

On the Relationships between the First and Second Zagreb Indices

Tamás Réti

Széchenyi István University, Egyetem tér 1, H-9026 Győr, Hungary

reti@sze.hu

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Abstract

The Zagreb indices introduced by Gutman and Trinajstić more than thirty years ago are graph-based molecular structure descriptors. The research and application of the Zagreb indices appears mainly in mathematical chemistry. In this paper we present some new inequalities related to the first and the second Zagreb indices. By introducing the notions of P-dominant graphs, and of the valency-functions of a graph, correspondences with the novel inequalities presented are discussed. In addition, we find examples characterizing the validity of Zagreb indices inequality and equalities for some particular classes of connected biregular graphs.

1. Introduction

The graph based molecular descriptors called Zagreb indices were introduced more than thirty years ago by Gutman and Trinajstić [1]. Since then, several results concerning both Zagreb indices have been communicated in the chemical and mathematical literature [2,3].

In this study some novel inequalities established between the first and the second Zagreb indices will be presented. This work was motivated primarily by the intensive research focused on the comparison of Zagreb indices M_1 and M_2 [4-14].

This paper is organized as follows. In section 2 graph theoretical notions and relations are introduced. In section 3 some known results based on the comparative study of Zagreb indices

are summarized. In section 4, novel relationships between M_1 and M_2 are presented. In section 5, a possible reformulation of the Zagreb indices inequalities is given. Finally, in section 6, open problems and conjectures are outlined.

2. Basic notions and relations

We consider only finite connected graphs without loops and multiple edges. For a connected graph G , $V(G)$ and $E(G)$ denote the set of vertices and edges, and $n=|V|$ and $m=|E|$ the numbers of vertices and edges, respectively. An edge of G connecting vertices u and v is denoted by (u,v) . A vertex u is called an r -vertex if the degree $d(u)$ of u is equal to r . The number of r -vertices in G is denoted by n_r .

We denote by $D(G)$ the finite set of degrees of G , and by $\Delta= \Delta(G)$ and $\delta= \delta(G)$ the maximum and the minimum degrees, respectively, of vertices of G . To avoid trivialities we always assume that $n \geq 3$, and $d(u) \geq 1$. A graph is called R -regular if all its vertices have the same degree R .

We distinguish between two types of connected simple graphs: Connected graphs of class 1 have the property that no two vertices of the same degree are adjacent. In connected graphs of class 2 at least one edge connects vertices of equal degree. Connected regular graphs belong to class 2. It follows that a connected graph G belongs to the class 1, if and only if $g(r,r)=0$ for any $r \in D(G)$. In this case it is easy to see that a connected graph G belongs to class 1, if and only if $\sum_{r \neq s} g(r,s) = 2n_r r$ holds for any $r \in D(G)$.

Based on the considerations in Ref. [15], a graph G denoted by (Δ, δ) is said to be biregular (bidegred) with degrees Δ and δ ($\Delta > \delta \geq 1$), if at least one vertex of G has degree Δ and at least one vertex has degree δ , and if no vertex of G has degree different from Δ or δ . The complete bipartite graphs belong to the biregular graphs of class 1. Connected biregular graphs of class 1, which are necessarily bipartite, are sometimes called semiregular graphs.

Let a, b and c be three positive integers, $1 \leq a \leq b \leq c \leq n-1$. The graph G denoted by (a,b,c) is said to be triregular if for $i=1,2,\dots,n$, either $d_i = a$ or $d_i = b$ or $d_i = c$, and there exists at least one vertex of degree a , at least one vertex of degree b , and at least one vertex of degree c .

A subdivision graph $S_1(G)$ of a graph G is obtained by inserting a new vertex (of degree 2) on every edge of G .

