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Global Forcing Number of Some Chemical Graphs*

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Abstract

A global forcing set S of a simple connected graph G with a perfect matching is a set of edges such that no two different perfect matchings of G coincide on S. The minimum size of global forcing sets of G is called the global forcing number of G. In this paper we give two characterizations for a set of edges to be a global forcing set of a graph. As their applications, we obtain explicit formulas for the global forcing numbers of two kinds of hexagonal systems, parallelogram $B_{p,q}$ and zigzag multiple chain $Z(k,l)$, and we prove that the global forcing number of a catafused coronoid with n hexagons is n or $n-1$, correcting the mistake of a known result. Finally, we obtain a sharp lower bound $\lceil \frac{2f}{3} \rceil$ on the global forcing number of a boron-nitrogen fullerene graph B by showing that any two adjacent faces of B form a nice subgraph, where f is the face number of B.

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1 Introduction

The concept of innate degree of freedom of a Kekulé structure was introduced by Klein and Randić [8], which coincides with the concept of forcing number of a perfect matching introduced by Harary et al. in [6]. The forcing number of a perfect matching is the smallest cardinality of edge sets that is contained only in this perfect matching. Later, there are some research papers concerning forcing number, see [1, 2, 7, 9, 10, 12, 13, 19].

In [15, 16], Vukičević et al. classified all the Kekulé structures of C_{60} into six classes according to their innate degree of freedom. But there are still considerable different structures among the Kekulé structures with the same innate degree of freedom. In order to deal with so many Kekulé structures in fullerene graphs, Vukičević [17] recently introduced a further forcing concept under the name "global (or total) forcing set". Let G be a graph with a perfect matching. A global forcing set of G is a set of edges $S \subseteq E(G)$ that can distinguish all perfect matchings of G completely, *i.e.*, no two different matchings of G coincide on S. Different from forcing set, global forcing set is defined globally in a graph, *i.e.*, without reference to a particular perfect matching. Hence studying global forcing set is a significant problem. In [14, 17], Vukičević gave an explicit formula for the global forcing number of the grid graphs and established a bound on the global forcing number of the triangular grid with equal number of vertices in each row and column. Došlić [5] proved that the global forcing number of all cata-condensed benzenoids with n hexagons is n. Vukičević and Došlić provided a method to study global forcing set of a graph in terms of nice cycles of the graph.

In this paper we shall characterize global forcing sets of a graph, and then as applications, we obtain the global forcing numbers of some chemical graphs. The paper is organized as follows. In Section 2, we give two characterizations for a set of edges to be a global forcing set. Accordingly, we provide a way to find a global forcing set of a graph. In Section 3, we give explicit formulas for the global forcing numbers of two kinds of hexagonal systems, parallalogram $B_{p,q}$ and zigzag multiple chain $Z(k, l)$. Došlić [5] obtained that the global forcing number of any catafused coronoid CC_n with n hexagons is n. In Section 4, however we found that it is wrong and show that the global forcing number of CC_n is $n - 1$ if CC_n has two adjacent L_2 mode hexagons on its main ring, and n otherwise. In Section 5, we obtain a sharp lower bound $\lceil \frac{2f}{3} \rceil$ on the global forcing number of a boron-nitrogen fullerene graph B by showing that any two adjacent faces of

B form a nice subgraph, where f is the face number of B. Furthermore, we point out that the lower bound can be achieved by any 3-resonant BN-fullerene graph.

2 Definitions and some results

We use [11] for terminology and notation not defined here. Let G be a graph. A set of independent edges of a graph G is called a *matching* of G. A *perfect matching* (or *Kekulé structure* in chemical literature) M of G is a matching such that every vertex is incident with exactly one edge in M. An edge of G is *allowed* if it lies in some perfect matching of G and *forbidden* otherwise. A graph is said to be *elementary* if its allowed lines form a connected subgraph of G. For a connected bipartite graph, the elementary property is equivalent to the property that all edges are allowed [11].

A subgraph H of G is said to be *nice* if $G - V(H)$ has a perfect matching. Let G be a bipartite graph with a perfect matching M and C a cycle of G. We call C an M-*alternating* cycle if the edges of C appear alternately in M and $E(G) \setminus M$. It is obvious that an even cycle C of G is nice if and only if there is a perfect matching M of G such that C is M-alternating. Let G be a plane bipartite graph and C the boundary of a face f of G. If G has a perfect matching M such that C is an M-alternating cycle, then f is called a *resonant face*. It is obvious that a face is resonant if and only if the boundary of it is a nice cycle.

The *symmetric difference* of two sets A and B is the set of elements belonging to exactly one of A and B, denoted by $A \triangle B$.

Let $G = (V, E)$ be a graph with a perfect matching and $\mathcal{M}(G)$ the set of all perfect matchings in G. Let $S \subseteq E(G)$ and $S = \{e_1, e_2, \ldots, e_{|S|}\}$. Let $\mathcal{M}(G)|_S = \{M|_S : M \in \mathcal{M}(G)\}$, where $M|_S$ denotes the restriction of M on S, *i.e.*, $M|_S = M \cap S$. Define f_S: $\mathcal{M}(G)|_S \to \{0,1\}^{|S|}$ by

$$[f_S(M|_S)]_i = \begin{cases} 1, & \text{if } e_i \in M|_S, \\ 0, & \text{otherwise.} \end{cases}$$

If f_S is an injection, then S is called a *global forcing set* of G. The smallest cardinality of global forcing sets of G is called the *global forcing number* of G, denoted by $\gamma(G)$. If G has no perfect matchings, then we regard $\emptyset$ as its global forcing set. Hence, all the graphs we considered in this paper have a perfect matching.

For example, we consider the graph Naphthalene shown in Fig. 1. Naphthalene contains three different perfect matchings, $M_1 = \{e_1, e_4, e_6, e_9, e_{11}\}$, $M_2 = \{e_2, e_4, e_6, e_8, e_{10}\}$, $M_3 = \{e_1, e_3, e_5, e_7, e_9\}$. Let $S = \{e_6, e_9\}$. Since $M_1 \cap S = \{e_6, e_9\}$, $M_2 \cap S = \{e_6\}$, $M_3 \cap S = \{e_9\}$, we know that f_S is an injection. Hence S is a global forcing set of Naphthalene.

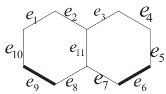


Fig. 1. $\{e_6, e_9\}$ is a global forcing set.

Let G be a simple connected graph with the cyclomatic number $c(G) = |E(G)| - |V(G)| + 1$. In [5], Došlić established the following bounds on the global forcing number of a graph.

Proposition 2.1. *[5] Let G be a simple connected graph with a perfect matching. Then $\lceil \log_2 K(G) \rceil \leq \gamma(G) \leq c(G)$, where $K(G) = |\mathcal{M}(G)|$. Moreover, $\gamma(G) = c(G)$ if and only if all cycles of G are nice.*

Since the global forcing number of a general graph may not equal the bounds, and since the enumeration of perfect matchings in general graphs is a #P-complete problem, it makes sense to search for an approach to determine the global forcing number of G. In the following theorem, we give a characterization for a global forcing set of a graph.

