})$. *If* $\mathbf{X}$, $\mathbf{Y}$ *and* $\mathbf{Z}$ *are all real symmetric matrices of order* n*, and* $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$, *then*

$$\max_{r+s=j+n}\{\lambda_r(\mathbf{X}) + \lambda_s(\mathbf{Y})\} \leq \lambda_j(\mathbf{Z}) \leq \min_{r+s=j+1}\{\lambda_r(\mathbf{X}) + \lambda_s(\mathbf{Y})\}.$$

Theorem 2.3 *Let* $G_n(p)$ *be a random graph of* $\mathscr{G}_n(p)$. *Then* $\mathbb{EE}(G_n(p))$ *enjoys the following*

$$\mathbb{EE}(G_n(p)) = e^{np}\left(e^{O(\sqrt{n})} + o(1)\right) \quad a.s.$$

Proof. Recalling the definition of $\mathbb{EE}(G_n(p))$, we just need to formulate an estimate to $\mathbb{EE}(\mathbf{A}_n)$. For the spectral distribution of $\mathbf{A}_n$, one can derive from Lemma 2.2 that

$$p\lambda_i(\mathbf{J}_n - \mathbf{I}_n) + \lambda_n(\overline{\mathbf{A}}_n) \leq \lambda_i(\mathbf{A}_n) \leq p\lambda_i(\mathbf{J}_n - \mathbf{I}_n) + \lambda_1(\overline{\mathbf{A}}_n).$$

Obviously, by Lemma 2.1

$$-2\sqrt{p(1-p)}+o(1)\le\frac{\lambda_i(\overline{\mathbf{A}}_n)}{\sqrt{n}}\le 2\sqrt{p(1-p)}+o(1)\quad a.s.$$

Together with the fact that $\lambda_1(\mathbf{J}_n-\mathbf{I}_n)=n-1$, $\lambda_2(\mathbf{J}_n-\mathbf{I}_n)=\cdots=\lambda_n(\mathbf{J}_n-\mathbf{I}_n)=-1$, we obtain

$$\lambda_1(\mathbf{A}_n)=np+O(\sqrt{n})\quad a.s., \tag{2.1}$$

and for $i=2,\ldots,n$,

$$\lambda_i(\mathbf{A}_n)=O(\sqrt{n})\quad a.s. \tag{2.2}$$

It is easily seen that

$$\begin{aligned}\mathbb{EE}(\mathbf{A}_n)&=\sum_{i=1}^{n}e^{\lambda_i(\mathbf{A}_n)}\\&=e^{\lambda_1(\mathbf{A}_n)}+\sum_{i=2}^{n}e^{\lambda_i(\mathbf{A}_n)}\\&=e^{np+O(\sqrt{n})}+\sum_{i=2}^{n}e^{O(\sqrt{n})}\\&=e^{np+O(\sqrt{n})}+(n-1)e^{O(\sqrt{n})}\quad a.s.\end{aligned}$$

Therefore

$$\frac{\mathbb{EE}(\mathbf{A}_n)}{e^{np}}=e^{O(\sqrt{n})}+(n-1)\frac{e^{O(\sqrt{n})}}{e^{np}}=e^{O(\sqrt{n})}+o(1)\quad a.s.$$

Thus

$$\mathbb{EE}(\mathbf{A}_n)=e^{np}\left(e^{O(\sqrt{n})}+o(1)\right)\quad a.s. \tag{2.3}$$

as desired. ■

3 Estrada index of the random multipartite graphs

We begin with the definition of random multipartite graphs. We use $K_{n;\nu_1,\nu_2,\ldots,\nu_m}$ to denote the complete m-partite graph with vertex set V whose parts $V_1,\ldots,V_m$ ($2\le m=m(n)<n$) are such that $|V_i|=n\nu_i=n\nu_i(n)$, $i=1,\ldots,m$. Let $\mathscr{G}_{n;\nu_1,\ldots,\nu_m}(p)$ be the random m-partite graphs with vertex set V in which the edges are chosen independently with probability p from the set of edges of $K_{n;\nu_1,\ldots,\nu_m}$. In this section, we denote by $\mathbf{A}_{n,m}$ the adjacency matrix $\mathbf{A}(G_{n;\nu_1,\ldots,\nu_m}(p))=(x_{ij})_{n\times n}$ of random m-partite graph $G_{n;\nu_1,\ldots,\nu_m}(p)\in\mathscr{G}_{n;\nu_1,\ldots,\nu_m}(p)$. It is not difficult to verify that $\mathbf{A}_{n,m}$ satisfies the following properties

- $x_{ij}s$, $1 \le i < j \le n$, are independent random variables with $x_{ij} = x_{ji}$;
- $x_{ij} = 1$ with probability p and $x_{ij} = 0$ with probability $1-p$ if $i \in V_l$ and $j \in V \backslash V_l$, while $x_{ij} = 0$ if i and $j \in V_l$, where $V_1, \ldots, V_m$ are the parts of V and l is an integer with $1 \le l \le m$.