3. Some known results relating to the relations between the first and second Zagreb indices

The first Zagreb index $M_1(G)$ is equal to the sum of squares of the degrees of the vertices, and the second Zagreb index $M_2(G)$ is equal to the sum of products of the degrees of pairs of adjacent vertices of the graph G .

$$M_1 = M_1(G) = \sum_{u \in V(G)} d^2(u) = \sum_r \sum_{s \leq r} g(r,s)(r+s)$$

and

$$M_2 = M_2(G) = \sum_{(u,v) \in E(G)} d(u)d(v) = \sum_r \sum_{s \leq r} g(r,s)rs$$

where $g(r,s)$ is the total number of edges in G with end-vertex degrees r and s , where we do not distinguish $g(r,s)$ and $g(s,r)$. The number $g(r,s)$ are sometimes denoted by $m_{r,s}$.

In the past few years several research studies have been focused on the comparisons of the Zagreb indices $M_1(G)$ and $M_2(G)$.

a) One of the open graph theoretical problems concerns the investigation of the topological parameter represented by $M_2/m - M_1/n$. Numerous results have been reported on this [4-16]. Based on the AutoGraphiX conjecture-generating system it was conjectured [4,15] that for all simple connected graphs the inequality

$$\frac{M_1}{n} \leq \frac{M_2}{m} \tag{1}$$

holds, and the bound is tight for complete graphs. The relation (1) is usually referred to as the *Zagreb indices inequality*. If the equality case is excluded, then we speak of the *strict Zagreb indices inequality* [15].

It was shown in [4,7,9] that this conjecture is not true and counterexamples were given for connected and disconnected graphs. Curiously, it was demonstrated that the number of connected graphs for which inequality $M_1/n > M_2/m$ holds is infinite [10-16].

The relation $M_1/n \leq M_2/m$ has been proven [4-7] to hold for trees, unicyclic graphs and chemical graphs (i.e. connected graphs in which no vertex has degree greater than 4).

Andova and Cohen [10] verified that Zagreb indices inequality holds for any connected biregular graph. Sun and Chen proved the following theorem [8]: If G is connected simple graphs with n vertices, and m edges, and i) $\Delta(G) - \delta(G) \leq 2$ or ii) $\Delta(G) - \delta(G) \leq 3$ and $\delta(G) \neq 2$, then $M_1/n \leq M_2/m$.

Recently, it was verified [15,16] that the Zagreb indices inequality (1) holds for the subdivision graph $S_1(G)$ of any graph G , biregular graph of class 1 (strict inequality holds for biregular graphs of class 2), triregular graphs of class 1 (strict inequality holds for connected triregular (a,b,c) graphs of class 2).

It is interesting to note that little attention was paid to the equality case, i.e. for which

$$\frac{M_1}{n} = \frac{M_2}{m} \quad (2)$$

holds. In what follows, we call (2) *the Zagreb indices equality* [15].

It is easy to show that the Zagreb indices equality holds for regular graphs. Hansen and Vukićević verified [4] that for a connected biregular graph G , the Zagreb indices equality holds if and only if G is a graph of class 1 (i.e. all edges of G have the same pair (Δ, δ) of degrees).

Recently it has been verified [15] that there is no connected triregular graph of class 1 that satisfies the Zagreb indices equality.

Moreover it was shown [15] that if a connected graph G has maximal degree 4, then G satisfies that Zagreb indices equality if and only if G is regular or biregular graph of class 1.

There, it was also verified [15] that there exist infinitely many connected graphs of maximal degree $\Delta=5$ that are neither regular nor biregular of class 1, which satisfy the Zagreb indices equality. Recently Abdo et al. has proved [16], that there exist infinitely many connected graphs of maximal degree $\Delta \geq 5$ that neither regular nor biregular of class 1 which satisfy the Zagreb indices equality.

b) For a simple connected graph G (without loops and multiple edges) Das and Gutman proved [17] that

$$M_1 + 2M_2 \leq 4m^2$$

with equality if and only if G is the complete graph on n vertices. Moreover, they verified [17] that for a simple connected graph G

$$M_2 \leq 2m^2 - (n-1)m\delta + \frac{1}{2}(\delta-1)M_1$$

with equality if G is a star graph or a regular graph.

