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Construction of Bipartite Graphs Having the Same Randić Energy*

Oscar Rojo[a] and Luis Medina[b]
[a] *Universidad Católica del Norte, Antofagasta, Chile*
e-mail: orojo@ucn.cl
[b] *Universidad de Antofagasta, Antofagasta, Chile*
e-mail: luis.medina@uantof.cl

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Abstract

We recall the notion of the Randić energy of a simple undirected graph. Given a bipartite graph $\mathcal{G}$, we construct a sequence of bipartite graphs having the same Randić energy of $\mathcal{G}$.

1 Introduction

Let $\mathcal{G} = (V, E)$ be simple undirected graph on n vertices. Let $M(\mathcal{G})$ be a matrix representation of $\mathcal{G}$. Some examples of $M(\mathcal{G})$ are the adjacency matrix $A(\mathcal{G})$, the Laplacian matrix $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$ and the signless Laplacian $Q(\mathcal{G}) = D(\mathcal{G}) + L(\mathcal{G})$ where $D(\mathcal{G})$ is the diagonal matrix of vertex degrees. It is well known that $L(\mathcal{G})$ and $Q(\mathcal{G})$ are positive semidefinite matrices and that $(0, \mathbf{e})$ is an eigenpair of $L(\mathcal{G})$ where $\mathbf{e}$ is the all ones vector.

We consider here the normalized Laplacian matrix and the Randić matrix of $\mathcal{G}$. Let $v_1, v_2, \ldots, v_n$ be the vertices of $\mathcal{G}$. Denote by $d(v_1), d(v_2), \ldots, d(v_n)$ the degree of $v_1, v_2, \ldots, v_n$, respectively. Let $D^{-\frac{1}{2}}(\mathcal{G})$ be the diagonal matrix whose diagonal entries are

$$\frac{1}{\sqrt{d(v_1)}}, \frac{1}{\sqrt{d(v_2)}}, \ldots, \frac{1}{\sqrt{d(v_n)}}$$

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whenever $d(v_i) \neq 0$. If $d(v_i) = 0$ for some i then the corresponding diagonal entry of $D^{-\frac{1}{2}}(\mathcal{G})$ is defined to be 0. The normalized Laplacian matrix of $\mathcal{G}$, denoted by $\mathcal{L}(\mathcal{G})$, was introduced by F. Chung [10] as

$$\mathcal{L}(\mathcal{G}) = D^{-\frac{1}{2}} L(\mathcal{G}) D^{-\frac{1}{2}}(\mathcal{G}) = I - D^{-\frac{1}{2}}(\mathcal{G}) A(\mathcal{G}) D^{-\frac{1}{2}}(\mathcal{G}). \tag{1}$$

The eigenvalues of $\mathcal{L}(\mathcal{G})$ are called the normalized Laplacian eigenvalues of $\mathcal{G}$. From (1), we have

$$D^{\frac{1}{2}}(\mathcal{G}) \mathcal{L}(\mathcal{G}) D^{\frac{1}{2}}(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G}) = L(\mathcal{G})$$

and thus

$$D^{\frac{1}{2}}(\mathcal{G}) \mathcal{L}(\mathcal{G}) D^{\frac{1}{2}}(\mathcal{G}) \mathbf{e} = L(\mathcal{G}) \mathbf{e} = 0\mathbf{e}.$$

Hence 0 is an eigenvalue of $\mathcal{L}(\mathcal{G})$ with eigenvector $D^{\frac{1}{2}}(\mathcal{G}) \mathbf{e}$. It is known that the eigenvalues of $\mathcal{L}(\mathcal{G})$ lie in the interval $[0, 2]$ and 0 is a simple eigenvalue if and only if $\mathcal{G}$ is connected. Among papers on $\mathcal{L}(\mathcal{G})$ we mention [3], [4], [8] and [9].

