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The HOMFLY Polynomial for a Family of Polyhedral Links

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Abstract

A mathematical methodology for understanding the construction of polyhedral links has been developed in this paper. A family of polyhedral links is generated based on the geometry of polyhedron by using the operation of 'Tangle Covering'. We then show that the HOMFLY polynomial for this family of links can be derived from the Z^w-polynomial of the original polyhedral graph by a simple substitution rule. The result thus generalizes the computation of the HOMFLY polynomial to the family of nearly arbitrary polyhedral links, which complements our previous research on semi-regular case. In addition, our work also gives the HOMFLY polynomials of rational links, a typical link family which could facilitate a number of important problems in knot theory.

1. Introduction

A link [1,2], a set of knotted loops all tangled up together, is the main study object of knot theory. In 1961, the first topological catenane [3] was synthesized in laboratory, which immediately attracts the continuous interests of chemists. With the development of nanotechnology, a variety of polyhedral catenates such as DNA tetrahedron [4–6], DNA cube [7, 8], DNA truncated octahedron [9], DNA octahedron [10, 11], DNA dodecahedron [6, 12], DNA icosahedron [13, 14] and DNA bipyramid [15], have

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been synthesized by using DNA blocks. These exciting results trigger scientist to study the properties of these topological nontrivial structures embedding in 3D space, and also open a new area for knot theory.

Polyhedral links, a mathematical model of DNA polyhedra, model edges of interlinked and interlocked architectures as two strands of DNA chains [16-26]. Qiu's group has constructed some graceful links from Goldberg polyhedra and carbon nanotubes by the means of 'three cross-curves and double lines covering' [16, 17]. Subsequently, other construction methods have been developed based on the various polyhedrons [18–26]. However, due to the structure diversity of knots and catenanes [27], some new methods to describe them may have to be considered. In this paper, a general operation 'Tangle Covering' is proposed to generate a family of polyhedral links including regular polyhedral links and semi-regular polyhedral links discussed in our previous work [23]. This work provides a sound basis for the understanding of structure, properties and, further, the molecular design of DNA polyhedral catenanes.

The HOMFLY polynomial, a powerful invariant of oriented links, can distinguish most links from their mirror images [28], which is crucial in biomolecular system [29, 30]. It is known that the computation of HOMFLY polynomial is believed to be #P hard [31, 32]. For the links with small crossing number, we can resort to some software packages [33]. However, in the case of the links with large crossing number, the computation becomes a difficult problem. This paper identifies a large family of links with special structure such that their HOMFLY polynomial can be easily obtained only by computing the Z^w–polynomial, a weighted dichromatic polynomial, of original graph. The general result greatly improves our previous work [23], and is expected to facilitate the subsequent identification of the topological link type and chirality of polyhedral links.

2. The construction for a family of polyhedral links

Some basic definitions, notations and operations are given in advance.

In graph theory, *a planar graph* is a graph which can be embedded in the plane or the sphere. A planar graph already drawn in the plane without edge intersections is called *a plane graph.* All convex polyhedrons are 3-connected planar graphs [34], hence any one of them has a plane graph. While in this paper, '*polyhedral graph*' means one of its plane graphs. *An isthmus* of a graph is an edge whose deletion increases the number of components. *A loop* of a graph is an edge whose endpoints are the same.

A tangle is defined to be two strands twisted around each other, which is an ideal building block for the construction of knots and links [35, 36]. Four basic blocks used in this paper are *a-tangle*, *b-tangle*, *c-tangle* and *d-tangle* (see Figure 1). The length of each tangle is defined as one half of its crossing number. *X-tangle* denotes one of four tangles throughout this paper.

We shall now construct a new link from any connected plane graph G as follow: cover each edge with a tangle of length *n* (*a-tangle*, *b-tangle*, *c-tangle* or *d-tangle*), and connect the ends of tangles along each edge in a face of *G*. The resulting link is denoted by $D(G)$ and is called *polyhedral link* if *G* is a polyhedral graph. This operation is called '*Tangle Covering*' which generalizes Jaeger link [35] and also the construction method in Ref. [36], and produces a family of links $\overline{D(G)}$ by changing the type and length of tangle for each edge.

In particular, the above operation will be called '*X-Tangle Covering*' if we use only *X-tangle* to cover each edge. Also, it will be called '*X^n-Tangle Covering*' if all tangles have the same length *n* for each edge. We write $D_X(G)$ and $D_X^n(G)$ for the two links obtained from *G* by using the operations '*X-Tangle Covering*' and '*X^n-Tangle Covering*', respectively (see Figure 1).

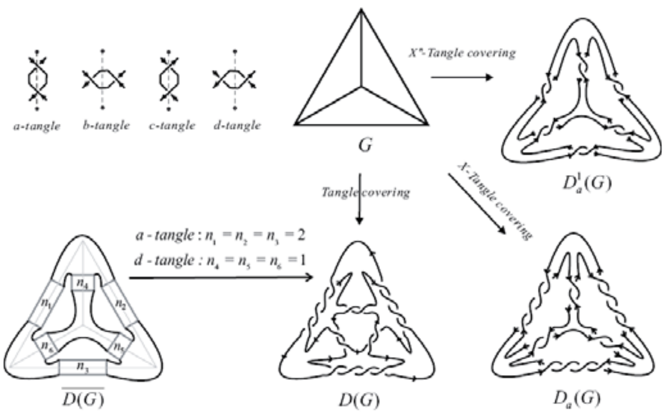

Figure 1. Three operations on tetrahedron: *Tangle Covering*, *X-Tangle Covering and X^n-Tangle Covering*.
(Each box in $\overline{D(G)}$ contains a tangle with length n_i *f*or i=1…6.)

Some interesting results can be obtained if G only consists of one edge e. If e is a loop, then $D_a^1(G)$ and $D_a^n(G)$ will be two torus links and $D_b^1(G)$ and $D_b^n(G)$ will be two trivial knots. Otherwise, the situation is reversed (see Figure 2).

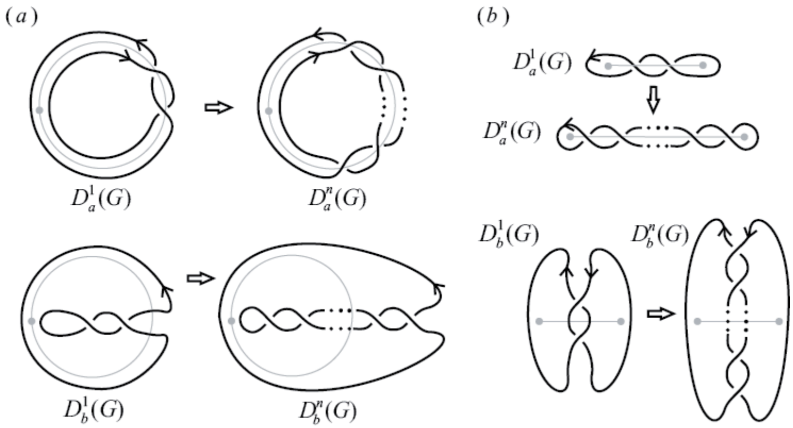

Figure 2. (a) Four links derived from a loop. (b) Four links derived from an isthmus.

Semi-regular polyhedral links are obtained from any polyhedral graph G by using the operation "*X-Tangle Covering*", which have the same type of tangles for each edge. They are classified as four classes of links according to the type of tangle used in '*X-Tangle Covering*', and denoted by $D_A(G), D_B(G),\ D_C(G)$ and $D_D(G)$. The four classes of links are just the links constructed in Ref. [23].

Regular polyhedral links are obtained from any polyhedral graph G by using the operation '*X^n-Tangle Covering*'. They are also classified as four classes of links according to the type of tangle used in '*X^n-Tangle Covering*', and denoted by $D_A^n(G), D_B^n(G), D_C^n(G)$ and $D_D^n(G)$. In contrast to semi-regular links, regular links have more special structure from which we can obtain the relationship between Tutte polynomial and HOMFLY polynomial as discussed in Section 4.

Remark. Note that all polyhedral graphs are connected. However, the operation '*Tangle Covering*' can be also extended to any plane graph G. In the case that G is disconnected, its each connected component will produce a link as above, then $D(G)$ will be the collection of such links. We take this convention that $D(G)$ will be an isolated Jordan curves if G consists of an isolated vertex.