c) Caporossi et al. proved [18] that

$$M_2 - M_1 \geq 11m - 12n \tag{3}$$

holds for simple connected graphs. The bound is tight and attained if G is a simple connected graph with vertices $\Delta=3$ and $\delta=2$ only, $m \geq 6n/5$ and no pair of vertices of degree 2 are adjacent. Inequality (3) holds for polyhedral graphs as well. Because for a polyhedral graph G_P the inequality $m \geq 3n/2$ is valid, it follows that $m \geq 6n/5$. This implies that for G_P the equality $M_2 - M_1 = 11m - 12n$ holds if and only if G_P is a simple (3-regular) polyhedral graph.

In what follows some novel relations related to first and second Zagreb indices are presented, where in particular cases equalities are fulfilled for regular or biregular graphs of class 1.

4. Main results

4.1 Theorems characterizing the relations between M_1 and M_2

Proposition 1. If G is a simple connected graph then

$$M_1 \geq \frac{M_2}{\Delta} + \delta m \tag{4}$$

with equality if G is regular.

Proof. Let u and v be vertices with degrees $d(u)=r$ and $d(v)=s$, and consider the following inequalities:

$$(r - \delta)(\Delta - s) \geq 0$$

Therefore

$$r + s \geq \frac{r\Delta + s\delta}{\Delta} \geq \frac{rs}{\Delta} + \delta$$

From the inequalities above, we get

$$M_1 = \sum_r \sum_r g(r,s)(r+s) \geq \frac{1}{\Delta} \sum_r \sum_{s \leq r} g(r,s)rs + \delta \sum_r \sum_{s \leq r} g(r,s)$$

It is easy to see that equality holds in (4) if and only if, G is regular graph. ■

Proposition 2. If G is a simple connected graph then

$$M_1 \leq \frac{M_2}{\delta} + \delta m \tag{5}$$

and

$$M_1 \leq \frac{M_2}{\Delta} + \Delta m \tag{6}$$

In both cases, equality holds if and only if, G is regular or a connected biregular graph of class 1.

Proof. Let u and v be vertices with degrees $d(u)=r$ and $d(v)=s$, and consider the following inequalities:

$$(r - \delta)(s - \delta) \geq 0$$

$$(\Delta - r)(\Delta - s) \geq 0$$

Therefore

$$\frac{rs}{\delta} + \delta \geq r + s$$

$$\frac{rs}{\Delta} + \Delta \geq r + s$$

Based on the previous inequalities, we obtain

$$M_1 = \sum_r r^2 n_r = \sum_r \sum_{s \leq r} g(r,s)(r+s) \leq \sum_r \sum_{s \leq r} g(r,s) \left(\frac{rs}{\delta} + \delta \right)$$

Consequently,

$$M_1 \leq \frac{1}{\delta} \sum_r \sum_{s \leq r} g(r,s)rs + \delta \sum_r \sum_{s \leq r} g(r,s) = \frac{M_2}{\delta} + \delta m$$

Similarly, we can prove inequality (6). It is trivial that for regular graphs, where $\delta=\Delta=R$, equality holds in (5) and (6).

Moreover, it is easy to see that equality holds in (5) and (6) if G is a connected biregular graph of class 1. In this case, $M_1=m(\Delta+\delta)$ and $M_2=m\Delta\delta$, consequently,

$$\frac{M_2}{\delta} + \delta m = \frac{m\Delta\delta}{\delta} + \delta m = m(\Delta + \delta) = M_1$$

and

$$\frac{M_2}{\Delta} + \Delta m = \frac{m\Delta\delta}{\Delta} + \Delta m = m(\Delta + \delta) = M_1.$$

■

Corollary 3. Since $\delta \leq [d] = 2m/n \leq \Delta$, from (5) we have

$$M_1 \leq \frac{M_2}{\delta} + [d]m = \frac{M_2}{\delta} + \frac{2m^2}{n}$$

Similarly, from (6) we have

$$M_1 \leq \frac{M_2}{[d]} + \Delta m = \frac{nM_2}{2m} + \Delta m$$

It is obvious that equality holds for regular graphs.