From now on, we assume that $\mathcal{G}$ is connected graph. Then $d(v_i) > 0$ for all i. We observe that the matrix $R(\mathcal{G}) = D^{-\frac{1}{2}}(\mathcal{G}) A(\mathcal{G}) D^{-\frac{1}{2}}(\mathcal{G})$ in (1) is the Randić matrix of $\mathcal{G}$ in which the (i, j)-entry is

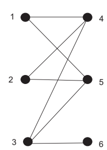
$$R_{i,j} = \begin{cases} \frac{1}{\sqrt{d(v_i)d(v_j)}} & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } v_i \text{ is not adjacent to } v_j \\ 0 & \text{if } i = j \end{cases}$$

Moreover

$$I - \mathcal{L}(\mathcal{G}) = R(\mathcal{G}). \tag{2}$$

The eigenvalues of $R(\mathcal{G})$ are called the Randić eigenvalues of $\mathcal{G}$. The Randić matrix was earlier studied in connection with the Randić index [1], [2], [14] and [15].

Example 1. *Let $\mathcal{G}$ be the graph*



Then

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 1 & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 1 & -\frac{1}{\sqrt{9}} & -\frac{1}{\sqrt{9}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{9}} & 1 & 0 & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{9}} & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 1 \end{bmatrix}$$

and

$$R(\mathcal{G}) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{9}} & \frac{1}{\sqrt{9}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{9}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{9}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \end{bmatrix}$$

If M is a Hermitian matrix, let

$$E(M) = \sum_{j=1}^{n} \left| \lambda_j(M) - \frac{tr(M)}{n} \right|$$

where $\lambda_1(M), \lambda_2(M), \ldots, \lambda_n(M)$ are the eigenvalues of M and $tr(M)$ is the trace of M.

In particular, if $M = A(\mathcal{G})$ then $E(M)$ is denoted by $E(\mathcal{G})$. That is

$$E(\mathcal{G}) = \sum_{j=1}^{n} |\lambda_j(A(\mathcal{G}))| .$$

$E(\mathcal{G})$ is known as the energy of the graph $\mathcal{G}$ and it was introduced by Gutman in 1978, it is studied in Chemistry and used to to approximate the total π-electron energy of a molecule [11, 12].

If $M = \mathcal{L}(\mathcal{G})$ then $E(M)$ is denoted by $\mathcal{E}(\mathcal{G})$. That is

$$\mathcal{E}(\mathcal{G}) = \sum_{j=1}^{n} |\lambda_j(\mathcal{L}(\mathcal{G})) - 1| .$$

$\mathcal{E}(\mathcal{G})$ is called the normalized Laplacian energy of $\mathcal{G}$.

If $M = R(\mathcal{G})$ then $E(M)$ is denoted by $RE(\mathcal{G})$. That is

$$RE(\mathcal{G}) = \sum_{j=1}^{n} |\lambda_j(R(\mathcal{G}))| .$$

$RE(\mathcal{G})$ is called the Randić energy of $\mathcal{G}$. Using (2), we obtain

$$RE(\mathcal{G}) = E(R(\mathcal{G})) = \mathcal{E}(\mathcal{G}) .$$

Therefore the Randić energy of $\mathcal{G}$ is the same as the normalized Laplacian energy of $\mathcal{G}$. The Randić energy of $\mathcal{G}$ is the interest for Mathematical Chemistry, recent articles on this energy are [5] and [6].

2 Bipartite graphs with the same Randić energy

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are nonnegative numbers then $\sum \alpha_j$ denotes the sum over the all positive α_j.

Let 0 and I be the all zeros matrix and the identity matrix of the appropriate sizes, respectively.

Let $r \geq 1$ be an integer. Given an $m \times n$ complex matrix B, we denote by $B^{(r+1)}$ the $(r+1) \times (r+1)$ block bordered matrix

$$B^{(r+1)} = \begin{bmatrix} 0 & \frac{1}{\sqrt{r}}B & \cdots & \frac{1}{\sqrt{r}}B \\ \frac{1}{\sqrt{r}}B^* & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{r}}B^* & 0 & \cdots & 0 \end{bmatrix}$$

where B^* denotes the conjugate transpose matrix of B. Observe that $B^{(r+1)}$ is an Hermitian matrix of order $(m+rn)$ in which there are r copies de $\frac{1}{\sqrt{r}}B$. In particular

$$B^{(2)} = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}.$$

Lemma 1.