Theorem 2.2. *Let G be a graph with a perfect matching. Then $S \subseteq E(G)$ is a global forcing set of G if and only if S intersects each nice cycle of G.*

Proof. Let C be any nice cycle of G. Then $G - V(C)$ has a perfect matching. Therefore, we can find two perfect matchings M_1, M_2 of G such that $M_1 \triangle M_2 = C$. If S is a global forcing set of G, then $M_1|_S \neq M_2|_S$. Since M_1 and M_2 have the same restriction on $E(G) \setminus E(C)$, we know that $S \nsubseteq E(G) \setminus E(C)$. That implies that $S \cap C \neq \emptyset$.

Conversely, suppose that S intersects each nice cycle of G. If S is not a global forcing set of G, then G has two different perfect matchings M_1 and M_2 such that $M_1|_S = M_2|_S$. Then $S \cap (M_1 \triangle M_2) = \emptyset$. Note that $M_1 \triangle M_2$ consists of at least one M_1 and M_2 alternating cycle. Let C be any such cycle. Then C is a nice cycle. Hence $S \cap C \neq \emptyset$. But $C \subseteq (M_1 \triangle M_2)$, a contradiction. $\square$

The following corollary is of fundamental importance. It will play a crucial role in determining the global forcing number of a graph.

Corollary 2.3. *Let G be a connected graph with a perfect matching, and $\mathbf{T} = \{T|T$ is a connected spanning subgraph of G without any nice cycle of $G\}$. Then*

$$\gamma(G) = c(G) - \max_{T \in \mathbf{T}}\{c(T)\}$$

Proof. For each $T \in \mathbf{T}$, since T is a connected spanning subgraph of G without any nice cycle of G, $E(G) \setminus E(T)$ intersects each nice cycle of G. Since $V(G) = V(T)$, by Theorem 2.2, we know that

$$\gamma(G) = |E(G)| - \max_{T \in \mathbf{T}}\{|E(T)|\} = c(G) - \max_{T \in \mathbf{T}}\{c(T)\}$$

□

Corollary 2.3 shows that one can always demonstrate a minimum global forcing set by exhibiting a maximum connected spanning subgraph without any nice cycle of G. In the following, we always assume that T is a connected spanning subgraph without any nice cycle of a graph.

Theorem 2.4. *S is a global forcing set of G if and only if for each nice subgraph H, $S|_H$ is a global forcing set of H.*

Proof. Suppose for each nice subgraph H of G, $S|_H$ is a global forcing set of H. In particular, since G is a nice subgraph of itself, S is a global forcing set of G.

Conversely, suppose S is a global forcing set of G and H is any nice subgraph of G. Let $S' := S|_H$. If H has at most one perfect matching, then $\emptyset$ is a global forcing set of H. Since $\emptyset \subseteq S'$, S' is also a global forcing set of H. If H has at least two perfect matchings, let M_1', M_2' be any two perfect matchings of H. Since H is a nice subgraph, $G - V(H)$ has a perfect matching M_0. Let $M_1 = M_1' \cup M_0$, $M_2 = M_2' \cup M_0$. Then M_1 and M_2 are perfect matchings of G. Because S is a global forcing set of G, we have $M_1|_S \neq M_2|_S$. Hence $M_1'|_{S'} \neq M_2'|_{S'}$ and $S|_H$ is a global forcing set of H. □

The following corollary is a direct consequence of Theorem 2.4.

Corollary 2.5. *Let G be a simple connected graph with a perfect matching. If $G_1, G_2, \ldots, G_k$ are nice subgraphs of G which are pairwise disjoint, then $\gamma(G) \geq \gamma(G_1) + \ldots + \gamma(G_k)$.*

3 Hexagonal systems

A *hexagonal* (or *benzenoid*) *system* is a finite connected planar graph without cut vertices in which every interior region is bounded by a regular hexagon of unit side length. A hexagonal system (HS) H is said to be *cata-condensed* if every vertex of H is on the boundary of H. Otherwise, the graph is *peri-condensed*.

Theorem 3.1. *[5] Let H be a cata-condensed HS with n hexagons. Then $\gamma(H) = n$.*

Lemma 3.2. *[5] Let H be an elementary peri-condensed HS with n hexagons. Then $\gamma(H) \leq n - 1$.*

It can be seen that the global forcing number of a cata-condensed HS can be obtained by Theorem 3.1. For a peri-condensed HS, there are few results. In this section, we will consider two kinds of peri-condensed hexagonal systems and give explicit formulas for the global forcing numbers of them.

A straight line segment C with end points P_1, P_2 is called a *cut segment* of a hexagonal system H if

(a) C is orthogonal to one of the three edge directions ,

(b) each of P_1, P_2 is the center of an edge lying on the contour of H,

(c) the graph obtained from H by deleting all edges intersected by C has exactly two components.

As an illustrative example, P_1P_2 is a cut segment of the hexagonal system shown in Fig. 2.

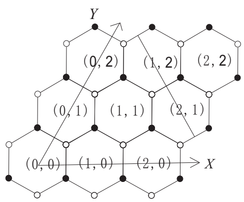


Fig. 2. Labels for the hexagons of $B_{3,3}$ and a cut segment P_1P_2 of $B_{3,3}$.

Let **C** denote the set of edges of H intersected by C. Then **C** is called an (*elementary, orthogonal*) cut of H. Color the vertices of H with black and white. As indicated in Fig. 2, every cut **C** has the property that all vertices next to the cut segment on the one side of the segment are black and on the other side are white. Therefore, the components of $H-C$ will be called the *black bank* $B(\mathbf{C})$ and the *white bank* $W(\mathbf{C})$, respectively.

Benzenoid parallelogram $B_{p,q}$ is a hexagonal system which consists of $p\times q$ hexagons, arranged in q rows, each row consisting of p hexagons. For convenience, the hexagons of $B_{p,q}$ $(p,q\geq 2)$ are labeled as followed. Firstly, we establish an affine coordinate system XOY (see Fig. 2): take the bottom side as the x-*axis*, a lateral side as y-*axis*, their intersection as the origin O such that both sides form an angle of $60°$. For any positive integer n, let Z_n be the finite set $\{0,1,\ldots,n-1\}$. The distance between a pair of parallel edges in a hexagon is a unit length. Each hexagon is labeled by the coordinates (x,y) of its center, where $x\in Z_p$, $y\in Z_q$. Hence such a hexagon is denoted by $h_{x,y}$, $H_y=\bigcup_{x=0}^{p-1}h_{x,y}$ is called the yth layer $(0\leq y\leq q-1)$.

For a bipartite graph G=(B,W), we use $B(G)$ and $W(G)$ to denote the sets of black vertices and white vertices of G respectively. Let $b(G)$= $|B(G)|$ and $w(G)$=$|W(G)|$. For $Q\subseteq V(G)$, denote its neighborhood by $N_G(Q)$. The following is the famous Hall's Theorem.

Theorem 3.3. *(Hall's Theorem) A bipartite graph G=(B,W) has a matching that saturates B if and only if for all $Q\subseteq B$, $|N_G(Q)|\geq|Q|$.*

We call two cycles *tangent* if they share exactly one path. We call a cycle a *k-face cycle* if the interior region surrounded by the cycle contains k faces. A k-face cycle of H is *skew* if the line connected the centers of the k faces is a straight line which is not parallel to x-*axis* and y-*axis*.