3. Dichromatic polynomial and HOMFLY polynomial

In this section, we establish the relationship between the HOMFLY polynomial [28] of a family of links obtained in section 2 and the dichromatic polynomial [36] of the original graph, which is stimulated by the earlier work of Traldi. From this result, we derive the connection between polyhedral links and the associated polyhedral graph. Hereinafter we use $G-e$ and $G \cdot e$ to denote the graphs obtained from graph G by deleting and contracting edge e respectively. $|V(G)|$ denotes the number of vertices of a graph G, and $|E(G)|$ the number of edges.

A graph G is a *weighted graph* if each edge e is given a label $w(e)$.

Definition 3.1 *The dichromatic polynomial* $Z^w(G) = Z^w(G;x,y) \in \mathbb{Z}[x,y]$ *for a weighted graph G is defined by the following rules:*

(1) *If G is an isolated vertex, then*

$$Z^w(G;x,y) = x.$$

(2) *If* $G \cup H$ *is the disjoint union of graphs G and H, then*

$$Z^w(G \cup H;x,y) = Z^w(G;x,y)Z^w(H;x,y).$$

(3) *If an edge e of G is a loop, then*

$$Z^w(G;x,y) = (1+w(e)y)Z^w(G-e;x,y).$$

Otherwise,

$$Z^w(G;x,y) = Z^w(G-e;x,y)+w(e)Z^w(G\cdot e;x,y).$$

Definition 3.2 *The HOMFLY polynomial* $H(L) = H(L;x,y,z) \in \mathbb{Z}[x,y,z]$ *for an oriented link L is defined by the following rules:*

(1) *If L is a trivial knot, then*

$$H(L;x,y,z) = 1.$$

(2) *If two links* L_1 *and* L_2 *are equivalent under ambient isotopic, then*

$$H(L_1;x,y,z) = H(L_2;x,y,z).$$

(3) *Suppose that three link diagrams* L_+, L_- *and* L_0 *are different only on a local region, as shown in Figure 3. Then*

$$xH(L_+;x,y,z)+yH(L_-;x,y,z)+zH(L_0;x,y,z)=0.$$

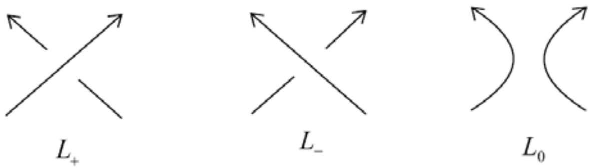

Figure 3. Three link diagrams: L_+, L_- and L_0.

Here, we can obtain the HOMFLY polynomial in two variables:

$$H(L;v,z)=H(L;v^{-1},-v,-z).$$

The HOMFLY polynomial has the following properties:

(1) If L is the connected sum of L_1 and L_2, denoted by $L_1\#L_2$, then

$$H(L;x,y,z)=H(L_1;x,y,z)H(L_2;x,y,z).$$

(2) If L is the disjoint union of L_1 and L_2, denoted by $L_1\cup L_2$, then

$$H(L;x,y,z)=(-\frac{x+y}{z})H(L_1;x,y,z)H(L_2;x,y,z).$$

(3) If L^* is the mirror image of L, then

$$H(L^*;v,z)=H(L;v^{-1},z).$$

It shows that the HOMFLY polynomial of an achiral link must satisfy:

$$H(L;v,z)=H(L;v^{-1},z).$$

Lemma 3.3 *Let e be an edge of a plane graph G which is covered by a-tangle of the length n. If e is a loop, then*

$$H\big(D(G)\big)=\left[-\left(-\frac{y}{x}\right)^n\left(\frac{x+y}{z}\right)+\left(\left(-\frac{y}{x}\right)^n-1\right)\left(\frac{z}{x+y}\right)\right]H\big(D(G-e)\big). \quad (1)$$

Otherwise,

$$H\big(D(G)\big)=\left(-\frac{y}{x}\right)^n H(D(G\cdot e);x,y,z)+\left(\left(-\frac{y}{x}\right)^n-1\right)\left(\frac{z}{x+y}\right)H\big(D(G-e)\big). \quad (2)$$

Proof We proceed by induction on the length of *a-tangle*, and split into two cases according to whether *e* is a loop or not.

(i) We first assume that the length of a-tangle is one.

If e is a loop, applying the definition (3) of HOMFLY polynomial to one of the crossings of $D(G)$ associated to e, we can obtain

$$H\big(D(G)\big) = \left[\left(-\frac{y}{x}\right)\left(-\frac{x+y}{z}\right) - \frac{z}{x}\right] H\big(D(G-e)\big).$$

Formula (1) can be easily checked by using the above equation.

If e is not a loop, applying the definition (3) of HOMFLY polynomial to one of the crossings of $D(G)$ associated to e, we can obtain

$$H\big(D(G)\big) = -\frac{y}{x} H\big(D(G \cdot e)\big) - \frac{z}{x} H\big(D(G-e)\big).$$

Formula (2) can be easily checked by using the above equation.

(ii) We now assume that the length of a-tangle is at least two. Let the link $D_{e:n}(G)$ be the same as $D(G)$ only except the length of tangle covering the edge e is n.

If e is a loop, applying the definition (3) of HOMFLY polynomial to one of the crossings of $D(G)$ associated to e, two new links $D_{e:n-1}(G)$ and $D(G-e)$ can be obtained as depicted pictorially in Figure 4, and

$$H(D(G); x, y, z) = \left(-\frac{y}{x}\right) H\big(D_{e:n-1}(G)\big) + \left(-\frac{z}{x}\right) H\big(D(G-e)\big).$$

Hence we can obtain the following equation by induction hypothesis.

$$\begin{aligned} H\big(D(G)\big) &= \left(-\frac{z}{x}\right) H\big(D(G-e)\big) + \left(-\frac{y}{x}\right) \\ &\times \left[-\left(-\frac{y}{x}\right)^{n-1}\left(\frac{x+y}{z}\right) + \left(\left(-\frac{y}{x}\right)^{n-1} - 1\right)\left(\frac{z}{x+y}\right)\right] H\big(D_{e:n-1}(G-e)\big). \end{aligned}$$

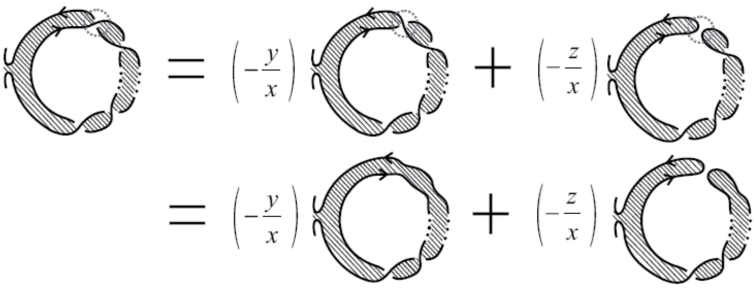

Figure 4. Five diagrams in the equation which differ only on a local region.
(Each diagram L stands for the value of $H(L, x, y, z)$.)

Because $D_{e:\,n-1}(G-e)$ and $D(G-e)$ are isotopic, we obtain

$$H\big(D(G)\big)=\left[-\left(-\frac{y}{x}\right)^{n}\left(\frac{x+y}{z}\right)-\frac{z}{x}+\left(\left(-\frac{y}{x}\right)^{n}+\frac{y}{x}\right)\left(\frac{z}{x+y}\right)\right]H\big(D(G-e)\big).$$

Formula (1) follows immediately from the above equation.

On the other hand, suppose that *e* is not a loop. Similarly, applying the definition (3) of HOMFLY polynomial to one of the crossings of *D*(*G*) associated to *e*, two links $D_{e:\,n-1}(G-e)$ and $D(G-e)$ can be obtained as depicted pictorially in Figure 5(a), then

$$H\big(D(G)\big)=\left(-\frac{z}{x}\right)H\big(D(G-e)\big)+\left(-\frac{y}{x}\right)H\big(D_{e:n-1}(G)\big).$$

By induction hypothesis, we have

$$\begin{aligned}H\big(D(G)\big)&=\left(-\frac{z}{x}\right)H\big(D(G-e)\big)+\left(-\frac{y}{x}\right)^{n}H\big(D_{e:n-1}(G\cdot e)\big)\\&+\left(-\frac{y}{x}\right)\left(\frac{z}{x+y}\right)\left[\left(-\frac{y}{x}\right)^{n-1}-1\right]H\big(D_{e:n-1}(G-e)\big).\end{aligned}$$

Since $D_{e:n-1}(G-e)$ and $D(G-e)$ are isotopic, and $D_{e:n-1}(G\cdot e)$ and $D(G\cdot e)$ are isotopic, we have

$$H(D(G);x,y,z)=\left(-\frac{y}{x}\right)^{n}H\big(D(G\cdot e)\big)+\left[\left(\frac{z}{x+y}\right)\left[\left(-\frac{y}{x}\right)^{n}+\frac{y}{x}\right]-\frac{z}{x}\right]H\big(D(G-e)\big).$$

Formula (2) follows immediately from the above equation.