$$E\left(B^{(r+1)}\right) = E\left(B^{(2)}\right). \tag{3}$$

Proof. Since $tr\left(B^{(r+1)}\right) = tr\left(B^{(2)}\right) = 0$, it is sufficient to prove that $\sum_j \left|\lambda_j\left(B^{(r+1)}\right)\right| = \sum_j \left|\lambda_j\left(B^{(2)}\right)\right|$. We have

$$\begin{aligned} B^{(r+1)}B^{(r+1)} &= \begin{bmatrix} BB^* & 0 & \cdots & 0 \\ 0 & \frac{1}{r}B^*B & \cdots & \frac{1}{r}B^*B \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{r}B^*B & \cdots & \frac{1}{r}B^*B \end{bmatrix} \\ &= \begin{bmatrix} BB^* & 0 \\ 0 & F \end{bmatrix} \end{aligned}$$

where

$$F = \frac{1}{r}\begin{bmatrix} B^*B & \cdots & B^*B \\ \vdots & \ddots & \vdots \\ B^*B & \cdots & B^*B \end{bmatrix}$$

We recall that the Kronecker product of two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ of sizes $m \times m$ and $n \times n$, respectively, is defined to be the $(mn) \times (mn)$ matrix $A \otimes B = (a_{i,j}B)$. It is known that the eigenvalues of $A \otimes B$ are $\lambda_i(A)\lambda_j(B)$ with $1 \leq i \leq m$ and $1 \leq j \leq n$. We have

$$\begin{bmatrix} B^*B & \cdots & B^*B \\ \vdots & \ddots & \vdots \\ B^*B & \cdots & B^*B \end{bmatrix} = (B^*B) \otimes \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}.$$

The eigenvalues of all ones matrix of order $r \times r$ are the simple eigenvalue r and 0 with multiplicity $(r-1)$. Then the positive eigenvalues of F are the eigenvalues of B^*B. Hence the positive eigenvalues of $B^{(r+1)}B^{(r+1)}$ are the positive eigenvalues of the matrices BB^* and B^*B. This is also the case for $B^{(2)}B^{(2)}$. In fact

$$B^{(2)}B^{(2)} = \begin{bmatrix} BB^* & 0 \\ 0 & B^*B \end{bmatrix}.$$

Therefore the semipositive definite matrices $B^{(r+1)}B^{(r+1)}$ and $B^{(2)}B^{(2)}$ have the same positive eigenvalues. Finally, using the fact that the absolute value of the eigenvalues of $B^{(r+1)}$ and $B^{(2)}$ are the square roots of the eigenvalues of $B^{(r+1)}B^{(r+1)}$ and $B^{(2)}B^{(2)}$, respectively, we obtain

$$E\left(B^{(r+1)}\right) = \sum \left|\lambda_j\left(B^{(r+1)}\right)\right| = \sum \left|\lambda_j\left(B^{(2)}\right)\right| = E\left(B^{(2)}\right).$$

The proof is complete. □

From now on, let $\mathcal{G}$ a given bipartite graph on n vertices. The vertex set of $\mathcal{G}$ can be divided into two disjoint sets V_1 with n_1 vertices and V_2 with n_2 vertices such that every edge connects a vertex in V_1 to one in V_2. Clearly $n = n_1 + n_2$. Labelling the vertices in V_1 by $1, 2, \ldots, n_1$ and the vertices in V_2 by $n_1+1, n_1+2, \ldots, n_1+n_2$, the Randić matrix of $\mathcal{G}$ becomes

$$R(\mathcal{G}) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} = B^{(2)} \qquad (4)$$

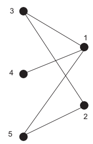
where B is an $n_1 \times n_2$ matrix. Similarly, labelling the vertices in V_2 by $1, 2, \ldots, n_2$ and the vertices in V_1 by $n_2+1, n_2+2, \ldots, n_2+n_1$, the Randić matrix of $\mathcal{G}$ becomes

$$R(\mathcal{G}) = \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}$$

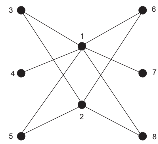
where B^T is the transpose of the matrix B in (4).