Corollary 4. Using (5) and (6) one obtains directly

$$M_1 \leq \frac{1}{2} \left\{ M_2 \left(\frac{1}{\Delta} + \frac{1}{\delta} \right) + m(\Delta + \delta) \right\} = \frac{\Delta + \delta}{2} \left\{ \frac{M_2}{\Delta\delta} + m \right\} \tag{7}$$

and

$$M_1 \leq \sqrt{\left(\frac{M_2}{\delta} + \delta m \right) \left(\frac{M_2}{\Delta} + \Delta m \right)} \tag{8}$$

From the previous considerations it follows that equality holds in (7) and (8) not only for regular but for connected biregular graphs of class 1 as well.

4.2 Domination degree and the valency function of a graph

Let r and $s \in D(G)$ be degrees of graph G with $1 \leq \delta \leq s \leq r \leq \Delta$. A degree $P \in D(G)$ is called a domination degree of a connected graph G , if for any r and s degrees, $g(r,s) = g(s,r) = 0$ if $r \neq P$.

From this definition it follows that $P \in D(G)$ is a domination degree of G , if and only if

$$\sum_s g(P,s) = \sum_r g(r,P) = m$$

Since for any R-regular graph $g(R,R)=m$ holds, this implies that each regular graph has only one domination degree ($P=R$), exactly. Moreover, from the definition it follows that the number of different domination degrees is at most 2. The connected biregular graphs of class 1 with vertices of degrees δ and Δ (where $\Delta>\delta\geq 1$) have always 2 distinct domination degrees $P_1=\delta$ and $P_2=\Delta$, for which $g(\Delta,\delta)=g(P_2,P_1)=m$ holds.

In order to characterize the “impact” of different degree types on the structure of a graph G, we introduced a polynomial function of second degree $Z_G(X)$ defined as

$$Z(X) = Z_G(X) = mX^2 - M_1X + M_2$$

The function $Z(X)$ where $X \geq 0$ is called the “valency function” of graph G. In the sequel, we will discuss some properties of function $Z(X)$.

The discriminant of $Z(X)$ is given as $DIS = M_1^2 - 4mM_2$. If it is negative, there are no real roots. If DIS is positive, then $Z(X)$ has two distinct positive roots, $X_1 \leq X_2$, for which

$$X_1 + X_2 = M_1 / m$$

$$X_1 X_2 = M_2 / m$$

Additionally, $Z(0)=M_2 > 0$.

If $DIS=0$, then there is exactly one positive root X_A , for which $X_A=X_1=X_2=M_1/(2m)$. Valency function $Z(X)$ has a minimum value at $X_{\min}=M_1/(2m)$ and $Z(X_{\min})=M_2 - M_1^2/(4m) = -DIS/(4m)$. This means that the value of $Z(X_{\min})$ is determined by the discriminant.

Proposition 5. If G is a simple connected graph then

$$Z(\delta) = m\delta^2 - M_1\delta + M_2 \geq 0 \tag{9}$$

$$Z(\Delta) = m\Delta^2 - M_1\Delta + M_2 \geq 0$$

Proof. These inequalities are the consequence of equations (5) and (6). ■

Remark 6. If function $Z(X)$ has one or two positive roots (where $X_1 \leq X_2$), this implies that

$$1 \leq \delta \leq X_1 \leq X_2 \leq \Delta$$

Remark 7. If G is a simple connected graph then

$$\frac{M_1(G)}{m} - \frac{M_2(G)}{m} \leq 1$$

Since $\delta \geq 1$ and $Z(\delta) \geq 0$, from (9) it follows that

$$Z(1) = m + M_2 - M_1 \geq 0$$

It is easy to see that equality holds if G is a star graph with $n=m+1$ vertices where $\delta=1$ and $\Delta=g(\Delta,1)=m$ hold, or G is a path Π_1 having only one edge. It should be noted that if G is a triangle-free connected graph, then $Z(1)=p_3(G)$, where $p_3(G)$ is the number of paths of length 3 in G [19].