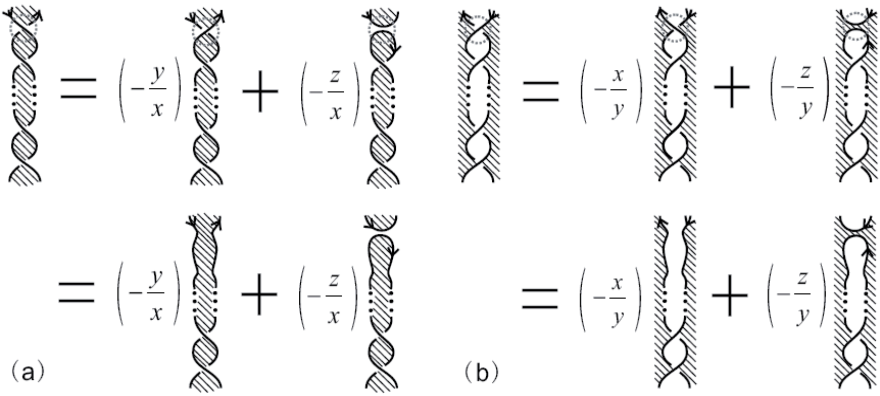

Figure 5. Five diagrams in each of the equations (a) and (b) which differ only on a local region. (Each diagram *L* stands for the value of of *H(L, x, y, z)*.)

Lemma 3.4 *Let e be an edge of a plane graph G which is covered by b-tangle of the length n. If e is a loop, then*

$$H(D(G)) = H\big(D(G-e)\big). \tag{3}$$

Otherwise,

$$H\big(D(G)\big) = \left(-\frac{x}{y}\right)^n H\big(D(G-e)\big) + \left(\left(-\frac{x}{y}\right)^n - 1\right)\left(\frac{z}{x+y}\right) H\big(D(G\cdot e)\big). \tag{4}$$

Proof We proceed by induction on the length of *b-tangle*, and split into two cases according to whether *e* is a loop or not.

(i) We first assume that the length of b-tangle is one.

If *e* is a loop, then $D(G)$ and $D(G-e)$ are ambient isotopic. Hence

$$H\big(D(G)\big) = H\big(D(G-e)\big).$$

If *e* is not a loop, applying the definition (3) of HOMFLY polynomial to one of the crossings of $D(G)$ associated to *e*, we can obtain

$$H\big(D(G)\big) = \left(-\frac{x}{y}\right) H\big(D(G-e)\big) + \left(-\frac{z}{y}\right) H\big(D(G\cdot e)\big).$$

The formula (3) can be easily checked by using the above equation.

(ii) We now assume that the length of b-tangle is at least two. Let the link $D_{e:n}(G)$ *be the same as* $D(G)$ *only except the length of tangle covering the edge e is n.*

If *e* is a loop, then $D(G)$ and $D(G-e)$ are ambient isotopic as above.

Otherwise, applying the definition (3) of HOMFLY polynomial to one of the crossings of $D(G)$ associated to *e*, two links $D_{e:\,n-1}(G)$ and $D(G\cdot e)$ can be obtained as depicted pictorially in Figure 5(*b*), and

$$H\big(D(G)\big) = \left(-\frac{x}{y}\right) H\big(D_{e:n-1}(G)\big) + \left(-\frac{z}{y}\right) H\big(D(G\cdot e)\big).$$

Hence we can obtain the following equation by our induction hypothesis.

$$\begin{aligned} H(D(G);x,y,z) = {} & \left(-\frac{x}{y}\right)^n H\big(D_{e:n-1}(G-e)\big) + \left(-\frac{z}{y}\right) H\big(D(G\cdot e)\big) \\ & + \left(-\frac{x}{y}\right)\left(\frac{z}{x+y}\right)\left[\left(-\frac{x}{y}\right)^{n-1} - 1\right] H\big(D_{e:n-1}(G\cdot e)\big). \end{aligned}$$

Since $D_{e:\,n-1}(G-e)$ and $D(G-e)$ are isotopic, and $D_{e:\,n-1}(G\cdot e)$ and $D(G\cdot e)$ are isotopic, we have

$$
\begin{aligned}
H(D(G);x,y,z) = &\left(-\frac{x}{y}\right)^n H\big(D(G-e)\big) \\
&+\left[\left(\frac{z}{x+y}\right)\left(\left(-\frac{x}{y}\right)^n+\frac{x}{y}\right)-\frac{z}{y}\right] H\big(D(G\cdot e)\big).
\end{aligned}
$$

The formula (4) follows immediately from the equation above. □

Similarly, the Lemmas 3.5 and 3.6 are given below, their proofs are omitted here.

Lemma 3.5 *Let e be an edge of a plane graph G which is covered by c-tangle of the length n. If e is a loop, then*

$$H\big(D(G)\big)=\left[-\left(-\frac{x}{y}\right)^n\left(\frac{x+y}{z}\right)+\left(\left(-\frac{x}{y}\right)^n-1\right)\left(\frac{z}{x+y}\right)\right]H\big(D(G-e)\big).$$

Otherwise,

$$H(D(G);x,y,z)=\left(-\frac{x}{y}\right)^n H\big(D(G\cdot e)\big)+\left(\left(-\frac{x}{y}\right)^n-1\right)\frac{z}{x+y}H\big(D(G-e)\big).$$

Lemma 3.6 *Let e be an edge of a plane graph G which is covered by d-tangle of the length n. If e is a loop, then*

$$H(D(G))=H\big(D(G-e)\big).$$

Otherwise,

$$H\big(D(G)\big)=\left(-\frac{y}{x}\right)^n H\big(D(G-e)\big)+\left(\left(-\frac{y}{x}\right)^n-1\right)\frac{z}{x+y}H\big(D(G\cdot e)\big).$$

Based on the Lemmas 3.3-3.6, we can obtain the following main theorem.

Theorem3.7 *Let G be a weighted plane graph, and let E_a, E_b, E_c and E_d be the numbers of the edges covered by a-tangle, b-tangle, c-tangle and d-tangle, respectively. For any edge e in G, it will be weighted with $w_{a_i}(e)$ for $1\le i\le E_a$, $w_{b_i}(e)$ for $1\le i\le E_b$, $w_{c_i}(e)$ for $1\le i\le E_c$ or $w_{d_i}(e)$ for $1\le i\le E_d$ if the edge e is covered by a-tangle of the length a_i, b-tangle of the length b_i, c-tangle of the length c_i or d-tangle of the length d_i. Then*

$$H(D(G);x,y,z)=-\left(\frac{z}{x+y}\right)^{E_a+E_c+1}\left(-\frac{x}{y}\right)^{\sum_{i=1}^{E_b}b_i-\sum_{i=1}^{E_d}d_i}\prod_{i=1}^{E_a}\left[\left(-\frac{y}{x}\right)^{a_i}-1\right]$$

$$\times\prod_{i=1}^{E_c}\left[\left(-\frac{x}{y}\right)^{c_i}-1\right]Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right), \tag{5}$$

where $$w_{a_i}(e)=\left(\frac{x+y}{z}\right)\left[1-\left(-\frac{x}{y}\right)^{a_i}\right]^{-1},\quad w_{b_i}(e)=\left(\frac{z}{x+y}\right)\left[1-\left(-\frac{y}{x}\right)^{b_i}\right],$$
$$w_{c_i}(e)=\left(\frac{x+y}{z}\right)\left[1-\left(-\frac{y}{x}\right)^{c_i}\right]^{-1} \text{ and } w_{d_i}(e)=\left(\frac{z}{x+y}\right)\left[1-\left(-\frac{x}{y}\right)^{d_i}\right].$$

Proof We proceed by induction on the number of edges of *G*. We consider only two cases when an edge *e* of *G* is covered by *a-tangle* and *b-tangle.* For *c-tangle* and *d-tangle* cases, we can obtain formula (5) by exchanging the variables *x* and *y* in above cases, respectively.

(i) Suppose first that G has exactly one edge e.

Case 1 *The edge e is covered by a-tangle with length* a_1.