Following [7] and [13], let $\mathcal{G}_1^{(2)}$ be the graph obtained from 2 copies of $\mathcal{G}$ by identifying the vertices in V_1. In this case, we label the vertices in V_1 by $1, 2, \ldots, n_1$. Similarly, let $\mathcal{G}_2^{(2)}$ be the graph obtained from 2 copies of $\mathcal{G}$ by identifying the vertices in V_2. In this last case, we label the vertices in V_2 by $1, 2, \ldots, n_2$.

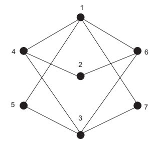
Example 2. *Let $\mathcal{G}$ be the bipartite graph in which V_1 has 2 vertices and V_2 has* 3 *vertices as shown below:*



Then $\mathcal{G}_1^{(2)}$:



and $\mathcal{G}_2^{(2)}$:



Observe that $\mathcal{G}_1^{(2)}$ is a bipartite graph on n_1+2n_2 vertices and $\mathcal{G}_2^{(2)}$ is a bipartite graph on n_2+2n_1 vertices. Labelling the vertices as in Example 2, we have

$$R\left(\mathcal{G}_1^{(2)}\right)=\begin{bmatrix} 0 & \frac{1}{\sqrt{2}}B & \frac{1}{\sqrt{2}}B \\ \frac{1}{\sqrt{2}}B^T & 0 & 0 \\ \frac{1}{\sqrt{2}}B^T & 0 & 0 \end{bmatrix}$$

and

$$R\left(\mathcal{G}_2^{(2)}\right)=\begin{bmatrix} 0 & \frac{1}{\sqrt{2}}B^T & \frac{1}{\sqrt{2}}B^T \\ \frac{1}{\sqrt{2}}B & 0 & 0 \\ \frac{1}{\sqrt{2}}B & 0 & 0 \end{bmatrix}$$

where $B=\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{4}} & 0 & \frac{1}{\sqrt{4}} \end{bmatrix}$.

Definition 1. *Let* $\mathcal{G}_1^{(r)}$ *be the graph obtained from* r *copies of* $\mathcal{G}$ *by identifying the vertices in* $V_1=\{1,2,\ldots,n_1\}$ *and let* $\mathcal{G}_2^{(r)}$ *be the graph obtained from* r *copies of* $\mathcal{G}$ *by identifying the vertices in* $V_2=\{1,2,\ldots,n_2\}$.

Observe that $\mathcal{G}_1^{(r)}$ is a bipartite graph on n_1+rn_2 vertices and $\mathcal{G}_2^{(r)}$ is a bipartite graph on n_2+rn_1 vertices.

As we illustrated in Example 2, there is a labelling for the vertices of $\mathcal{G}_1^{(r)}$ such that

$$R\left(\mathcal{G}_1^{(r)}\right)=\begin{bmatrix} 0 & \frac{1}{\sqrt{r}}B & \cdots & \frac{1}{\sqrt{r}}B \\ \frac{1}{\sqrt{r}}B^T & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{r}}B^T & 0 & \cdots & 0 \end{bmatrix} \tag{5}$$

and there is a labelling for the vertices of $\mathcal{G}_2^{(r)}$ such that

$$R\left(\mathcal{G}_2^{(r)}\right) = \begin{bmatrix} 0 & \frac{1}{\sqrt{r}}B^T & \cdots & \frac{1}{\sqrt{r}}B^T \\ \frac{1}{\sqrt{r}}B & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{r}}B & 0 & \cdots & 0 \end{bmatrix}. \tag{6}$$

Theorem 1. *Let $\mathcal{G}$ be a bipartite graph. Then*

$$RE\left(\mathcal{G}_1^{(r)}\right) = RE\left(\mathcal{G}_2^{(r)}\right) = RE\left(\mathcal{G}\right). \tag{7}$$