Apparently, in order to evaluate $\mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(p))$, we only need to investigate the spectral distribution of the random matrix $\mathbf{A}_{n,m}$. The readers can refer to Anderson et al. [1] for more details on spectral distribution of the random matrix $\mathbf{A}_{n,m}$. Unfortunately, we haven't found concise results like Theorem 2.3 on the exact form of estimates for $\mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(p))$, which appears to be rather hard.

However, in this section we shall establish an upper and lower bound to $\mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(p))$ for almost all m-partite graphs via the results on spectral distribution of $\mathbf{A}_n, \mathbf{A}_{n\nu_1}, \ldots, \mathbf{A}_{n\nu_m}$, respectively, constructed from $G_n(p), G_{n\nu_1}(p), \ldots, G_{n\nu_m}(p)$.

Theorem 3.1 *Let $G_{n;\nu_1,\ldots,\nu_m}(p)$ be a random graph of $\mathscr{G}_{n;\nu_1,\ldots,\nu_m}(p)$. Then $\mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(p))$ enjoys the following*

$$e^{np(1-\max\{\nu_1,\nu_2,\ldots,\nu_m\})}\left(e^{O(\sqrt{n})}+o(1)\right) \le \mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(p)) \le e^{np}\left(e^{O(\sqrt{n})}+o(1)\right) \quad a.s.$$

Proof. From the definition of the random m-partite graph $G_{n;\nu_1,\ldots,\nu_m}(p)$, suppose that $G_{n;\nu_1,\ldots,\nu_m}(p)$ has vertex set parts $V_1, V_2, \ldots, V_m$ satisfying $|V_i| = n\nu_i$, $i = 1, 2, \ldots, m$. Obviously, the adjacency matrix $\mathbf{A}_{n,m}$ of the random m-partite graph $G_{n;\nu_1,\ldots,\nu_m}(p)$ satisfies

$$\mathbf{A}_{n,m} + \mathbf{A}'_{n,m} = \mathbf{A}_n,$$

where

$$\mathbf{A}'_{n,m} = \begin{pmatrix} \mathbf{A}_{n\nu_1} & & & \\ & \mathbf{A}_{n\nu_2} & & \\ & & \ddots & \\ & & & \mathbf{A}_{n\nu_m} \end{pmatrix}_{n\times n}.$$

In order to establish a bound to $\mathbb{EE}(\mathbf{A}_{n,m})$, we fist present bounds for each eigenvalue of $\mathbf{A}_{n,m}$ according to the spectral distribution of $\mathbf{A}_n$ and $\mathbf{A}'_{n,m}$.

For $\lambda_1(\mathbf{A}_{n,m})$, it follows from Lemma 2.2 that

$$\lambda_1(\mathbf{A}_n) - \lambda_1(\mathbf{A}'_{n,m}) \le \lambda_1(\mathbf{A}_{n,m}) \le \lambda_1(\mathbf{A}_n) - \lambda_n(\mathbf{A}'_{n,m}).$$

By Eq. (2.1) and Eq. (2.2),

$$np(1-\max\{\nu_1,\ldots,\nu_m\})+O(\sqrt{n})\leq\lambda_1(\mathbf{A}_{n,m})\leq np+O(\sqrt{n})\ \ a.s. \tag{3.1}$$

Thus

$$e^{np(1-\max\{\nu_1,\ldots,\nu_m\})+O(\sqrt{n})}\leq e^{\lambda_1(\mathbf{A}_{n,m})}\leq e^{np+O(\sqrt{n})}\ \ a.s. \tag{3.2}$$

For $\lambda_i(\mathbf{A}_{n,m})$, $i=2,3,\ldots,n-m$, due to Lemma 2.2, one can deduce that

$$\lambda_{i+m}(\mathbf{A}_n)-\lambda_{m+1}(\mathbf{A}^{'}_{n,m})\leq\lambda_i(\mathbf{A}_{n,m})\leq\lambda_i(\mathbf{A}_n)-\lambda_n(\mathbf{A}^{'}_{n,m}).$$