Considering the correspondences between the domination degrees and the roots of valency function, the following conclusions can be drawn:

- i) If the valency function $Z(X)$ of graph G has no real roots (i.e. the discriminant is negative), then G has no domination degree.
- ii) A necessary condition for the existence of only one domination degree is the fulfillment of equality $DIS=0$.
- iii) Similarly, a necessary condition for the existence of two distinct domination degrees is that the valency functions have a positive discriminant.
- iv) If graph G has domination degrees (one or two), these should be necessarily identical to the roots of function $Z(X)$. The converse of this statement is false.

v) The quantity $Z(X_{\min})=M_2-M_1 \cdot M_1/(4m) = -DIS/(4m)$ can be positive, or negative number or zero. Depending on the values $Z(X_{\min})$, graphs can be classified into three disjoint sets. It is easy to construct three different connected planar graphs for which $Z(X_{\min})$ is equal to 1, 0 and -1, respectively.

4.3 P-dominant graphs

Let $P \geq 1$ be a positive integer. A connected graph G is called P -dominant if G has a $P \in D(G)$ domination degree.

Proposition 8. If a connected graph G is a P -dominant graph (i.e. G has a domination degree P), then

$$M_1(G) = \frac{M_2(G)}{P} + mP \tag{10}$$

Proof. Since for a graph with a domination degree P , all $g(r,s) = 0$ if $r \neq P$, it follows that

$$\begin{aligned} M_1(G) &= \sum_r \sum_{s \leq r} g(r,s)(r+s) = \sum_s g(P,s)(P+s) = \\ & P \sum_s g(P,s) + \frac{1}{P} \sum_r \sum_{s \leq r} g(r,s)rs = \frac{M_2(G)}{P} + mP \end{aligned}$$

■

Corollary 9. The converse of this statement is false. To demonstrate this, consider the triregular graph G_D having the degree set $D(G_D) = \{3,4,5\}$ on Figure 1. For G_D , we have $m(G_D)=12$, $M_1(G_D)=86$ and $M_2(G_D)=152$. It is easy to check that graph G_D has no domination degree. In spite of this fact, for degree 4 the equality $M_1(G_D) = M_2(G_D)/4 + 4m(G_D) = 152/4 + 4 \cdot 12 = 86$ holds.

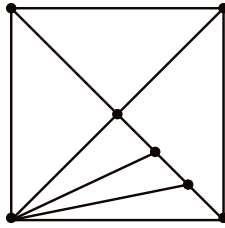


Figure1. The graph G_D demonstrating that the converse of Proposition 8 is false.

Corollary 10. If $S_1(G)$ is a subdivision graph of G then

$$M_1(S_1(G)) = \frac{M_2(S_1(G))}{2} + 2m(S_1(G))$$

Proof. Graph $S_1(G)$ is a connected 2-dominant graph having a domination degree 2. ■

4.4 Construction of 2-dominant graphs of class 1

First of all, we introduce the notions of the K -subdivision graph and of the “bunch graph of size K ”, where $K \geq 2$ is a positive integer. This latter one is a connected multigraph which contains $K \geq 2$ edges meeting at two vertices.

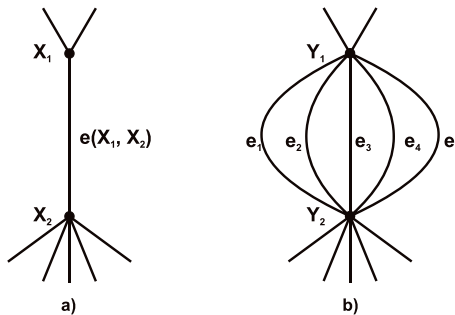


Figure 2. Replacing the edge $e(X_1, X_2)$ of the parent graph G by a bunch graph of size K (case of $K=5$)

The K -subdivision operation, that is the construction of the K -subdivision graph $S_K(G)$ of a connected graph G is based on following concept (See Figure 2.).