If *e* is a loop, $D(G)$ can be described as $D_a^n(G)$ in Figure 2 (a). Applying Lemma 3.3 to *e*, we can obtain

$$H(D(G);x,y,z)=\left[\left(-\frac{y}{x}\right)^{a_1}\left(-\frac{x+y}{z}\right)+\left(\left(-\frac{y}{x}\right)^{a_1}-1\right)\left(\frac{z}{x+y}\right)\right]H(D(G-e);x,y,z).$$
$$=\left(-\frac{y}{x}\right)^{a_1}\left(-\frac{x+y}{z}\right)+\left(\left(-\frac{y}{x}\right)^{a_1}-1\right)\left(\frac{z}{x+y}\right)$$

Since $D(G-e)$ is a trivial knot.

On the other hand, by (1) and (3) of Def. 3.1, we can obtain

$$Z^w(G;x,y)=(1+w_{a_1}(e)y)x.$$

By variable substitution, we have

$$Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right)=\left[1-\left(\frac{x+y}{z}\right)^2\left(1-\left(-\frac{x}{y}\right)^{a_1}\right)^{-1}\right]\left(-\frac{x+y}{z}\right).$$

Hence

$$H(D(G);x,y,z)=-\left(\frac{z}{x+y}\right)^2\left[\left(-\frac{y}{x}\right)^{a_1}-1\right]Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right).$$

If *e* is not a loop, then $D(G)$ is a trivial knot. Hence

$$H(D(G);x,y,z)=1.$$

On the other hand, by Def. 3.1, we can obtain

$$Z^w(G;x,y)=x^2+w_{a_1}(e)x.$$

By variable substitution, we have

$$Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right)=\left[1-\left[1-\left(-\frac{x}{y}\right)^{a_1}\right]^{-1}\right]\left(\frac{x+y}{z}\right)^2.$$

So it is easy to check formula (5) follows from the above equations.

Case 2 *The edge e is covered by b-tangle with length* b_1.

If *e* is a loop, then $D(G)$ is a trivial knot and can be described as $D_b^n(G)$ in Figure 2(a). Hence

$$H(D(G);x,y,z)=1.$$

On the other hand, we have

$$Z^w(G;x,y)=(1+w_{b_1}(e)y)x.$$

By variable substitution, we can obtain

$$Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right)=\left(-\frac{y}{x}\right)^{b_1}\left(-\frac{x+y}{z}\right).$$

So it is easy to check formula (5) from the above equations.

If *e* is not a loop, then $D(G)$ can be described as $D_b^n(G)$ in Figure 2(b). Applying Lemma 3.4 to *e*, we have

$$\begin{aligned}H(D(G);x,y,z)&=\left(-\frac{x}{y}\right)^{b_1}H(D(G-e);x,y,z)+\left(\left(-\frac{x}{y}\right)^{b_1}-1\right)\left(\frac{z}{x+y}\right)H(D(G\cdot e);x,y,z)\\&=\left(\frac{z}{x+y}\right)\left[\left(-\frac{x}{y}\right)^{b_1}-1\right]+\left(-\frac{x}{y}\right)^{b_1}\left(-\frac{x+y}{z}\right)\end{aligned}$$

Since $D(G\cdot e)$ is a trivial knot and $D(G-e)$ is a trivial link with two components.

On the other hand, we have

$$Z^w(G;x,y)=x^2+w_{b_1}(e)x.$$

By variable substitution, we can obtain

$$Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right)=\left[-\frac{x+y}{z}+\left(\frac{z}{x+y}\right)\left[1-\left(-\frac{y}{x}\right)^{b_1}\right]\right]\left(-\frac{x+y}{z}\right).$$

So it is easy to check formula (5) from the above equations.

(ii) We now assume that G has at least two edges, and also consider two cases as above.

Case 1 *The edge e is covered by a-tangle with the length* a_j *for* $1\le j\le |E_a|$.

If *e* is a loop, applying Lemma 3.3 to *e*, we can obtain

$$H(D(G);x,y,z)=\left[-\left(-\frac{y}{x}\right)^{a_j}\left(\frac{x+y}{z}\right)+\left[\left(-\frac{y}{x}\right)^{a_j}-1\right]\left(\frac{z}{x+y}\right)\right]H(D(G-e);x,y,z).$$

Hence, by induction,

$$H(D(G);x,y,z) = -\left(\frac{z}{x+y}\right)\left[\left(-\frac{y}{x}\right)^{a_j} - 1\right]\left[\left(\frac{x+y}{z}\right)^2\left(\left(-\frac{x}{y}\right)^{a_j} - 1\right)^{-1} + 1\right]$$

$$\times\left(\frac{z}{x+y}\right)^{E_a+E_c}\left(-\frac{x}{y}\right)^{\sum_{i=1}^{E_b} b_i - \sum_{i=1}^{E_d} d_i}\prod_{\substack{i=1\\ i\neq j}}^{E_a}\left[\left(-\frac{y}{x}\right)^{a_i} - 1\right]$$

$$\times\prod_{i=1}^{E_c}\left[\left(-\frac{x}{y}\right)^{c_i} - 1\right] Z^w\left(G-e;-\frac{x+y}{z},-\frac{x+y}{z}\right).$$

On the other hand, by Def 3.1 (3), we can obtain

$$Z^w(G;x,y) = (1 + w_{a_j}(e)y)Z^w(G-e;x,y).$$

By variable substitution method, we have

$$Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right) = \left[1 + \left(\frac{x+y}{z}\right)^2\left(\left(-\frac{x}{y}\right)^{a_j} - 1\right)^{-1}\right] Z^w\left(G-e;-\frac{x+y}{z},-\frac{x+y}{z}\right).$$

Hence

$$H(D(G);x,y,z) = -\left(\frac{z}{x+y}\right)\left[\left(-\frac{y}{x}\right)^{a_j} - 1\right]\left(\frac{z}{x+y}\right)^{E_a+E_c}\left(-\frac{x}{y}\right)^{\sum_{i=1}^{E_b} b_i - \sum_{i=1}^{E_d} d_i}$$

$$\times\prod_{\substack{i=1\\ i\neq j}}^{E_a}\left[\left(-\frac{y}{x}\right)^{a_i} - 1\right]\prod_{i=1}^{E_c}\left[\left(-\frac{x}{y}\right)^{c_i} - 1\right] Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right).$$

If e is not a loop, applying the lemma 3.3 to e, we can obtain

$$H(D(G);x,y,z) = \left(-\frac{y}{x}\right)^{a_j} H(D(G\cdot e);x,y,z) + \left(\left(-\frac{y}{x}\right)^{a_j} - 1\right)\left(\frac{z}{x+y}\right) H(D(G-e);x,y,z).$$

Applying the induction hypothesis to $G\cdot e$ and $G-e$, we have

$$H(D(G);x,y,z)$$

$$= -\left(\frac{z}{x+y}\right)^{E_a+E_c+1}\left(-\frac{x}{y}\right)^{\sum_{i=1}^{E_b} b_i - \sum_{i=1}^{E_d} d_i}\prod_{i=1}^{E_a}\left[\left(-\frac{y}{x}\right)^{a_i} - 1\right]\prod_{i=1}^{E_c}\left[\left(-\frac{x}{y}\right)^{c_i} - 1\right]$$

$$\times\left[\frac{x+y}{z}\left(1 - \left(-\frac{x}{y}\right)^{a_j}\right)^{-1} Z^w\left(G\cdot e;-\frac{x+y}{z},-\frac{x+y}{z}\right) + Z^w\left(G-e;-\frac{x+y}{z},-\frac{x+y}{z}\right)\right].$$

On the other hand, by Def 3.1 (3), we can obtain

$$Z^w(G;x,y) = Z^w(G-e;x,y) + w_{a_j}(e)\ Z^w(G\cdot e;x,y).$$

By variable substitution, we have

$$Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right) = Z^w\left(G-e;-\frac{x+y}{z},-\frac{x+y}{z}\right)$$

$$+\frac{x+y}{z}\left(1-\left(-\frac{x}{y}\right)^{a_j}\right)^{-1} Z^w\left(G\cdot e;-\frac{x+y}{z},-\frac{x+y}{z}\right).$$

Hence formula (5) follows immediately from the above equations.

Case 2 *The edge e is covered by b-tangle with the length* b_j *for* $1 \le j \le |E_b|$.