By Eq. (2.2),

$$\lambda_i(\mathbf{A}_{n,m})=O(\sqrt{n})\ \ a.s.$$

Hence

$$\sum_{i=2}^{n-m}e^{\lambda_i(\mathbf{A}_{n,m})}=(n-m-1)e^{O(\sqrt{n})}\ \ a.s. \tag{3.3}$$

For $\lambda_j(\mathbf{A}_{n,m})$, $j=n-m+1,\ldots,n$, owing to Lemma 2.2, we obtain

$$\lambda_j(\mathbf{A}_n)-\lambda_1(\mathbf{A}^{'}_{n,m})\leq\lambda_j(\mathbf{A}_{n,m})\leq\lambda_j(\mathbf{A}_n)-\lambda_n(\mathbf{A}^{'}_{n,m}).$$

One can derive from Eq. (2.1) and Eq. (2.2) that

$$-np\max\{\nu_1,\ldots,\nu_m\}+O(\sqrt{n})\leq\lambda_j(\mathbf{A}_{n,m})\leq O(\sqrt{n})\ \ a.s.$$

Therefore

$$me^{-np\max\{\nu_1,\ldots,\nu_m\}+O(\sqrt{n})}\leq\sum_{j=n-m+1}^{n}e^{\lambda_j(\mathbf{A}_{n,m})}\leq me^{O(\sqrt{n})}\ \ a.s.,$$

which is equivalent to

$$o(1)\leq\sum_{j=n-m+1}^{n}e^{\lambda_j(\mathbf{A}_{n,m})}\leq me^{O(\sqrt{n})}\ \ a.s. \tag{3.4}$$

We proceed to establish the bound for $\mathbb{EE}(\mathbf{A}_{n,m})$ according to (3.2), (3.3) and (3.4),

$$\begin{aligned}\frac{\mathbb{EE}(\mathbf{A}_{n,m})}{e^{np(1-\max\{\nu_1,\ldots,\nu_m\})}}&=\frac{e^{\lambda_1(\mathbf{A}_{n,m})}}{e^{np(1-\max\{\nu_1,\ldots,\nu_m\})}}+\frac{\sum_{i=2}^{n-m}e^{\lambda_i(\mathbf{A}_{n,m})}}{e^{np(1-\max\{\nu_1,\ldots,\nu_m\})}}+\frac{\sum_{j=n-m+1}^{n}e^{\lambda_j(\mathbf{A}_{n,m})}}{e^{np(1-\max\{\nu_1,\ldots,\nu_m\})}}\\&\geq\frac{e^{np(1-\max\{\nu_1,\ldots,\nu_m\})+O(\sqrt{n})}}{e^{np(1-\max\{\nu_1,\ldots,\nu_m\})}}+\frac{(n-m-1)e^{O(\sqrt{n})}}{e^{np(1-\max\{\nu_1,\ldots,\nu_m\})}}+\frac{o(1)}{e^{np(1-\max\{\nu_1,\ldots,\nu_m\})}}\\&=e^{O(\sqrt{n})}+o(1)\ \ a.s.\end{aligned} \tag{3.5}$$

Meanwhile

$$\begin{aligned}\frac{\mathbb{EE}(\mathbf{A}_{n,m})}{e^{np}} &= \frac{e^{\lambda_1(\mathbf{A}_{n,m})}}{e^{np}} + \frac{\sum_{i=2}^{n-m} e^{\lambda_i(\mathbf{A}_{n,m})}}{e^{np}} + \frac{\sum_{j=n-m+1}^{n} e^{\lambda_j(\mathbf{A}_{n,m})}}{e^{np}} \\ &\leq \frac{e^{np+O(\sqrt{n})}}{e^{np}} + \frac{(n-m-1)e^{O(\sqrt{n})}}{e^{np}} + \frac{me^{O(\sqrt{n})}}{e^{np}} \\ &= e^{O(\sqrt{n})} + o(1) \quad a.s. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$e^{np(1-\max\{\nu_1,\nu_2,\ldots,\nu_m\})}\left(e^{O(\sqrt{n})}+o(1)\right) \leq \mathbb{EE}(\mathbf{A}_{n,m}) \leq e^{np}\left(e^{O(\sqrt{n})}+o(1)\right) \quad a.s. \tag{3.7}$$

The theorem follows by (3.7). ■

The order of limiting behavior of $\mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(p))$ is bounded by (3.7). Furthermore, we would like to know in what cases the equalities in (3.7) hold.