As a first step, we generate the graph $G(K)$ by replacing each edge of the parent graph G by a bunch graph of size K . (See Figure 2.) In the resulting graph $G(K)$, all neighbor vertices are joined by K parallel edges. The number of vertices remains the same, but the degrees of every vertex will be equal to $Kd(u)$, where $d(u)$ stands for the degree of vertex u in the parent graph G . In the second step, we perform a subdivision operation on graph $G(K)$. Consequently, the K -subdivision graph $S_K(G)$ is obtained from $G(K)$ by inserting a new vertex (of degree 2) on every edge of $G(K)$.

Remark 11. From the concept of construction it is obvious that every graph $S_K(G)$ is a connected 2-dominant graph of class 1. Moreover, it is easy to see that considering the case of $K=1$, the graph $S_1(G)$ is identical to the traditional subdivision graph of G .

Lemma 12. [15] If G is a connected biregular graph of class 1, then the Zagreb indices equality (2) holds.

Lemma 13. [9] For any connected graph G ,

$$M_1(G) \geq \frac{4m^2}{n}$$

with equality if and only if G is regular.

Proposition 14. The Zagreb indices inequality is obeyed by the K -subdivision graph $S_K(G)$ of any connected graph G . The Zagreb indices equality for $S_K(G)$ holds if and only if the parent graph G is regular of degree $R \geq 1$.

Proof. If G is a regular graph, then $S_K(G)$ is biregular of class 1. According to Lemma 12 for $S_K(G)$ the Zagreb indices equality is valid.

It remains to verify that if G is not regular, then the strict Zagreb indices inequality holds. If G has n vertices and m edges, then $S_K(G)$ has $n + Km$ vertices and $2Km$ edges. If the vertex degrees of G are d_1, d_2, \dots, d_n then

$$M_1(S_K(G)) = (d_1^2 K^2 + d_2^2 K^2 \dots, d_n^2 K^2) + 4mK = K^2 M_1(G) + 4mK \quad (11)$$

On the other hand, since 2 is a domination degree of $S_K(G)$, from (10) and (11) it follows that

$$M_2(S_K(G)) = 2M_1(S_K(G)) - 4m(S_K(G)) = 2K^2 M_1(G) + 8mK - 8mK = 2K^2 M_1(G)$$

In the case of graph $S_K(G)$, inequality (1) will hold if

$$\frac{K^2 M_1(G) + 4mK}{n + Km} \leq \frac{2K^2 M_1(G)}{2Km}$$

which is directly transformed into

$$M_1(G) \geq \frac{4m^2}{n} \quad (12)$$

According to the Lemma 13, equality holds in (12) if and only if G is regular. This implies that inequality (12) and therefore also inequality (1) are strict. ■

Proposition 15. Let G be a connected regular of degree $R=3$ without bridges. Based on the concept of perfect matching, from G it is possible to construct one or more connected biregular graph of class 1.

Proof. It is known that every cubic (3-regular) graph G without bridges has a perfect matching [20,21]. Consider the finite edge set $\{e_1, e_2, \dots\}$ of G generated by perfect matching and, as a first step, replace all these edges by bunch graphs of size K . As a second step, perform a subdivision operation on each edge. The resulting graph will be a 2-dominant biregular graph of class 1, having vertices with degrees of $\delta = 2$ and $\Delta = R+K-1=K+2$. ■

As an example, different perfect matchings of the triangular prism graph are shown in Figure 3. Starting with two possible perfect matchings of the triangular prism graph, (Figure 3a₁ and Figure.3b₁), both resulting graphs (Figure 3a₃ and Figure 3b₃) will be non-isomorphic biregular graphs of class 1.