Lemma 3.4. *Let $\mathcal{S}$ be any set of skew 2-face cycles of $B_{p,q}$ which are pairwise tangent or disjoint. Then the union U of the cycles of $\mathcal{S}$ does not contain any nice cycle of $B_{p,q}$.*

Proof. Suppose, to the contrary, that U contains a nice cycle C_0 of $B_{p,q}$. Then the interior region of C_0 is composed of several skew 2-faces. Suppose C is the leftmost cut segment of $B_{p,q}$ such that C_0 is on the left bank of the corresponding elementary cut of C, say $\mathbf{C}$ (see Fig. 3). Such a segment must exist, since $h_{p-1,q-1}$ is not contained in any skew 2-face cycle. Let $B(\mathbf{C})$ be the upper component. If $|\mathbf{C}|=l+1$, we claim that $b(B(\mathbf{C}))-w(B(\mathbf{C}))=l$. In fact, let $\{x_1,\ldots,x_{l+1}\}$ be the set of all the black vertices of $\mathbf{C}$, where x_1 is the upmost vertex. Then for x_1, there is a zigzag path with x_1 and a white vertex as its two end vertices along the upper side of H_{q-1}. For each other vertex x_i $(i=2,\ldots,l+1)$, there is a zigzag path with x_i and a black vertex as its two end vertices along the lower side of H_{q+1-i} (see Fig. 3). Those paths are disjoint and cover all the vertices of $B(\mathbf{C})$. Then $b(B(\mathbf{C})) - w(B(\mathbf{C})) = l$. Hence the claim holds.

Since C_0 is a nice cycle of $B_{p,q}$, $G := B_{p,q} - V(C_0)$ has a perfect matching. Let Q be the set of all the black vertices of $B(\mathbf{C})$. Since $l+1 \geq 2$, removing C_0 from $B_{p,q}$ will reduce at least two neighbor vertices of Q. Thus $|N_G(Q)| \leq |N_{B_{p,q}}(Q)| - 2 = w(B(\mathbf{C})) + |\mathbf{C}| - 2 = b(B(\mathbf{C})) - 1 = |Q| - 1$. By Hall's Theorem, $B_{p,q} - V(C_0)$ has no perfect matchings, a contradiction. □

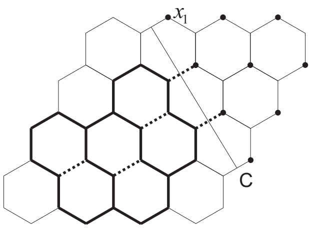


Fig. 3. Illustration for the proof of Lemma 3.4.

Theorem 3.5. *Let $q \leq p$. Then*

$$\gamma(B_{p,q}) = \begin{cases} \frac{pq+q}{2}, & \text{if } q \text{ is even,} \\ \frac{pq+p}{2}, & \text{otherwise.} \end{cases}$$

Proof. We first consider even q. A hexagonal chain is called a k-chain if it consists of k hexagons. Then $B_{p,q}$ is composed of $p-q+1$ skew q-chains, 2 skew $(q-i)$-chains for $i = 1, \ldots, q-1$ (see Fig. 4(a)). Every k-chain ($k = 1, \ldots, q$) contains $\lfloor \frac{k}{2} \rfloor$ pairwise tangent or disjoint skew 2-face cycles. Since $\lfloor \frac{q}{2} \rfloor \times (p-q+1) + 2 \times (\lfloor \frac{q-1}{2} \rfloor + \lfloor \frac{q-2}{2} \rfloor + \ldots + \lfloor \frac{2}{2} \rfloor + \lfloor \frac{1}{2} \rfloor) = \frac{q}{2} \times (p-q+1) + 4 \times [(\frac{q}{2} - 1) + \ldots + 1] = \frac{pq-q}{2}$, it follows that $B_{p,q}$ has $\frac{pq-q}{2}$ pairwise tangent or disjoint skew 2-face cycles. By Lemma 3.4, the union U of those cycles does not contain any nice cycle of $B_{p,q}$. By adding as many edges as possible to U on condition that they cannot generate a new cycle, we can obtain a spanning subgraph T of $B_{p,q}$ (see Fig. 4(b)). By Corollary 2.3, we know that $E(B_{p,q}) \setminus E(T)$ is a global forcing set of $B_{p,q}$. Therefore, $\gamma(B_{p,q}) \leq |E(B_{p,q})| - |E(T)| = c(B_{p,q}) - c(T) = pq - \frac{pq-q}{2} = \frac{pq+q}{2}$.

On the other hand, let $G_i = (\bigcup_{y=0}^{q-1-i} h_{i,y}) \cup (\bigcup_{x=i+1}^{p-1} h_{x,q-1-i})$ $(i = 0, \ldots, q-1)$. We claim that each G_i is a nice subgraph of $B_{p,q}$. In fact, for any G_i, $B_{p,q} - G_i$ has one or two components. Let the upper component be K_1 (when $i = 0, K_1 = \emptyset$), the other one

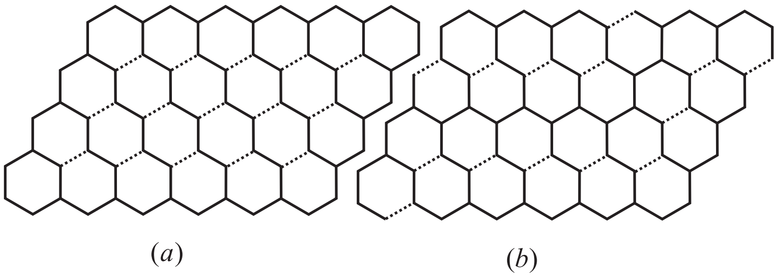


Fig. 4.(a) The skew chains in $B_{6,4}$;(b) A connected spanning subgraph T without any nice cycle of $B_{6,4}$.

K_2 (when $i = q-1, K_2 = \emptyset$). As illustrated in Fig. 5, $V(K_1)$, $V(K_2)$ can be covered by some disjoint odd paths. Then K_1 and K_2 have perfect matchings. Hence each G_i is a nice subgraph of $B_{p,q}$.

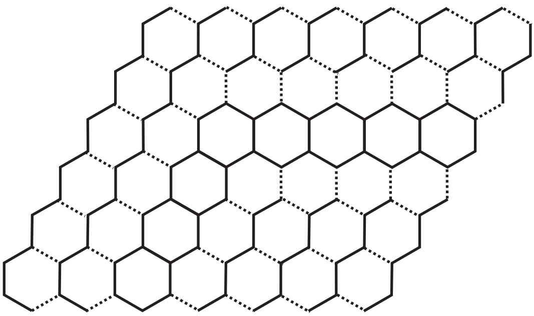


Fig. 5. G_2 in $B_{7,6}$ and the covered odd paths in $B_{7,6} - G_2$.

In particular, G_0, G_2,. . . ,G_{q-2} are disjoint nice subgraphs of $B_{p,q}$, $\gamma(G_i) = p+q-1-2i$, for $i = 0, 2, \ldots, q-2$. By Corollary 2.5, $\gamma(B_{p,q}) \geq \gamma(G_0) + \gamma(G_2) + \ldots + \gamma(G_{q-2}) = (p+q-1) + (p+q-5) + \ldots + (p-q+3) = \frac{pq+q}{2}$. So we have that $\gamma(B_{p,q}) = \frac{pq+q}{2}$.