Proof. Let $C = B^T$. From (5) and (6)

$$R\left(\mathcal{G}_1^{(r)}\right) = B^{(r+1)} \text{ and } R\left(\mathcal{G}_2^{(r)}\right) = C^{(r+1)}.$$

We have $RE\left(\mathcal{G}_1^{(r)}\right) = E\left(R\left(\mathcal{G}_1^{(r)}\right)\right)$ and $RE\left(\mathcal{G}_2^{(r)}\right) = E\left(R\left(\mathcal{G}_2^{(r)}\right)\right)$. We apply Lemma 1 to obtain

$$E\left(R\left(\mathcal{G}_1^{(r)}\right)\right) = E\left(B^{(r+1)}\right) = E\left(B^{(2)}\right)$$

and

$$E\left(R\left(\mathcal{G}_2^{(r)}\right)\right) = E\left(C^{(r+1)}\right) = E\left(C^{(2)}\right).$$

Then

$$RE\left(\mathcal{G}_1^{(r)}\right) = E\left(B^{(2)}\right) \text{ and } RE\left(\mathcal{G}_2^{(r)}\right) = E\left(C^{(2)}\right). \tag{8}$$

Finally, using the equalities $RE\left(\mathcal{G}\right) = E\left(R\left(\mathcal{G}\right)\right) = E\left(B^{(2)}\right)$ and $RE\left(\mathcal{G}\right) = E\left(R\left(\mathcal{G}\right)\right) = E\left(C^{(2)}\right)$ in (8), the result follows. □

Given any bipartite graph $\mathcal{G}$ and $r \geq 2$, we have constructed two graphs $\mathcal{G}_1^{(r)}$ and $\mathcal{G}_2^{(r)}$ with the same Randić energy of $\mathcal{G}$. Clearly, if $n_1 \neq n_2$ then $\mathcal{G}_1^{(r)}$ and $\mathcal{G}_2^{(r)}$ are graphs of different orders.

Corollary 1. *If $n_1 = n_2$, then $R\left(\mathcal{G}_1^{(r)}\right)$ and $R\left(\mathcal{G}_2^{(r)}\right)$ are cospectral.*

Proof. Since $n_1 = n_2$, $R\left(\mathcal{G}_1^{(r)}\right)$ and $R\left(\mathcal{G}_2^{(r)}\right)$ are matrices of the same order. Then

$$\begin{bmatrix} 0 & \cdots & 0 & I \\ \vdots & 0 & \cdots & 0 \\ 0 & I & \ddots & \vdots \\ I & 0 & \cdots & 0 \end{bmatrix} R\left(\mathcal{G}_2^{(r)}\right) \begin{bmatrix} 0 & \cdots & 0 & I \\ \vdots & 0 & \cdots & 0 \\ 0 & I & \ddots & \vdots \\ I & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & I \\ \vdots & 0 & \cdots & 0 \\ 0 & I & \ddots & \vdots \\ I & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & B^T & \cdots & B^T \\ B & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & I \\ \vdots & 0 & \cdots & 0 \\ 0 & I & \ddots & \vdots \\ I & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & B & \cdots & B \\ B^T & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B^T & 0 & \cdots & 0 \end{bmatrix} = R\left(\mathcal{G}_1^{(r)}\right).$$

Therefore $R\left(\mathcal{G}_1^{(r)}\right)$ and $R\left(\mathcal{G}_2^{(r)}\right)$ are cospectral. □

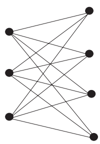
3 Some examples

It is known that if $\mathcal{G}$ is a graph of order n with no isolated vertices then

$$2 \leq RE\,(\mathcal{G}) \leq 2\left\lfloor \frac{n}{2} \right\rfloor. \tag{9}$$