If e is a loop, then $D(G)$ and $D(G-e)$ are ambient isotopic. Hence

$$H(D(G);x,y,z) = H(D(G-e);x,y,z).$$

Applying our induction hypothesis to $D(G-e)$, we have

$$H(D(G-e);x,y,z) = -\left(\frac{z}{x+y}\right)^{E_a+E_c+1}\left(-\frac{x}{y}\right)^{\sum_{i=1}^{E_b} b_i-\sum_{i=1}^{E_d} d_i}\prod_{i=1}^{E_a}\left[\left(-\frac{y}{x}\right)^{a_i}-1\right]$$

$$\times\prod_{i=1}^{E_c}\left[\left(-\frac{x}{y}\right)^{c_i}-1\right]\left[\left(-\frac{y}{x}\right)^{b_j} Z^w\left(G-e;-\frac{x+y}{z},-\frac{x+y}{z}\right)\right].$$

On the other hand, we have

$$Z^w(G;x,y) = (1+w_{b_j}(e)y)Z^w(G-e;x,y).$$

By variable substitution, we can obtain

$$Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right) = \left(-\frac{y}{x}\right)^{b_j} Z^w\left(G-e;-\frac{x+y}{z},-\frac{x+y}{z}\right).$$

Hence formula (5) follows immediately from the above equations.

If e is not a loop, by using Lemma 3.4, we can obtain

$$H(D(G);x,y,z) = \left(-\frac{x}{y}\right)^{b_j} H(D(G-e);x,y,z) + \left(\left(-\frac{x}{y}\right)^{b_j}-1\right)\left(\frac{z}{x+y}\right)H(D(G\cdot e);x,y,z).$$

Applying our induction hypothesis to $G-e$ and $G\cdot e$, we can obtain

$$H(D(G);x,y,z)$$

$$= -\left(\frac{z}{x+y}\right)^{E_a+E_c+1}\left(-\frac{x}{y}\right)^{\sum_{i=1}^{E_b} b_i-\sum_{i=1}^{E_d} d_i}\prod_{i=1}^{E_a}\left[\left(-\frac{y}{x}\right)^{a_i}-1\right]\prod_{i=1}^{E_c}\left[\left(-\frac{x}{y}\right)^{c_i}-1\right]$$

$$\times\left[Z^w\left(G-e;-\frac{x+y}{z},-\frac{x+y}{z}\right)+\frac{z}{x+y}\left(1-\left(-\frac{y}{x}\right)^{b_j}\right)Z^w\left(G\cdot e;-\frac{x+y}{z},-\frac{x+y}{z}\right)\right].$$

On the other hand, we have

$$Z^w(G;x,y) = Z^w(G-e;x,y) + w_{b_j}(e)Z^w(G\cdot e;x,y).$$

By variable substitution, we have

$$Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right) = Z^w\left(G-e;-\frac{x+y}{z},-\frac{x+y}{z}\right)$$

$$+\frac{z}{x+y}\left(1-\left(-\frac{y}{x}\right)^{b_j}\right)Z^w\left(G\cdot e;-\frac{x+y}{z},-\frac{x+y}{z}\right).$$

Hence formula (5) follows immediately from the above equations. □

Remark. Theorem 3.7 holds for all links obtained from the plane graphs, hence also for polyhedral links. Moreover, we have the following corollary for regular polyhedral links, which can be obtained immediately from Theorem 3.7.

Corollary 3.8 *Let G be a weighted polyhedral graph, and each edge e will be given a weight* $w_a(e)$, $w_b(e)$, $w_c(e)$ *or* $w_d(e)$ *if it is covered by a-tangle, b-tangle, c-tangle or d-tangle of the length n. Then*

(1) $$H(D_A^n(G);x,y,z)=-\left(\frac{z}{x+y}\right)^{|E(G)|+1}\left(\left(-\frac{y}{x}\right)^n-1\right)^{|E(G)|}Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right);$$

(2) $$H(D_B^n(G);x,y,z)=-\left(\frac{z}{x+y}\right)\left(-\frac{x}{y}\right)^{n|E(G)|}Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right);$$

(3) $$H(D_C^n(G);x,y,z)=-\left(\frac{z}{x+y}\right)^{|E(G)|+1}\left(\left(-\frac{x}{y}\right)^n-1\right)^{|E(G)|}Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right);$$

(4) $$H(D_D^n(G);x,y,z)=-\left(\frac{z}{x+y}\right)\left(-\frac{y}{x}\right)^{n|E(G)|}Z^w\left(G;-\frac{x+y}{z},-\frac{x+y}{z}\right),$$

where $$w_a(e)=\left(\frac{x+y}{z}\right)\left[1-\left(-\frac{x}{y}\right)^n\right]^{-1},\quad w_b(e)=\left(\frac{z}{x+y}\right)\left[1-\left(-\frac{y}{x}\right)^n\right],$$
$$w_c(e)=\left(\frac{x+y}{z}\right)\left[1-\left(-\frac{y}{x}\right)^n\right]^{-1}\ \text{and}\ \ w_d(e)=\left(\frac{z}{x+y}\right)\left[1-\left(-\frac{x}{y}\right)^n\right].$$

4. Applications

4.1 HOMFLY polynomial and *W*-polynomial

Using Theorem 3.7, we establish the relationship between HOMFLY polynomial and W-polynomial [36, 39, 40], which generalizes the connection between polyhedral links and polyhedral graphs, and greatly improves the existing result [23]. In addition, this work also gives a connection between rational links and their associated plane graphs.

We define a graph *G* as *a colored graph*, if there is a function *f* from edge set *E* to color set Λ.

Definition 4.1.1 *The W-polynomial* $W(G)=W(G;d,t_1,t_2)\in\mathbb{Z}[t_1,t_2,d]$ *for a colored graph G is* *defined by the following recursion formulas:*

(1) If E_n be a graph which is composed of n isolated vertexes, then

$$W(E_n) = d^{n-1}.$$

(2) Let e be an edge of G. We use c(e) to denote the color of edge e and assume that $c(e) = \lambda$.

If e is an isthmus, then

$$W(G) = (x_\lambda + y_\lambda t_1)W(G \cdot e).$$

If e is a loop, then

$$W(G) = (y_\lambda + x_\lambda t_2)W(G - e).$$

Otherwise,

$$W(G) = x_\lambda W(G \cdot e) + y_\lambda W(G - e).$$

The following Lemma can be easily obtained by the skein relations of Z^w-polynomial and W-polynomial, from which any weighted graph G can be considered as a colored graph.

Lemma 4.1.2 *Let G be a weighted and colored graph, and each edge e will be given a weight* $w(e)$ *if and only if it is colored with* $\lambda = c(e)$. *In W(G), if* $x_\lambda = w(e_\lambda), y_\lambda = 1,$ *then*

$$Z^w(G; x, y) = xW(G; x, x, y).$$

The following Theorem can be followed directly from Theorem 3.7 and Lemma 4.1.2, and hence its proof is omitted here.

Theorem 4.1.3 *Let G be a weighted plane graph as defined in theorem 3.7, and hence also a colored graph. In W(G), if*

$$x_{a_i} = w_{a_i}(e) = \left(\frac{x+y}{z}\right)\left[1 - \left(-\frac{x}{y}\right)^{a_i}\right]^{-1}, y_{a_i} = 1,$$
$$x_{b_i} = w_{b_i}(e) = \left(\frac{z}{x+y}\right)\left[1 - \left(-\frac{y}{x}\right)^{b_i}\right], y_{b_i} = 1,$$
$$x_{c_i} = w_{c_i}(e) = \left(\frac{x+y}{z}\right)\left[1 - \left(-\frac{y}{x}\right)^{c_i}\right]^{-1}, y_{c_i} = 1,$$
$$x_{d_i} = w_{d_i}(e) = \left(\frac{z}{x+y}\right)\left[1 - \left(-\frac{x}{y}\right)^{d_i}\right], y_{d_i} = 1,$$

Then,

$$H(D(G);x,y,z)=\left(\frac{z}{x+y}\right)^{E_a+E_c}\left(-\frac{x}{y}\right)^{\sum_{i=1}^{E_b}b_i-\sum_{i=1}^{E_d}d_i}\prod_{i=1}^{E_a}\left[\left(-\frac{y}{x}\right)^{a_i}-1\right]$$

$$\times\prod_{i=1}^{E_c}\left[\left(-\frac{x}{y}\right)^{c_i}-1\right]W\left(G;-\frac{x+y}{z},-\frac{x+y}{z}-\frac{x+y}{z}\right).$$

Remark. Theorem 4.1.3 clearly holds for all polyhedral links obtained in section 2. For rational links, we will show that they can be also constructed from some plane graphs in the next section, and hence their HOMLY polynomial can be derived by using this theorem.