If q is odd, then $\lfloor \frac{q}{2} \rfloor \times (p-q+1) + 2 \times (\lfloor \frac{q-1}{2} \rfloor + \lfloor \frac{q-2}{2} \rfloor + \ldots + \lfloor \frac{2}{2} \rfloor + \lfloor \frac{1}{2} \rfloor) = \frac{q-1}{2} \times (p-q+3) + 4 \times [(\frac{q-1}{2} - 1) + \ldots + 1] = \frac{pq-p}{2}$. By the similar discussion, we have $\gamma(B_{p,q}) \leq \frac{pq+p}{2}$. On the other hand, since $G_0, G_2, \ldots, G_{q-1}$ are disjoint nice subgraphs, it follows that $\gamma(B_{p,q}) \geq \gamma(G_0) + \gamma(G_2) + \ldots + \gamma(G_{q-1}) = (p+q-1) + (p+q-5) + \ldots + (p-q+1) = \frac{pq+p}{2}$. Hence $\gamma(B_{p,q}) = \frac{pq+p}{2}$. $\square$

In the following, we consider the global forcing number of zigzag multiple chain $Z(k,l)$

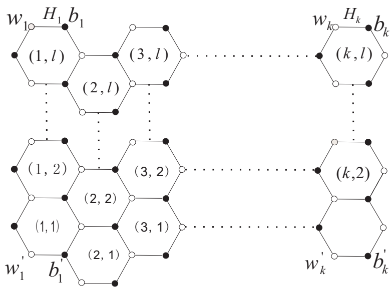


Fig. 6. Label the center of each hexagon of zigzag multiple chain $Z(k,l)$.

[3] which is illustrated in Fig. 6. For convenience, we label the center of each hexagon of $Z(k,l)$ with an ordered pair (x,y) (see Fig. 6) and denote each hexagon of $Z(k,l)$ by $h_{x,y}$ ($x = 1,2,\ldots,k$, $y = 1,2,\ldots,l$). Color the vertices of $Z(k,l)$ with white and black. Let $H_i = \bigcup_{j=1}^{l} h_{i,j}$, P_i (Q_i) be the zigzag path along the left (right) side of H_i ($i = 1,\ldots,k$). Denote the end vertices of P_i (resp. Q_i) by w_i, w_i' (resp. b_i, b_i') (see Fig. 6).

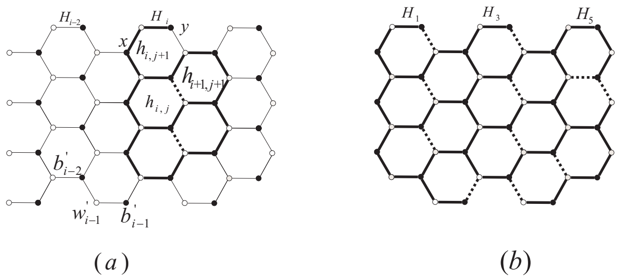


Fig. 7. Illustration for the proof of Theorem 3.6.

Theorem 3.6. $\gamma(Z(k,l)) = l\lceil \frac{k}{2} \rceil + \lfloor \frac{k}{2} \rfloor$.

Proof. The argument is similar to the proof of Theorem 3.5. We only prove the situation of old k here.

Without loss of generality, we always assume that w_1b_1 is higher than w_2b_2. Let $H = H_1 \cup h_{2,1} \cup H_3 \cup \ldots \cup h_{k-1,1} \cup H_k$. Then H is a cata subgraph of $Z(k,l)$ covering all the vertices of $Z(k,l)$. Therefore H is a nice subgraph of $Z(k,l)$. By Theorems 2.4 and 3.1, $\gamma(Z(k,l)) \geq \gamma(H) = l\lceil \frac{k}{2} \rceil + \lfloor \frac{k}{2} \rfloor$.

On the other hand, let $\mathcal{C} = \{C_{i,j} | C_{i,j} = \partial(h_{i,j} \cup h_{i+1,j+1}), i = 1, 3, 5, \ldots, k-4, k-2, 1 \leq j \leq l-1\}$. Clearly, any two cycles of $\mathcal{C}$ are pairwise tangent or disjoint.

Claim. The union U of all the cycles in $\mathcal{C}$ does not contain any nice cycle of $Z(k,l)$.

Suppose, to the contrary, that U contains a nice cycle C_0 of $Z(k,l)$. Let P_i be the rightmost zigzag path such that the region surrounded by C_0 is on the right side of P_i. Let J denote the subgraph of $Z(k,l)$ induced by the edge set of $C_0 \cap P_i$. Then J is bipartite and we have $|W(J)| - |B(J)| \geq 1$. Since $Z(k,l)$ is a plane graph, the interior region of C_0 is composed of several 2-faces. Suppose $C_{i,j}$ is the upmost 2-face cycle which is in the interior region of C_0 and intersects P_i. There are two neighbor vertices of $C_{i,j}$ in $h_{i,j+1}$, say x and y. Without loss of generality, let $x \in P_i$, $y \in Q_i$. It is easy to see that x, y are black vertices (see Fig. 7(a)).

Let $P = P_i \cup \{w_i b_i\} \cup Q_{b_i y}$, where $Q_{b_i y}$ is a part of Q_i with b_i and y as its two end vertices. Since P is an odd path, we have $|W(P)| = |B(P)|$. $Z(k,l) - V(P)$ has two components, say K_1, K_2, where K_1 is the left component. If $i \neq 1$, then $K_1 = H_1 \cup \ldots \cup H_{i-2} \cup \{b'_{i-2} w'_{i-1}, w'_{i-1} b'_{i-1}\}$. If $i = 1$, then $K_1 = \emptyset$. In either case, $|B(K_1)| = |W(K_1)|$.

For convenience, let $G' = Z(k,l) - V(C_0)$. Let S be the set of all the black vertices of K_1 and P in graph G', *i.e.*, $S = B(K_1) \cup B(P) \setminus B(J)$. Since $N_{G'}(S) = W(K_1) \cup W(P) \setminus W(J)$, we have $|N_{G'}(S)| = |W(K_1)| + |W(P)| - |W(J)| = |B(K_1)| + |B(P)| - |W(J)| \leq |B(K_1)| + |B(P)| - |B(J)| - 1 = |S| - 1$. By Hall's Theorem, G' has no perfect matchings. Then C_0 is not a nice cycle of $Z(k,l)$, a contradiction. Consequently, the claim holds. By adding as many edges as possible to U on condition that they cannot generate a new cycle, we can obtain a spanning subgraph T of $Z(k,l)$ (see Fig. 7(b)). Therefore, $\gamma(Z(k,l)) \leq |E(Z(k,l))| - |E(T)| = c(Z(k,l)) - c(T) = l\lceil \frac{k}{2} \rceil + \lfloor \frac{k}{2} \rfloor$. □

4 Catafused coronoids

In this section, we shall consider the global forcing number of catafused coronoid graphs. A *coronoid graph* is a graph obtained from a peri-condensed benzenoid by deleting some internal vertices and/or edges in such a way that any remaining edge belongs to at least one hexagon and the remaining graph has only one bounded non-hexagonal face. A coronoid graph is *catafused* if it has no vertices shared between three hexagons. The *inner dual* of a coronoid graph G is obtained from its standard dual by deleting the vertices corresponding to the non-hexagonal faces and all incident edges (indicated in Fig. 8). A coronoid graph is catafused if and only if its inner dual is a unicyclic graph.

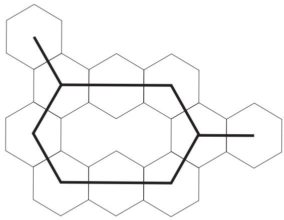


Fig. 8. Illustration for the inner dual of a catafused coronoid.

