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The Energy of Kneser Graphs

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Abstract

In this note, by proving two combinatorial identities, we compute the energy of Kneser graphs.

Let Γ be a graph with n vertices and eigenvalues $\lambda_1, \ldots, \lambda_n$. The *energy* of Γ is defined as $E(\Gamma) = |\lambda_1| + \cdots + |\lambda_n|$. A graph Γ is said to be *hyperenergetic* if $E(\Gamma) > 2n - 2$. The concept of hyperenergeticity was first introduced by Gutman in [3]. Hyperenergetic graphs are important because molecular graphs with maximum energy pertain to maximality stable π-electron systems.

The *Kneser graph* $K(v, k)$ is the graph with k-subsets of a fixed v-set as its vertices, with two vertices adjacent if they are disjoint. By [2, Theorem 9.4.3], if $v \geq 2k + 1$ then the eigenvalues of $K(v, k)$ are $(-1)^j \binom{v-k-j}{k-j}$ with multiplicity $\binom{v}{j} - \binom{v}{j-1}$, $j = 0, 1, \ldots, k$. Then the energy of $K(v, k)$ is

$$E(K(v,k)) = \sum_{j=0}^{k} \left(\binom{v}{j} - \binom{v}{j-1} \right) \binom{v-k-j}{k-j}.$$

Akbari [1] proved that $K(v, k)$ is hyperenergetic for any integers v and $k \geq 2$ with $v \geq 2k + 1$. In this note, we shall compute the energy of Kneser graphs.

We start with two combinatorial identities.

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Lemma 1 *For any odd number n and any integer k with $2k > n > 0$, we have*

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{2k-n}{k-j} = 0.$$

Proof. Note that

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{2k-n}{k-j} &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{n-j}\binom{2k-n}{k-n+j}\\ &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{2k-n}{k-j}.\end{aligned}$$

Since n is odd number, the desired result follows. □

Lemma 2 *For any integer r and positive integer k, we have*

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{r}{j}\binom{2k-r}{k-j} = \frac{(r-1)(r-3)\cdots(r-2k+1)2^k}{k!}.$$

Proof. Consider the polynomial

$$g(x) = \sum_{j=0}^{k}(-1)^{k-j}\binom{x}{j}\binom{2k-x}{k-j} - \frac{(x-1)(x-3)\cdots(x-2k+1)2^k}{k!}.$$

The degree of $g(x)$ is at most k. Lemma 1 implies that $1, 3, \ldots, 2k-1$ are k distinct roots of $g(x)$. Since $g(0) = 0$, we have $g(x) = 0$. Hence $g(r) = 0$, as desired. □

Theorem 1 *For $v \geq 2k+1$, the energy of $K(v, k)$ is*

$$E(K(v,k)) = \frac{(v-1)(v-3)\cdots(v-2k+1)2^k}{k!}.$$

Proof. Since

$$\begin{aligned}&\sum_{j=0}^{k}\left(\binom{v}{j} - \binom{v}{j-1}\right)\binom{v-k-j}{k-j}\\ =\ &\sum_{j=0}^{k}\binom{v}{j}\binom{v-k-j}{k-j} - \sum_{j=1}^{k}\binom{v}{j-1}\binom{v-k-j}{k-j}\\ =\ &\sum_{j=0}^{k}\binom{v}{j}\left(\binom{v-k-j-1}{k-j} + \binom{v-k-j-1}{k-j-1}\right) - \sum_{j=0}^{k-1}\binom{v}{j}\binom{v-k-j-1}{k-j-1}\\ =\ &\sum_{j=0}^{k}\binom{v}{j}\binom{v-k-j-1}{k-j}\\ =\ &\sum_{j=0}^{k}(-1)^{k-j}\binom{v}{j}\binom{2k-v}{k-j},\end{aligned}$$

by Lemma 2 the desired result follows. □

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