4.2 HOMFLY polynomial and Tutte polynomial

Using Corollary 3.8, we establish the relationship between the HOMFLY polynomial of regular polyhedral links and the Tutte polynomial [37, 38] of the origin graph, which generalizes the known result [21].

Definition 4.2.1 *The Tutte polynomial* $T(G;x,y)\in Z[x,y]$ *for a graph G is defined by the following recursion formulas:*

(1) If G is a graph with no edge, then

$$T(G;x,y)=1.$$

(2) Let e be an edge of G. If e is an isthmus, then

$$T(G;x,y)=xT(G\cdot e;x,y).$$

If e is a loop, then

$$T(G;x,y)=yT(G-e;x,y).$$

Otherwise,

$$T(G;x,y)=T(G-e;x,y)+T(G\cdot e;x,y).$$

Lemma 4.2.2 [21] *Let G be a connected graph, and each edge be given a weight w. Then*

$$Z^w(G;x,y)=w^{|V(G)|-1}xT\left(G;1+\frac{x}{w},1+wy\right).$$

The following Theorem can be immediately obtained from the Corollary 3.8 and lemma 4.2.2, and its proof is omitted here.

Theorem 4.2.3 *Let G be a weighted polyhedral graph, and each edge will be given a weight* $w_a(e)$, $w_b(e)$, $w_c(e)$ *or* $w_d(e)$ *if it is covered by a-tangle, b-tangle, c-tangle or d-tangle of the length n. Then*

$$(1)\ H(D_A^n(G);x,y,z)=\left(\frac{x+y}{z}\right)^{|V(G)|-|E(G)|-1}\left(\left(-\frac{y}{x}\right)^n-1\right)^{|E(G)|-|V(G)|+1}\left(-\frac{y}{x}\right)^{n(|V(G)|-1)}$$
$$\times T\left(G;\left(-\frac{x}{y}\right)^n,1-\left(\frac{x+y}{z}\right)^2\left(1-\left(-\frac{x}{y}\right)^n\right)^{-1}\right);$$

$$(2)\ H(D_B^n(G);x,y,z)=\left(\frac{z}{x+y}\right)^{|V(G)|-1}\left(1-\left(-\frac{y}{x}\right)^n\right)^{|V(G)|-1}\left(-\frac{x}{y}\right)^{n|E(G)|}$$
$$\times T\left(G;1-\left(\frac{x+y}{z}\right)^2\left(1-\left(-\frac{y}{x}\right)^n\right)^{-1},\left(-\frac{y}{x}\right)^n\right);$$

$$(3)\ H(D_C^n(G);x,y,z)=\left(\frac{x+y}{z}\right)^{|V(G)|-|E(G)|-1}\left(\left(-\frac{x}{y}\right)^n-1\right)^{|E(G)|-|V(G)|+1}\left(-\frac{x}{y}\right)^{n(|V(G)|-1)}$$
$$\times T\left(G;\left(-\frac{y}{x}\right)^n,1-\left(\frac{x+y}{z}\right)^2\left(1-\left(-\frac{y}{x}\right)^n\right)^{-1}\right);$$

$$(4)\ H(D_D^n(G);x,y,z)=\left(\frac{z}{x+y}\right)^{|V(G)|-1}\left(1-\left(-\frac{x}{y}\right)^n\right)^{|V(G)|-1}\left(-\frac{y}{x}\right)^{n|E(G)|}$$
$$\times T\left(G;1-\left(\frac{x+y}{z}\right)^2\left(1-\left(-\frac{x}{y}\right)^n\right)^{-1},\left(-\frac{x}{y}\right)^n\right).$$

5. Examples

5.1 The HOMFLY polynomial for tetrahedral links

Applying Theorem 3.7 to tetrahedral links, we compute their HOMFLY polynomials and explore their topological properties.

Tetrahedral links $D_T(G)$ are obtained from the tetrahedral graph *G* by using the operation 'Tangle Covering', where the edge e_i is covered by the *a-tangle* of the length a_i for i=1,2,3, and the edges e_4, e_5 and e_6 are covered by *b-tangle* of the length n_4, *c-tangle* of the length n_5 and *d-tangle* of the length n_6 respectively. These links can be also described as $\overline{D(G)}$ in Figure 1 when $n_1=a_1$, $n_2=a_2$, $n_3=a_3$, $n_4=b_4$, $n_5=a_5$ and $n_6=a_6$. Here, using Theorem 3.7, the HOMFLY polynomial of $D_T(G)$ can be described by the following formula:

$$H(T_L(G); v, z) = -\frac{1}{z}\, P_1 - zP_2 - z^3P_3 + z^5P_4,$$

Where

$$P_1 = v^{-1+2a_1+2a_2+2a_3-2b_4-2c_5}(v^2-1)\big(v^{2b_4} - v^{2d_6} + v^{2c_5+2d_6}\big),$$

$$P_2 = -\frac{v^{1-2b_4-2c_5}}{v^2-1}(v^{2a_2} - v^{2a_1+2a_2} - v^{2a_2+2a_3} - v^{2a_2+2d_6} - v^{2a_3+2d_6} + v^{2c_5+2d_6},$$
$$+v^{2a_1+2a_2+2a_3} + v^{2a_1+2a_2+2b_4} + v^{2a_1+2a_3+2b_4} + v^{2a_2+2a_3+2b_4} + v^{2a_2+2a_3+2d_6}$$
$$+v^{2a_3+2b_4+2d_6} - 4v^{2a_1+2a_2+2a_3+2b_4} + 2v^{2a_1+2a_2+2a_3+2d_6} - v^{2a_1+2a_3+2b_4+2d_6}$$
$$-v^{2a_2+2a_3+2b_4+2d_6} + v^{2a_1+2a_2+2a_3+2b_4+2c_5} + v^{2a_1+2a_2+2a_3+2b_4+2d_6}$$
$$-2v^{2a_1+2a_2+2a_3+2c_5+2d_6}),$$

$$P_3 = -\frac{v^{3-2b_4-2c_5}}{(v^2-1)^3}(-2v^{2a_2} - v^{2a_3} + v^{2c_5} + 3v^{2a_2+2a_3} + 2v^{2a_1+2a_2} + 2v^{2a_2+2d_6}$$
$$+2v^{2a_3+2d_6} + v^{2a_1+2a_3} - 2v^{2c_5+2d_6} + v^{2a_2+2b_4} + v^{2a_3+2b_4} - v^{2a_1+2c_5}$$
$$+v^{2a_1+2C_5+2d_6} + v^{2b_4+2C_5+2d_6} + v^{2a_1+2b_4+2C_5} - v^{2a_1+2a_2+2d_6} - v^{2a_1+2a_3+2d_6}$$
$$-v^{2a_2+2a_3+2C_5} - v^{2a_2+2b_4+2d_6} - 2v^{2a_1+2a_2+2b_4} - 2v^{2a_1+2a_3+2b_4}$$
$$-2v^{2a_3+2b_4+2d_6} - 3v^{2a_1+2a_2+2a_3} - 3v^{2a_2+2a_3+2b_4} - 3v^{2a_2+2a_3+2d_6}$$
$$+5v^{2a_1+2a_2+2a_3+2b_4} + 3v^{2a_2+2a_3+2b_4+2d_6} + 2v^{2a_1+2a_3+2b_4+2d_6}$$
$$+v^{2a_1+2a_2+2a_3+2d_6} + v^{2a_1+2a_2+2a_3+2c_5} + v^{2a_1+2a_2+2b_4+2d_6}$$
$$+v^{2a_2+2a_3+2c_5+2d_6} - v^{2a_1+2b_4+2c_5+2d_6} - v^{2a_2+2a_3+2b_4+2c_5+2d_6}$$
$$-2v^{2a_1+2a_2+2a_3+2b_4+2c_5} - 3v^{2a_1+2a_2+2a_3+2b_4+2d_6}$$
$$+v^{2a_1+2a_2+2a_3+2b_4+2c_5+2d_6}),$$

$$P_4 = \frac{v^{5-2b_4-2c_5}}{(v^2-1)^5}(v^{2a_1}-1)\big(v^{2b_4}-1\big)\big(v^{2d_6}-1\big)(v^{2a_2} + v^{2a_3} - 2v^{2a_2+2a_3}$$
$$-v^{2c_5} + v^{2a_2+2a_3+2c_5}).$$

(1) *Some chiral links* $L_{n,m}$ *are obtained from* $D_T(G)$ *for* $b_4 = c_5 = d_6 = m$.