Let CC_n denote a catafused coronoid with n hexagons. The hexagons that correspond to the vertices of the unique cycle in its inner dual graph form a ring in CC_n, which is called the *main ring* of CC_n. Some modes of hexagons are presented in Fig. 9, the reader may refer to [3] for the more concrete definition of modes of hexagons. It is obvious that a catafused coronoid may possess the modes L_1, L_2, A_2 and A_3. The A_2 and A_3 mode hexagons on the main ring are called *key* hexagons of CC_n. Let H be a subgraph of CC_n. Let $CC_n \ominus H$ be the remaining subgraph of CC_n after deleting all the vertices which belong only to H.

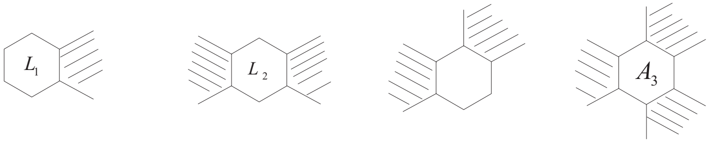


Fig. 9. Some modes of hexagons in a benzenoid.

In [5], Došlić obtained that $\gamma(CC_n) = n$. However, we have found a counterexample (see Fig. 10). Since the spanning subgraph T of the graph CC_{10} shown in bold has no nice cycles of CC_{10}, by Corollary 2.3, we have $\gamma(CC_{10}) \leq c(CC_{10}) - c(T) = 10+1-2 = 9$.

Lemma 4.1. *[5] CC_n is Kekuléan and elementary.*

Lemma 4.2. *Let C be a cycle of CC_n which does not contain the non-hexagonal face in its interior region. Then C is a nice cycle of CC_n if and only if C does not contain all the key hexagons in its interior region.*

Proof. Suppose that C contains all the key hexagons of CC_n in its interior region. Since C does not contain the non-hexagonal face in its interior region, there must be some L_2

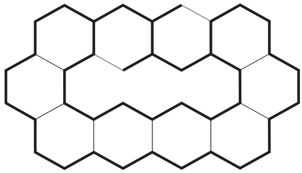


Fig. 10. A catafused coronoid CC_{10} whose global forcing number is not 10.

mode hexagons on the main ring of CC_n which are not contained in the interior region of C and form a linear hexagonal chain, say G_0 (see Fig. 11(a)). G_0 intersects C at two edges. Embed CC_n in the plane with the two edges vertical. Since G_0 is a linear hexagonal chain, any perfect matching of G_0 cannot contain two vertical edges. Hence $G_0 - V(G_0 \cap C)$ has no perfect matchings. Since $G_0 - V(G_0 \cap C)$ is a component of $CC_n - V(C)$, it follows that $CC_n - V(C)$ has no perfect matchings. This means that C is not a nice cycle.

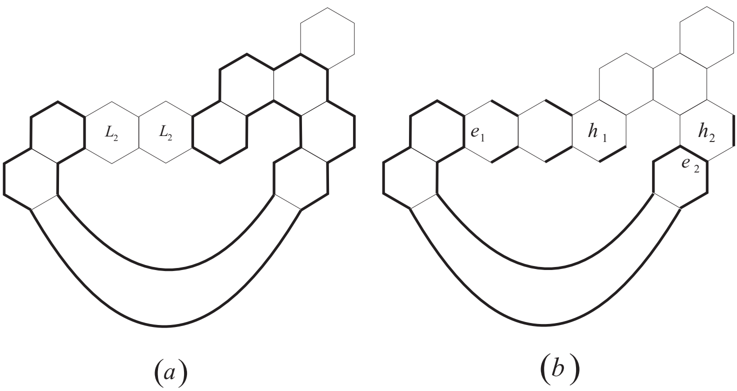


Fig. 11. Illustration for the proof of Lemma 4.2.

Conversely, suppose that C does not contain all the key hexagons in its interior region. If C does not contain any hexagon of the main ring in its interior region, then one component of $CC_n \ominus C$ is a catafused coronoid with less hexagons, and the other possible components are cata-condensed hexagonal systems, which are elementary. Hence C intersects each component of $CC_n \ominus C$ at only one edge, which belongs to a perfect matching of the component. Then each component contains a matching covering all the vertices of the component except the two endpoints of the joining edge. The union of such matchings in every components of $CC_n \ominus C$ is a perfect matching of $CC_n - V(C)$. Hence, C is a nice cycle. If C contains at least one hexagon of the main ring in its interior region, then each component of $CC_n \ominus C$ is a cata-condensed hexagonal system. There is only one component, say G_0, containing some hexagons of the main ring. Then G_0 intersects C at two edges, say e_1 and e_2. Since G_0 is a cata-condensed hexagonal system with at least one A_2 or A_3 mode hexagon, e_1 and e_2 are contained in a perfect matching of G_0. In fact, let h_1 (h_2) be the key hexagon of G_0 which is nearest to e_1 (e_2). Since G_0 contains at least one key hexagon, h_1 and h_2 must exist. Let e_1 (resp. e_2) be a matching edge. Then it determines a perfect matching in a linear sub-hexagonal chain of G_0 except h_1 (resp. h_2) (see Fig. 11(b)). But since h_1 and h_2 are A_2 or A_3 mode, e_1 and e_2 cannot force the edges of other hexagons to be matching edges. Hence the degree of all remaining vertices are at least two after deleting all the matching edges and their incident vertices. Thus the remaining subgraph is still a cata-condensed hexagonal system. Hence G_0 has a perfect matching containing e_1 and e_2. Note that the other components of $CC_n \ominus C$ are cata-condensed hexagonal systems intersecting C at only one edge. Then each component of $CC_n - V(C)$ has a perfect matching. Hence C is a nice cycle. □

Lemma 4.3. *Let G be a plane connected graph. If G has exactly k interior faces, then $c(G) = k$.*

Proof. Let T_0 be a spanning tree of G. Once we delete an edge $e \in E(G) \setminus E(T_0)$ from G, the number of interior faces of $G - e$ decreases by one. Note that T_0 has no interior faces. Hence when G turns into T_0, the number of interior faces decreases by k. That means that we need to delete k edges from G. Thus $c(G) = k$. □

Theorem 4.4.

$$\gamma(CC_n) = \begin{cases} n-1, & \textit{if } CC_n \textit{ has two adjacent } L_2 \textit{ mode hexagons on its main ring,} \\ n, & \textit{otherwise.} \end{cases}$$

Proof. Let f be the bounded non-hexagonal face of CC_n. If CC_n has an L_2 mode hexagon f_1 on its main ring, then $\partial(f \cup f_1)$ isolates one vertex in its interior region. If CC_n has no L_2 mode hexagons on is main ring, it still has two adjacent key hexagons f_2 and f_3 on its main ring, then $\partial(f \cup f_2 \cup f_3)$ isolates odd vertices in its interior region. In either case, CC_n has at least one non-nice cycle. Then $\gamma(CC_n) \leq n$.

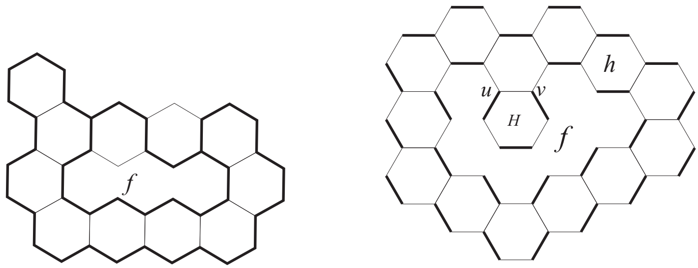


Fig. 12. Illustration for the proof of Theorem 4.4.