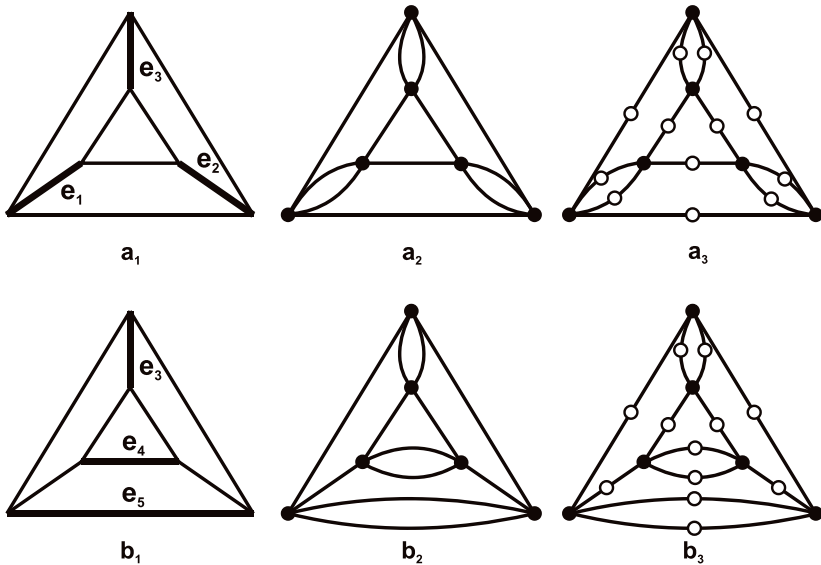


Figure 3. Generation of two biregular graphs of class 1 using the perfect matching on the triangular prism graph (case of K=2)

5. A possible quantitative characterization (reinterpretation) of Zagreb indices inequalities

There are several concepts to characterize the Zagreb indices inequality in a quantitative manner. For example, the topological descriptor $W(G)$ defined as

$$W(G) = \frac{nM_2(G) - mM_1(G)}{nm} = \frac{M_2(G)}{m} - \frac{M_1(G)}{n}$$

would be a possible solution for this purpose [10-13,16]. Unfortunately, the descriptor $W(G)$ is not sensitive enough to discriminate between the behaviors of graphs of different type. It is easy to construct an infinite sequence of connected graphs $\{G(q) : q = 1, 2, 3, \dots\}$, for which

$$\lim_{q \rightarrow \infty} W(G(q)) = \lim_{q \rightarrow \infty} \left\{ \frac{nM_2(G(q)) - mM_1(G(q))}{nm} \right\} = \infty$$

holds. It seems that a better choice is: the topological index $T(G)$ defined as

$$T = T(G) = \frac{mM_1(G)}{nM_2(G)} \tag{13}$$

It follows that $T \leq 1$ if and only if, $M_1/n \leq M_2/m$, and $T \geq 1$ if and only if, $M_1/n \geq M_2/m$. This implies that $T(G)=1$ if and only if the Zagreb indices equality holds.

Das showed [22] that for a simple connected graph

$$M_1 = \sum_{u \in V(G)} d^2(u) = \sum_{u \in V(G)} d(u)\mu(u)$$

$$M_2 = \sum_{(u,v) \in E(G)} d(u)d(v) = \frac{1}{2} \sum_{u \in V(G)} d^2(u)\mu(u)$$

where the positive quantity $\mu(u)$ stands for the average of the degrees of the vertices adjacent to $u \in V(G)$. Since $\delta \leq \mu(u) \leq \Delta$ for any $u \in V(G)$, we get

$$\delta \leq \frac{2M_2}{M_1} = \frac{\sum_{u \in V(G)} d^2(u)\mu(u)}{\sum_{u \in V(G)} d(u)\mu(u)} \leq \Delta$$

From this one obtains

$$\frac{[d(G)]}{\Delta} = \frac{2m}{\Delta n} \leq T(G) = \frac{mM_1}{nM_2} \leq \frac{2m}{\delta n} = \frac{[d(G)]}{\delta} \tag{14}$$

It follows that for a polyhedral graph G_P (where $\delta \geq 3$ and $[d] < 6$) the inequality $T(G_P) < 2$ holds. Currently, there is no information on the upper bounds of the topological descriptor $T(G)$. It can be verified that the lower bound for $T(G)$ is zero. It is easy to construct a connected triregular graph G_t for which $T(G_t) \leq \varepsilon$ where ε is an arbitrary small positive number.