Remark 1. *If $\mathcal{G}$ is a bipartite graph of order n such that $RE\,(\mathcal{G}) = 2\left\lfloor \frac{n}{2}\right\rfloor$ then the graphs $\mathcal{G}_1^{(r)}$ and $\mathcal{G}_2^{(r)}$ do not attain the upper bound in (9). In fact, $RE\left(\mathcal{G}_1^{(r)}\right) = RE\,(\mathcal{G}) = 2\left\lfloor \frac{n}{2}\right\rfloor < 2\left\lfloor \frac{n_1+rn_2}{2}\right\rfloor$ and $RE\left(\mathcal{G}_2^{(r)}\right) = RE\,(\mathcal{G}) = 2\left\lfloor \frac{n}{2}\right\rfloor < 2\left\lfloor \frac{n_2+rn_1}{2}\right\rfloor$.*

Example 3. *Let $m, n \geq 2$. Let P_n and S_n be the path and the star on n vertices, respectively. Let $K_{m,n}$ be the complete bipartite graph. Clearly S_n can be obtained from $n-1$ copies of P_2 identifying one of its vertices. Then, from Theorem 1, $RE\,(S_n) = RE\,(P_2)$. Since $R\,(P_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, its follows $RE\,(P_2) = 2$. Thus $RE\,(S_n) = 2$ for all $n \geq 2$. Moreover, it is easy to see that $K_{m,n}$ can be obtained from m copies of S_{n+1} identifying its pendant vertices. From Theorem 1, $RE\,(K_{m,n}) = RE\,(S_{n+1}) = 2$. For instance, $K_{3,4}$ is obtained from 3 copies of S_5 :*



Example 4. *Let $n \geq 2$. Let C_{2n} be the cycle on $2n$ vertices. Clearly, C_{2n} is a bipartite graph. Labelling the vertices as above the Randić of C_{2n} becomes*

$$R\,(C_{2n}) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

where B is an $n \times n$ circulant matrix of the form

$$B = \frac{1}{2}\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

An easy computation shows that

$$R(C_{2n})\,R(C_{2n})^T = \begin{bmatrix} BB^T & 0 \\ 0 & B^TB \end{bmatrix}$$

with

$$BB^T = B^TB = \frac{1}{4}\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 2 & 1 & \ddots & & 0 \\ 0 & 1 & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & 1 & 0 \\ 0 & & & 1 & 2 & 1 \\ 1 & 0 & \cdots & \cdots & 1 & 2 \end{bmatrix}.$$

The eigenvalues of the last matrix are $\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2k\pi}{n}\right)$ *for* $k = 1, \ldots, n$. *We have* $\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2k\pi}{n}\right) = \cos^2\left(\frac{k\pi}{n}\right)$. *Then*

$$\begin{aligned} RE(C_{2n}) &= 2\sum_{k=1}^{n}\sqrt{\cos^2\left(\frac{k\pi}{n}\right)} = 2\sum_{k=1}^{n}\left|\cos\left(\frac{k\pi}{n}\right)\right| \\ &= 2\left(1 + 2\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor}\cos\left(\frac{k\pi}{n}\right)\right) \end{aligned}$$

where $\lfloor \frac{n}{2} \rfloor$ *is the largest integer not exceeding* $\frac{n}{2}$. *Applying the Dirichlet kernel*

$$D_m = 1 + 2\sum_{k=1}^{m}\cos kx = \frac{\sin\left(\left(m+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}},$$

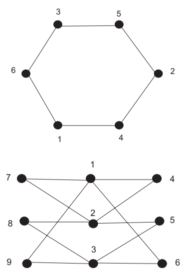
we get

$$RE(C_{2n}) = \frac{2\sin\left(\left(\lfloor\frac{n}{2}\rfloor + \frac{1}{2}\right)\frac{\pi}{n}\right)}{\sin\frac{\pi}{2n}}.$$

Let $(C_{2n})_1^{(r)}$ *be the graph obtained from* r *copies of* C_{2n} *identifying the vertices in* $V_1 = \{1, 2, \ldots, n\}$. *Applying Theorem 1,*

$$RE\left((C_{2n})_1^{(r)}\right) = \frac{2\sin\left(\left(\lfloor\frac{n}{2}\rfloor + \frac{1}{2}\right)\frac{\pi}{n}\right)}{\sin\frac{\pi}{2n}}$$

for any $r \geq 2$. *In particular,* $RE\left((C_6)_1^{(2)}\right) = RE(C_6) = 4:$



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