To form the main ring, CC_n has at least one key hexagon. Let h_0 be a key hexagon of CC_n. Then by Lemma 4.2, $CC_n \ominus h_0$ is a nice subgraph of CC_n. By Theorems 2.4 and 3.1, $\gamma(CC_n) \geq \gamma(CC_n \ominus h_0) = n - 1$.

Case 1. CC_n has two adjacent L_2 mode hexagons on its main ring, say h_1, h_2.

$\partial(f \cup h_1)$ is a non-nice cycle of CC_n, since it isolates a vertex of CC_n. On the other hand, by Lemma 4.2, $\partial(CC_n \ominus (h_1 \cup h_2))$ is a non-nice cycle. The union U of the two cycles generates a new cycle which is non-nice since it isolates a vertex of h_2 (see Fig. 12(a)). Hence, U does not contain any nice cycle of CC_n. By adding as many edges as possible to U on condition that they cannot generate a new cycle, we can obtain a spanning subgraph T with $c(T) = 2$. By Corollary 2.3, $\gamma(CC_n) \leq n + 1 - 2 = n - 1$.

Case 2. CC_n does not have two adjacent L_2 mode hexagons on its main ring.

It suffices to show that every global forcing set contains at least n edges. By Corollary 2.3, it is equivalent to prove that $c(T) \leq 1$ for any spanning subgraph T without any nice cycle of CC_n. Suppose, to the contrary, that $c(T) \geq 2$. By Lemma 4.3, T has at least two interior faces. Let $F_1, F_2, \ldots, F_k$ $(k \geq 2)$ be the interior faces of T. If all the interior faces do not contain f, then by Lemma 4.2, every interior face of T contains all the key hexagons. Since such faces are mutually disjoint except for their boundaries, we know that $k = 1$, a contradiction. Consequently, there is one interior face, say F_1, containing f, and one interior face, say F_2, containing all the key hexagons.

We claim that F_1 contains at least one hexagon of the main ring. Suppose, to the contrary, that F_1 does not contain any hexagon of the main ring. Since $V(CC_n)$ is covered by $\partial(f)$ and the other even cycle, $\partial(f)$ is a nice cycle. Then there is a perfect matching

M of CC_n such that $\partial(f)$ is an M-alternating cycle. If $\partial(F_1) = \partial(f)$, then $\partial(F_1)$ is a nice cycle, a contradiction. Hence, there are some cata-condensed sub-hexagonal systems in the interior region of $\partial(F_1)$. For any such cata-condensed sub-hexagonal system H, since F_1 does not contain any hexagon of the main ring, H intersects $\partial(F_1)$ at only one edge, say uv. If u and v are not covered by the edges of $M|_{F_1}$, then both of them must be covered by edges of $M|_H$. Consequently, $M \triangle \partial(H)$ is a perfect matching of CC_n containing uv (see Fig. 12(b)). Carry out the above procedure for each cata-condensed sub-hexagonal system in the interior region of $\partial(F_1)$, we can get a perfect matching M' of CC_n such that $\partial(F_1)$ is an M'-alternating cycle. Then $\partial(F_1)$ is a nice cycle, a contradiction. Hence F_1 contains at least one hexagon on the main ring. Let h be such a hexagon. Since F_2 contains all the key hexagons on the main ring, the hexagons of the main ring contained in F_1 are all L_2 mode. Hence h is L_2 mode. Since CC_n does not have two adjacent L_2 mode hexagons on its main ring, the two hexagons adjacent to h are key hexagons, which are thus contained in F_2. Since all the hexagons of the main ring not contained in F_2 are in a component of $CC_n \ominus F_2$, F_2 contains all the hexagons of the main ring except h. Since h is contained in F_1, $F_1 \cup F_2$ contains the main ring. Then all the hexagons contained in the interior region of F_1 and F_2 form a catafused coronoid whose boundary is a nice cycle. Hence $F_1 \cup F_2$ contains a nice cycle, a contradiction. Therefore $c(T) \leq 1$. $\square$

5 BN-fullerene graphs

In this section, we shall establish a sharp lower bound on the global forcing number of boron-nitrogen fullerenes. Boron-nitrogen fullerenes (or BN-fullerene graphs) are cubic plane graphs with only square and hexagonal faces. By a simple calculation using Euler's formula, we have that there are exactly 6 square faces and others hexagonal. Among all BN-fullerene graphs, there is a class of graphs, we denote it by $\mathcal{T} = \bigcup_{n\geq 1}\{T_n\}$, where T_n consists of n concentric layers of hexagons, capped on each end by a cap formed by three squares. In the degenerate case $n = 0$, we get the ordinary cube.

A graph G is *cyclically k-edge connected* if G cannot be separated into two components, each of which contains a cycle, by deleting fewer than k edges. The *cyclical edge-connectivity* of G, denoted by $c\lambda(G)$, is the greatest number k such that G is cyclically k-edge connected. Let $\mathcal{H}$ be a set of disjoint faces of G. If G has a perfect matching M such that the boundary of each face of $\mathcal{H}$ is an M-alternating cycle, then $\mathcal{H}$ is called a *resonant pattern* of G. For a positive integer k, if every i $(1 \leq i \leq k)$ disjoint faces of G form a resonant pattern, then G is called k-resonant.

In the following, we present some properties of BN-fullerene graphs.

Lemma 5.1. *[4] For every BN-fullerene graph B. If $B \in \mathcal{T}$, then $c\lambda(B) = 3$; otherwise, $c\lambda(B) = 4$.*

Lemma 5.2. *[18]If G is a cyclically 4-edge connected 3-regular plane graph, then there are neither three faces which are pairwise adjacent but do not share a common vertex, nor two faces sharing at least two disjoint edges in G.*

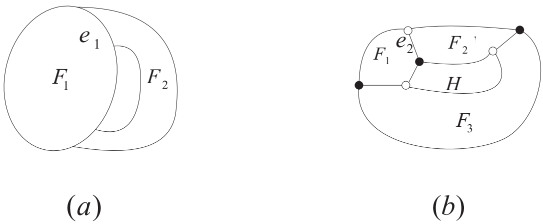


Fig. 13. Illustration for the proof of Corollary 5.3.

Let $C = v_1v_2 \ldots v_nv_1$ be a cycle. Let $d_C(v_i, v_j)$ denote the distance between v_i and v_j on C. Two edges v_iv_{i+1} and v_jv_{j+1} of C with $1 \leq i, j \leq k$ are *opposite* if $d_C(v_i, v_j) = d_C(v_{i+1}, v_{j+1})$ and equals half of the length of C, where $v_{k+1} = v_1$. We say the distance between v_iv_{i+1} and v_jv_{j+1} is 1 if $d_C(v_i, v_{j+1}) = 1$ or $d_C(v_{i+1}, v_j) = 1$.

Corollary 5.3. *Let B be a BN-fullerene graph. Then there are no two faces sharing at least two disjoint edges in B. Moreover, if F_1, F_2, F_3 are three different faces of B and the distance between $F_1 \cap F_2$ and $F_1 \cap F_3$ is 1 in F_1, then $F_2 \cap F_3 = \emptyset$.*

Proof. By Lemma 5.1, $c\lambda(B) = 3$ or 4.

If $c\lambda(B) = 4$, by Lemma 5.2, we can draw the conclusion.