$$H(L_{n,m}) = \frac{v^{-1+2a_1+2a_2+2a_3}}{z}(1-v^2) - \frac{zv^{1-4m}}{v^2-1}(v^{4m} + v^{2a_2} - v^{2m+2a_2} - v^{2a_1+2a_2}$$
$$+v^{2m+2a_1+2a_2} - v^{2m+2a_3} + v^{4m+2a_3} + v^{2m+2a_1+2a_3} - v^{4m+2a_1+2a_3} - v^{2a_2+2a_3}$$
$$+2v^{2m+2a_2+2a_3} - v^{4m+2a_2+2a_3} + 2v^{2a_1+2a_2+2a_3} - 2v^{2m+2a_1+2a_2+2a_3})$$
$$-\frac{z^3v^{3-4m}}{(v^2-1)^3}(v^{2m}-1)(v^{2a_1}-1)(v^{2m} - v^{4m} - 2v^{2a_2} + v^{2m+2a_2} - 2v^{2a_3}$$
$$-v^{2a_3} + 2v^{2m+2a_3} + 3v^{2a_2+2a_3} - 4v^{2m+2a_2+2a_3} + v^{4m+2a_2+2a_3})$$

$$-\frac{z^5 v^{5-4m}}{(v^2-1)^5}(v^{2m}-1)^2(v^{2a_1}-1)(-v^{2m}+v^{2a_2}+v^{2a_3}$$
$$-2v^{2a_2+2a_3}+v^{2m+2a_2+2a_3}).$$

If L_n is achiral, $H(L_n, v, z)$ must be symmetric in v, and hence $v^{-1+2a_1+2a_2+2a_3}(1-v^2)$ must also be symmetric in v. However, it is not symmetric in v. Therefore, all links from $D_T(G)$ are chiral for $b_4=c_5=d_6=m$.

(2) The HOMFLY polynomials of some links obtained from $D_T(G)$.

The links L_n are obtained from $D_T(G)$ for $a_1=a_2=a_3=b_4=c_5=d_6=n$. Their HOMFLY polynomials are given in the following.

$$H(L_n)=\frac{1}{z}(v^{-1+6n}-v^{1+6n})+\frac{zv^{1-2n}}{v^2-1}(v^{2n}-1)(1-2v^{2n}+4v^{4n})$$
$$-\frac{z^3v^{3-2n}}{(v^2-1)^3}(v^{2n}-1)^4(v^{2n}-2)+\frac{z^5v^{5-2n}}{(v^2-1)^5}(v^{2n}-1)^5.$$

The links $L^1_{n,m}$ are obtained from $D_T(G)$ for $a_1=a_2=a_3=n$ and $b_4=c_5=d_6=m$. Their HOMFLY polynomials are given in the following.

$$H(L^1_{n,m};x,y,z)=\frac{1}{z}(v^{-1+6n}-v^{1+6n})+\frac{zv^{1-4m}}{v^2-1}(v^{2n}-1)(v^{4m}+v^{2n}-v^{4n}-2v^{2m+2n}$$
$$+2v^{4m+2n}+2v^{2m+4n})-\frac{z^3v^{3-4m}}{(v^2-1)^3}(v^{2m}-1)^2(v^n-1)^2(v^{2m}-3v^{2n}+v^{2m+2n})$$
$$+\frac{z^5v^{5-4m}}{(v^2-1)^5}(v^{2m}-1)^2(v^{2n}-1)^2(v^{2m}-2v^{2n}+v^{2m+2n}).$$

The links $L^2_{n,m}$ are obtained from $D_T(G)$ for $a_1=a_2=n$ and $a_3=b_4=c_5=d_6=m$. Their HOMFLY polynomials are given in the following.

$$H(L^2_{n,m};x,y,z)=\frac{v^{-1+2m+4n}}{z}(1-v^2)-\frac{zv^{1-4m}}{v^2-1}(v^{6m}+v^{2n}-v^{4n}-2v^{2m+2n}$$
$$+3v^{4m+2n}-2v^{6m+2n}+2v^{2m+4n}-2v^{4m+4n})-\frac{z^3v^{3-4m}}{(v^2-1)^3}(v^{2m}-1)$$
$$\times(v^{2n}-1)(v^{4m}-2v^{2n}+4v^{2m+2n}-4v^{4m+2n}+v^{6m+2n})$$
$$+\frac{z^5v^{5-4m}}{(v^2-1)^5}(v^{2m}-1)^2(v^{2n}-1)(v^{2n}-2v^{2n+2m}+v^{4m+2n}).$$

5.2 The HOMFLY polynomial of rational links

Rational links is a very important and simple class of links which often leads to solving some important problems of knot theory. Here, by using Theorem 4.1.2, we give an explicit formula of their

HOMLY polynomial.

Let $D_{a,b}(G^h)$, $D_{c,d}(G^h)$, $D_{a,b}(G^v)$ and $D_{c,d}(G^v)$ be four families of rational links obtained from a connected plane graphs G^h and G^v by using the operation 'Tangle Covering' respectively (see Figure 6). According to the Theorem 3.7, we can obtained the corresponding four weighted graphs $G^h_{a,b}$, $G^h_{c,d}$, $G^v_{a,b}$ and $G^v_{c,d}$. Hence according to Theorem 4.1.2, the above weighted graphs are also considered as four colored graphs (see Figure 6).

Using Theorem 11 in Ref. [39], we can obtain the W-polynomials of $G^h_{a,b}$, $G^h_{c,d}$, $G^v_{a,b}$ *and* $G^v_{c,d}$.

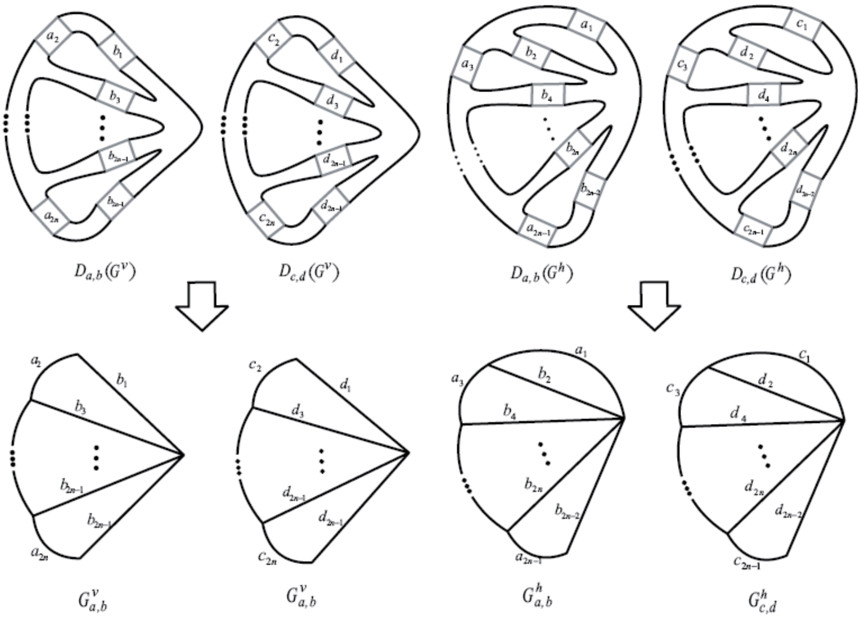

Figure 6. Four families of rational links $D_{a,b}(G^h)$, $D_{c,d}(G^h)$, $D_{a,b}(G^v)$ and $D_{c,d}(G^v)$ and the corresponding colored graph $G^h_{a,b}, G^h_{c,d}, G^v_{a,b}$ and $G^v_{c,d}$.
(Each box contains *a-tangle* with length a_i, *b-tangle* with length b_j, *c-tangle* with length c_i or *d-tangle* with length d_j, where $1 \leq i \leq 2n$ and $1 \leq j \leq 2n+1$ in $G^v_{a,b}$ and $G^v_{c,d}$, and $1 \leq i \leq 2n+1$ and $1 \leq j \leq 2n+2$ in $G^h_{a,b}$ and $G^h_{c,d}$.)