COROLLARY 3.2 *Let $G_{n;\nu_1,\ldots,\nu_m}(p)$ be a random graph of $\mathscr{G}_{n;\nu_1,\ldots,\nu_m}(p)$. Then*

$$\mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(p)) = e^{np}\left(e^{O(\sqrt{n})}+o(1)\right) \quad a.s. \tag{3.8}$$

if and only if $\max\{n\nu_1,\ldots,n\nu_m\} = O(\sqrt{n})$.

Proof. From (3.6) together with (3.3) and (3.4), the equality (3.8) holds if and only if

$$\lambda_1(\mathbf{A}_{n,m}) = np + O(\sqrt{n}) \quad a.s. \tag{3.9}$$

and from (3.1) the equality (3.9) holds if and only if $\max\{n\nu_1,\ldots,n\nu_m\} = O(\sqrt{n})$. ■

Finally we consider the equality case of the lower bound in (3.7). We have the following conjecture, but fail to prove it. However, we compare the numerical value of $\mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(\frac{1}{2}))$ obtained by the software MATLAB with the value in (3.10) of Conjecture 3.3 (see Table 3.1), where $\nu_i = \frac{1}{m}$ for $i = 1,2,\ldots,m$. From Table 3.1 we find that the numerical value of $\mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(\frac{1}{2}))$ is close to the estimate of our conjecture *a.s.*

CONJECTURE 3.3 *Let $G_{n;\nu_1,\ldots,\nu_m}(p)$ be a random graph of $\mathscr{G}_{n;\nu_1,\ldots,\nu_m}(p)$. Then $\mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(p))$ enjoys*

$$\mathbb{EE}(G_{n;\nu_1,\ldots,\nu_m}(p)) = e^{np(1-\max\{\nu_1,\nu_2,\ldots,\nu_m\})}\left(e^{O(\sqrt{n})}+o(1)\right) \quad a.s., \tag{3.10}$$

if and only if

$$\lim_{n\to\infty} \min\{n\nu_1,\ldots,n\nu_m\} \to \infty \quad and \quad \lim_{n\to\infty}\frac{\nu_i}{\nu_j} = 1.$$

n	m=2		m=3		m=5	
	Numerical Value	*Theoretical Value*	*Numerical Value*	*Theoretical Value*	*Numerical Value*	*Theoretical Value*
4900	$e^{1277.7813}$	$e^{1225}\left(e^{O(70.0000)}+o(1)\right)$	$e^{1693.3834}$	$e^{1633}\left(e^{O(70.0000)}+o(1)\right)$	$e^{2024.8205}$	$e^{1960}\left(e^{O(70.0000)}+o(1)\right)$
5000	$e^{1302.6120}$	$e^{1250}\left(e^{O(70.7107)}+o(1)\right)$	$e^{1727.3981}$	$e^{1667}\left(e^{O(70.7107)}+o(1)\right)$	$e^{2066.5741}$	$e^{2000}\left(e^{O(70.7107)}+o(1)\right)$
5100	$e^{1328.7111}$	$e^{1275}\left(e^{O(71.4143)}+o(1)\right)$	$e^{1761.7126}$	$e^{1700}\left(e^{O(71.4143)}+o(1)\right)$	$e^{2106.5623}$	$e^{2040}\left(e^{O(71.4143)}+o(1)\right)$
5200	$e^{1354.3134}$	$e^{1300}\left(e^{O(72.1110)}+o(1)\right)$	$e^{1794.6330}$	$e^{1733}\left(e^{O(72.1110)}+o(1)\right)$	$e^{2147.2056}$	$e^{2080}\left(e^{O(72.1110)}+o(1)\right)$
5300	$e^{1379.9684}$	$e^{1325}\left(e^{O(72.8011)}+o(1)\right)$	$e^{1828.4596}$	$e^{1767}\left(e^{O(72.8011)}+o(1)\right)$	$e^{2187.5057}$	$e^{2120}\left(e^{O(72.8011)}+o(1)\right)$
5400	$e^{1405.7796}$	$e^{1350}\left(e^{O(73.4847)}+o(1)\right)$	$e^{1862.8553}$	$e^{1800}\left(e^{O(73.4847)}+o(1)\right)$	$e^{2228.4379}$	$e^{2160}\left(e^{O(73.4847)}+o(1)\right)$
5500	$e^{1431.1669}$	$e^{1375}\left(e^{O(74.1620)}+o(1)\right)$	$e^{1896.3589}$	$e^{1833}\left(e^{O(74.1620)}+o(1)\right)$	$e^{2269.1265}$	$e^{2200}\left(e^{O(74.1620)}+o(1)\right)$

Table 3.1. Estrada index of a random m-partite graph

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