If $c\lambda(B) = 3$, then $B \in \mathcal{T}$. If there are two faces sharing at least two disjoint edges, then two such edges form a cyclically 2-edge cut (see Fig. 13(a)). This is a contradiction. Consequently, any two faces share at most one edge. Moreover, consider the three faces F_1, F_2, F_3 of B. Let $F_1 \cap F_3 = \{e_1\}$, $F_1 \cap F_2 = \{e_2\}$, where $e_i = b_iw_i$ $(i = 1, 2)$. Suppose to the contrary, that $F_2 \cap F_3 \neq \emptyset$. Since F_2 and F_3 share at most one edge, we suppose that $F_2 \cap F_3 = \{e_3\}$, where $e_3 = b_3w_3$. Since the distance between e_1, e_2 is 1, it follows that $w_1b_2 \in E(B)$ or $w_2b_1 \in E(B)$. Without loss of generality, suppose $w_1b_2 \in E(B)$. Note that $\{e_1, e_2, e_3\}$ is a cut. Suppose w_3 is in the component of $B - \{e_1, e_2, e_3\}$ which contains w_1, b_2 (see Fig. 13(b)). The boundary of the component is a cycle, denoted by C. Let the subgraph induced by the vertices in the interior region of C be H. Since C is an even cycle, it follows that $b(C) = w(C)$. Note that $b(C) - 1$ black vertices on C are connected to white vertices of H and $w(C) - 2$ white vertices on C are connected to black vertices of H. Hence $3b(H) - (w(C) - 2) = 3w(H) - (b(C) - 1)$. Thus $3b(H) + 1 = 3w(H)$. This is impossible. Hence, $F_2 \cap F_3 = \emptyset$. □

Theorem 5.4. *[18] Every BN-fullerene graph is 2-resonant.*

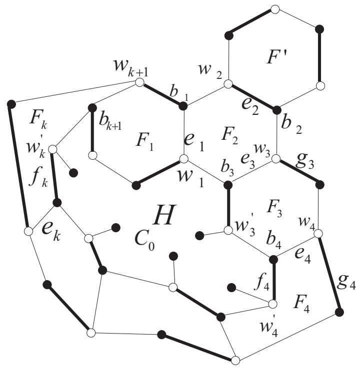


Fig. 14. Illustration for the proof of Lemma 5.5.

Basing on the above results, we have the following interesting property of BN-fullerene graphs.

Lemma 5.5. *For a BN-fullerene graph B, any two adjacent faces form a nice subgraph of B.*

Proof. Let F_1, F_2 be any two adjacent faces of B. If one of the two faces, say F_2, is a square, by Corollary 5.3, there is another face F_3 adjacent to F_2, but disjoint with F_1. By Theorem 5.4, $\{F_1, F_3\}$ is a resonant pattern. Thus we can find a perfect matching M such that $\partial(F_1 \cup F_2)$ is an M-alternating cycle. That is, $\partial(F_1 \cup F_2)$ is a nice cycle of B.

Suppose that both of F_1 and F_2 are hexagons. By Corollary 5.3, there is another face $F^{'}$ adjacent to F_2, but disjoint with F_1. Let $F_1 \cap F_2 = \{e_1\}$, $F^{'} \cap F_2 = \{e_2\}$. Since $\{F_1, F^{'}\}$ is a resonant pattern, we can get a perfect matching M of B such that F_1 and $F^{'}$ are M-alternating, $e_1 \notin M$ and $e_2 \in M$.

Color the vertices of B with white and black. Let $e_1{=}b_1w_1$, $e_2{=}b_2w_2$. If $e_3{=}b_3w_3 \in M$, then $\partial(F_1 \cup F_2)$ is an M-alternating cycle. If not, we claim that e_3 is contained in an M-alternating cycle C such that $C \cap \partial(F_1) = \emptyset$ and $C \cap \partial(F^{'}) = \emptyset$. Suppose, to the contrary, that any M-alternating cycle passing through e_3 intersects $\partial(F_1)$ or $\partial(F^{'})$. Since $e_3 \notin M$, b_3 and w_3 are incident with other matching edges, say f_3, g_3 respectively. Note that f_3, g_3 and e_3 are on the boundary of a face, say F_3. If F_3 is a square, then we obtain an M-alternating cycle $\partial(F_3)$ passing through e_3 and by Corollary 5.3, $\partial(F_3) \cap \partial(F_1) = \emptyset$, $\partial(F_3) \cap \partial(F^{'}) = \emptyset$, a contradiction. Hence, F_3 is a hexagon. The opposite edge of e_3 in F_3, denoted by $e_4 = b_4w_4$, is not a matching edge, otherwise it contradicts the hypothesis. So b_4, w_4 are incident with other matching edges, say f_4, g_4 respectively. Clearly, e_4, f_4, g_4 are on the boundary of a face, say F_4. By the similar discussion, F_4 is a hexagon. Go on with this procedure, we finally obtain a sequence of hexagonal faces $F_3, F_4, \ldots, F_k$ $(k \geq 4)$.

Let $G_i = \partial(F_1) \cup \partial(F_2) \cup \partial(F^{'}) \cup \partial(F_3) \cup \ldots \cup \partial(F_i)$ $(i = 3, \ldots, k)$. Suppose F_k is the first hexagon which intersects its former graph at not only one edge. Since $b_kw_k \notin M$, b_k and w_k are incident with other matching edges, say f_k, g_k respectively. Let $f_i = b_iw_i^{'}$ $(i = 3, \ldots, k)$. Since F_k is a hexagon and intersects G_{k-1} at not only the edge b_kw_k, $w_k^{'}$ is adjacent to a black vertex of G_{k-1} other than b_k. Note that all the black vertices of f_i $(i = 3, \ldots, k-1)$ are full degree, $w_k^{'}$ is not adjacent to any b_i. Since B is a plane graph, $w_k^{'}$ is not adjacent to any black endpoint of g_i. Hence, $w_k^{'}$ is adjacent to exactly one black vertex of F_1 and $F^{'}$. In either case, we can get an M-alternating cycle $C_0 \subset E(G)$ passing through all the matching edges $f_3, \ldots, f_k$ (see Fig. 14). Let H be the subgraph induced by the vertices in the interior region of C_0. Since $M|_H$ is a perfect matching of H, we have $b(H) = w(H)$. On the other hand, since B is a 3-regular bipartite graph, some vertices on C_0 are connected to some vertices of H. Let A be the set of all such vertices of C_0. If $w_k^{'}$ is adjacent to a black vertex of F_1, then $A = \{w_3^{'}, \ldots, w_k^{'}\} \cup (A \cap F_1)$. Suppose A has w white vertices and b black vertices. Since there are equal white vertices and black vertices in $A \cap F_1$, $w > b$. On the other hand, since H is a bipartite graph,

$|E(H)| = 3b(H) - w = 3w(H) - b$, which yields $w = b$, a contradiction. If w_k' is adjacent to a black vertex of F', then we can also draw a contradiction by the similar discussion. Consequently, e_3 is contained in an M-alternating cycle C_0 such that $C_0 \cap \partial(F_1) = \emptyset$ and $C_0 \cap \partial(F') = \emptyset$. Hence, we can obtain a perfect matching such that $\partial(F_1 \cup F_2)$ is an M-alternating cycle by switching the matching edges of C_0. Thus, $\partial(F_1 \cup F_2)$ is a nice cycle of B. $\square$

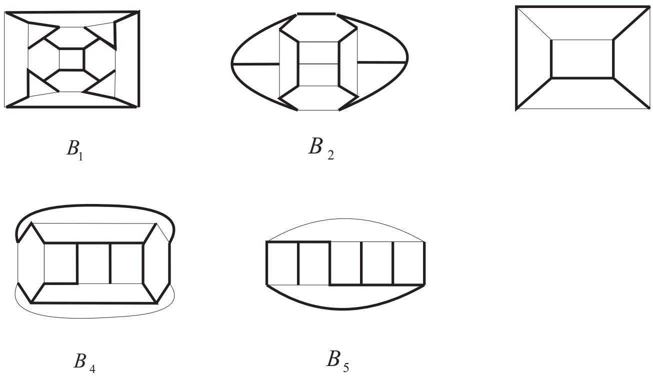


Fig. 15. The 3-resonant BN-fullerene graphs with cyclical edge connectivity 4.