Theorem 5.2.1 *Let* $G^h_{a,b}$, $G^h_{c,d}$, $G^v_{a,b}$ *and* $G^v_{c,d}$ *be four colored graph as described above. Then*

(1) $W(G^h_{a,b}, d, d, d) = (S^0_1, S^0_2)(\prod_{i=1}^n A_i)J$, (2) $W(G^v_{a,b}, d, d, d) = (S^0_3, S^0_4)(\prod_{i=1}^n A_i)J$,

(3) $W(G^h_{c,d}, d, d, d) = (S^0_5,\ S^0_6)(\prod_{j=1}^{n} A_j)J$ (4) $W(G^v_{c,d}, d, d, d) = (S^0_7,\ S^0_8)(\prod_{j=1}^{n} A_j)J,$

where

$$S^0_1 = y_{b_1}d^4, S^0_2 = x_{b_1}d^3, S^0_5 = y_{d_1}d^4, S^0_6 = x_{d_1}d^3,$$

$$S^0_3 = y_{a_1}y_{b_2}d^4, \qquad S^0_4 = x_{a_1}y_{b_2}d^3 + y_{a_1}x_{b_2}d^3 + x_{a_1}x_{b_2}d^4,$$

$$S^0_7 = y_{c_1}y_{d_2}d^4, \qquad S^0_8 = x_{c_1}y_{d_2}d^3 + y_{c_1}x_{d_2}d^3 + x_{c_1}x_{d_2}d^4,$$

$$a^i_{1,1} = y_{a_{2i}}y_{b_{2i+1}}d^2 + x_{a_{2i}}y_{b_{2i+1}}d,\ a^i_{1,2} = y_{a_{2i}}x_{b_{2i+1}}d + x_{a_{2i}}x_{b_{2i+1}},$$

$$a^i_{2,1} = y_{a_{2i}}y_{b_{2i+1}}d^2, a^i_{2,2} = x_{a_{2i}}y_{b_{2i+1}}d + y_{a_{2i}}x_{b_{2i+1}}\,d + x_{a_{2i}}x_{b_{2i+1}}d^2,$$

$$a^j_{1,1} = y_{c_{2j}}y_{d_{2j+1}}d^2 + x_{c_{2j}}y_{d_{2j+1}}d,\ a^j_{1,2} = y_{c_{2j}}x_{d_{2j+1}}d + x_{c_{2j}}x_{d_{2j+1}},$$

$$a^j_{2,1} = y_{c_{2j}}y_{d_{2j+1}}d^2, a^j_{2,2} = x_{c_{2j}}y_{d_{2j+1}}d + y_{c_{2j}}x_{d_{2j+1}}\,d + x_{c_{2j}}x_{d_{2j+1}}d^2.$$

Using Theorem 4.1.3 and Theorem 5.2.1, we can obtain the following theorem which give the HOMFLY polynomials of $D_{a,b}(G^h), D_{c,d}(G^v),\ D_{a,b}(G^v)$ and $D_{c,d}(G^v)$.

Theorem 5.2.2 *Let $D_{a,b}(G^h)$, $D_{c,d}(G^h)$, $D_{a,b}(G^v)$ and $D_{c,d}(G^v)$ be four families of rational links as constructed above. Then*

$$(1)\ H\big(D_{a,b}(G^h); v, z\big) = \left(\frac{vz}{v^2-1}\right)^{|E_a|} v^{-\sum_{i=1}^{|E_b|} 2b_i} \prod_{i=1}^{|E_a|} (v^{2a_i} - 1)\ (S^0_1,\ S^0_2)\left(\prod_{i=1}^{n} A_i\right)J,$$

$$(2)\ H\big(D_{a,b}(G^v); v, z\big) = \left(\frac{vz}{v^2-1}\right)^{|E_a|} v^{-\sum_{i=1}^{|E_b|} 2b_i} \prod_{i=1}^{|E_a|} (v^{2a_i} - 1)\ (S^0_3,\ S^0_4)\left(\prod_{i=1}^{n} A_i\right)J,$$

$$(3)\ H\big(D_{c,d}(G^h); v, z\big) = \left(\frac{vz}{v^2-1}\right)^{|E_c|} v^{\sum_{i=1}^{|E_d|} 2d_i} \prod_{i=1}^{|E_c|} (v^{-2c_i} - 1)\ (S^0_5,\ S^0_6)\left(\prod_{j=1}^{n} A_j\right)J,$$

$$(4)\ H\big(D_{c,d}(G^v); v, z\big) = \left(\frac{vz}{v^2-1}\right)^{|E_c|} v^{\sum_{i=1}^{|E_d|} 2d_i} \prod_{i=1}^{|E_c|} (v^{-2c_i} - 1)\ (S^0_7,\ S^0_8)\left(\prod_{j=1}^{n} A_j\right)J,$$

where

$$S^0_1 = S^0_3 = S^0_5 = S^0_7 = \left(\frac{1-v^2}{vz}\right)^4, S^0_2 = (v^{2b_1} - 1)\left(\frac{1-v^2}{vz}\right)^2, S^0_6 = (v^{-2d_1} - 1)\left(\frac{1-v^2}{vz}\right)^2,$$

$$S^0_4 = (v^{-2a_1} - 1)^{-1}\left(\frac{1-v^2}{vz}\right)^4 + (v^{2b_2} - 1)\left(\frac{1-v^2}{vz}\right)^2 + (1 - v^{-2a_1})^{-1}(1 - v^{2b_2})\left(\frac{1-v^2}{vz}\right)^4,$$

$$S_8^0 = (v^{2c_1} - 1)^{-1}\left(\frac{1-v^2}{vz}\right)^4 + \left(v^{-2d_2} - 1\right)\left(\frac{1-v^2}{vz}\right)^2 + (1 - v^{2c_1})^{-1}\left(1 - v^{-2d_2}\right)\left(\frac{1-v^2}{vz}\right)^4,$$

$$A_i = \begin{pmatrix} a_{1,1}^i & a_{1,2}^i \\ a_{2,1}^i & a_{2,2}^i \end{pmatrix} \ with$$

$$a_{1,1}^i = \left(\frac{1-v^2}{vz}\right)^2 + (v^{-2a_{2i}} - 1)^{-1}\left(\frac{1-v^2}{vz}\right)^2,$$

$$a_{2,1}^i = \left(\frac{1-v^2}{vz}\right)^2, a_{1,2}^i = \frac{1 - v^{2b_{2i+1}}}{1 - v^{-2a_{2i}}} + v^{2b_{2i+1}} - 1 \ \ and$$

$$a_{2,2}^i = (v^{-2a_{2i}} - 1)^{-1}\left(\frac{1-v^2}{vz}\right)^2 + (1 - v^{-2a_{2i}})^{-1}\left(1 - v^{2b_{2i+1}}\right)\left(\frac{1-v^2}{vz}\right)^2 + v^{2b_{2i+1}} - 1,$$

$$A_j = \begin{pmatrix} a_{1,1}^j & a_{1,2}^j \\ a_{2,1}^j & a_{2,2}^j \end{pmatrix} \ with$$

$$a_{1,1}^j = \left(\frac{1-v^2}{vz}\right)^2 + \left(v^{2c_{2j}} - 1\right)^{-1}\left(\frac{1-v^2}{vz}\right)^2,$$

$$a_{2,1}^j = \left(\frac{1-v^2}{vz}\right)^2, a_{1,2}^j = \frac{1 - v^{-2d_{2j+1}}}{1 - v^{2c_{2j}}} + v^{-2d_{2j+1}} - 1 \ and$$

$$a_{2,2}^j = (v^{2c_{2i}} - 1)^{-1}\left(\frac{1-v^2}{vz}\right)^2 + \left(\frac{1 - v^{-2d_{2j+1}}}{1 - v^{2c_{2j}}}\right)\left(\frac{1-v^2}{vz}\right)^2 + v^{-2d_{2j+1}} - 1.$$

6 Conclusions

Given any polyhedron, a large family of polyhedral links is obtained by applying the operation ‘Tangle Covering’, which enriches the topology of molecular links. Two important classes of links, named as regular links and semi-regular links, are derived from this family of links, where regular links have the same type and crossing number of tangles for each edge and semi-regular links only have the same type of tangles. These links have been regarded as mathematical model for molecular knots and links, and parts of them constructed on the regular polyhedron have been discussed in other papers [16-26].

Furthermore, we have given a generalized relationship between the HOMFLY polynomial of a family of polyhedral links and the Z^w-polynomial of an original graph. This results in two important relationships, one between the HOMFLY polynomial of regular links and the Tutte polynomial of the original graph, and the other between the HOMFLY polynomial of regular links and the W-polynomial of the original graph. Our result not only enriched the connections between graphs and links but also simplified the computation of HOMFLY polynomial, especially for the links with large crossing number. Therefore, this paper provides a possible approach to classifying and identifying the complex links, as well as the description and analysis of the chemical properties for molecular catenanes and knots.

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