6. Final remarks, open problems, conjectures

As we have already mentioned, complete bipartite graphs $K_{\alpha,\beta}$ on $\alpha + \beta$ vertices for which $1 \leq \alpha < \beta$ are positive integers, form a subset of connected biregular graphs of class 1.

It is easy to show that for any α, β positive integers, (where $2 \leq \alpha < \beta$) there exist infinitely many non-isomorphic 3-connected biregular graphs of class 1 which are not complete bipartite graphs. An infinite sequence of such graphs [23] (denoted by $H_{\alpha,\beta}(J)$ where $J \geq 3, 4, 5, \dots$) can be easily constructed from J complete bipartite graphs $K_{\alpha,\beta}$ as it is illustrated in Figure 4.

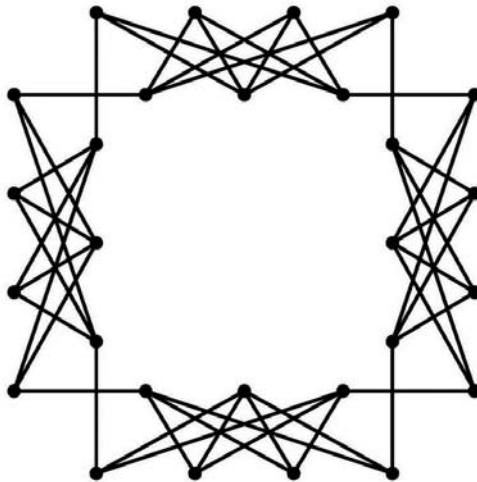


Figure 4. Construction of the biregular non-complete bipartite graph of class 1 denoted by $H_{3,4}(4)$ (case of $\alpha = 3, \beta = 4$ and $J = 4$)

It can be easily proved that there exist polyhedral graphs (3-connected planar graphs) with a domination degree P for which $3 \leq P \leq 10$ holds.

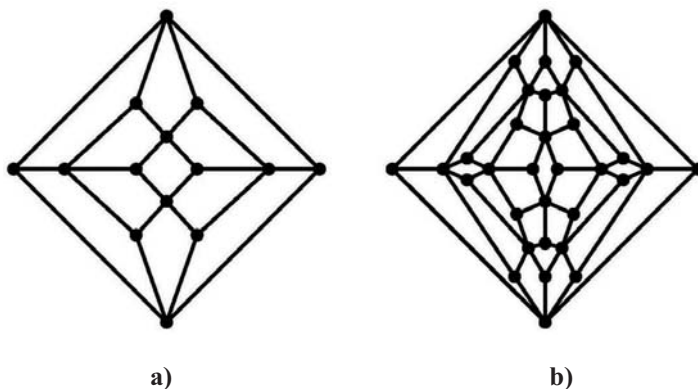


Figure 5. Polyhedral graphs belonging to the family of connected biregular graphs of class 1, the rhombic dodecahedron graph a) and the rhombic triacontahedron graph b)

Conjectures:

- i) There is no polyhedral graph with a domination degree larger than ten.
- ii) The only polyhedral graph with maximum domination degree 10 is the dual of the truncated dodecahedron graph having twenty vertices of degree 3 and twelve vertices of degree 10.
- iii) The set of biregular polyhedral graphs of class 1 (graphs having two distinct domination degrees) is finite.
- iv) There exist only two polyhedral graphs belonging to the family of connected biregular graphs of class 1: the graph of the rhombic dodecahedron and the graph of the rhombic triacontahedron. Both of them are non-complete bipartite graphs having rhombus faces, only. See Figure 5.

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