Theorem 5.6. *[18] A cyclically 4-edge connected BN-fullerene graph B is k-resonant ($k \geq 3$) if and only if B is one of B_1, B_2, B_3, B_4, B_5 in Fig. 15.*

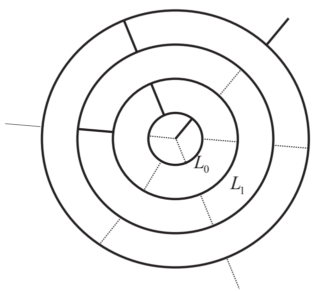


Fig. 16. A maximum spanning subgraph of T_n without any nice cycle of T_n.

Theorem 5.7. *For any BN-fullerene graph B with f faces, we have*

$$\gamma(B) \geq \lceil \frac{2f}{3} \rceil$$

and the bound can be achieved by the 3-resonant BN-fullerene graphs.

Proof. Since B is elementary, each face of B is resonant and any 2-face cycle is a nice cycle of B. Consequently, for any spanning subgraph T of B, each face of T must contain at least 3 faces of B. Since B has f faces, T has at most $\lfloor \frac{f}{3} \rfloor$ faces. By Lemma 4.3, $c(T) \leq \lfloor \frac{f}{3} \rfloor - 1$. Hence we have $\gamma(B) \geq f - 1 - (\lfloor \frac{f}{3} \rfloor - 1) = \lceil \frac{2f}{3} \rceil$.

We now show that this lower bound can be achieved when B is 3-resonant. If $c\lambda(B) = 4$, by Theorem 5.6, B is one of B_1, B_2, B_3, B_4, B_5 shown in Fig. 16. Their subgraphs T are all shown in bold in Fig. 16, which implies that $c(T) \geq \lfloor \frac{f}{3} \rfloor - 1$ for each B_i $(i = 1, \ldots, 5)$. Hence $\gamma(B) \leq \lceil \frac{2f}{3} \rceil$. If $B = T_n \in \mathcal{T}$, since there are three faces in each layer, the total face number of T_n is $3n+6$. Denote the boundary of the ith layer by L_i $(0 \leq i \leq n)$. There are odd vertices in the interior region of each L_i. Then L_i is not a nice cycle. Choose an arbitrary traverse edge e_j (the traverse edge is defined as the edge whose endpoints are on two layers) in each layer $(0 \leq j \leq n + 1)$. Let $T' = \bigcup_{i=0}^{n}(L_i \cup \{e_i\}) \cup \{e_{n+1}\}$. All L_i $(0 \leq i \leq n)$ are pairwise disjoint, then T' is a connected spanning subgraph of T_n without any nice cycle of T_n (see Fig. 16). By Corollary 2.3, we have

$$\gamma(T_n) \leq |c(T_n)| - |c(T')| = 3n + 6 - 1 - (n + 1) = 2n + 4 = \lceil \frac{2f}{3} \rceil$$

By the first statement of the theorem, we know that $\gamma(B) = \lceil \frac{2f}{3} \rceil$ for any 3-resonant BN-fullerene graph. □

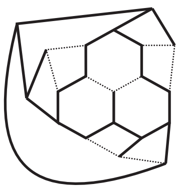
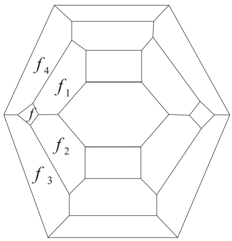


Fig. 17. G_1 with a spanning subgraph (in bold).

Remark 5.8. *For non-3-resonant BN-fullerene graphs B, $\gamma(B)$ may be equal to or strictly larger than $\lceil \frac{2f}{3} \rceil$.*



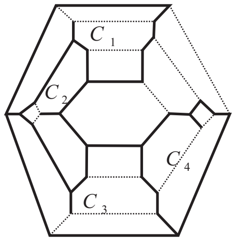


Fig. 18. (a) Labels for some faces of G_2;(b) A subgraph T_2 of G_2.

For example, we consider two non-3-resonant BN-fullerene graphs. For the graph G_1 shown in Fig. 17, a subgraph T is shown in bold. Since $c(T) = 3 = \lfloor \frac{f}{3} \rfloor - 1$, by Corollary 2.3, $\gamma(G_1) \leq c(G_1) - c(T) = 11 - 3 = 8$. By Theorem 5.7, we have $\gamma(G_1) = 8 = \lceil \frac{2f}{3} \rceil$. For the graph G_2 shown in Fig. 18(a) which has 18 faces, we have

Proposition 5.9. $\gamma(G_2) = \lceil \frac{2f}{3} \rceil + 1 = 13$.

Proof. By Theorem 5.7, $\gamma(G_2) \geq \lceil \frac{2f}{3} \rceil = 12$. We can show that the spanning subgraph T_1 of G_2 shown in bold in Fig. 18(b) does not contain any nice cycle of G_2. Since $c(T_1) = 4$, by Corollary 2.3, $\gamma(G_2) \leq 17 - 4 = 13$.

We now show that $\gamma(G_2) \neq 12$, which will establish the proposition. If $\gamma(G_2) = 12$, then by Corollary 2.3, we know that for any maximum spanning subgraph T_2 without any nice cycle of G_2, $c(T_2) = 5$. This implies that T_2 has 6 faces and each face contains exactly 3 faces of G_2. Since any two squares are disjoint, the face cycle of T_2 which contains a square in its interior region can only be the boundary of one of K_1, K_2, K_3, K_4 shown in Fig. 19. We can show that any 3-face cycle as one of $\partial(K_1)$, $\partial(K_2)$ and $\partial(K_3)$ is a nice cycle. Hence any face cycle of T_2 which contains a square in its interior region can only be as $\partial(K_4)$. Since G_2 has 6 pairwise disjoint squares, and since T_2 has 6 faces, we know that each face cycle of T_2 can only be as $\partial(K_4)$. Hence the square f of G_2 indicated in Fig. 18(a) is contained in the interior region of one of the four possible 3-face cycles: $\partial(f \cup f_1 \cup f_2)$, $\partial(f \cup f_2 \cup f_3)$, $\partial(f \cup f_3 \cup f_4)$ and $\partial(f \cup f_4 \cup f_1)$. We can deduce that T_2 contains a nice cycle of G_2 if f is contained in the interior region of any of the above four cycles, a contradiction. Hence $\gamma(G_2) \neq 12$. $\square$

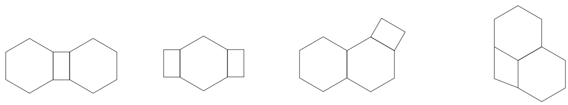


Fig. 19. 3-face cycles which contain a square in their interior regions.

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