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On Incidence Energy of Trees *

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Abstract

Let $G = (V, E)$ be a simple graph, $I(G)$ its incidence matrix. The incidence energy of G is the sum of the singular values of $I(G)$. It was shown that among all trees of order n, the trees with the largest and smallest incidence energy are the n-vertex path P_n and n-vertex star S_n, respectively. In this paper, we characterize the trees with the second greatest, the third greatest, the second smallest and the third smallest incidence energy among all trees on n vertices.

1 Introduction

The energy $E(G)$ of a graph $G = (V, E)$ is the sum of the absolute values of the eigenvalues of its adjacency matrix $A(G)$, and it has been researched extensively. For more details see [1, 2]. Nikiforov [3] recently extended the concept of energy to all (not necessarily square) matrices, defining the energy of a matrix M as the sum of the singular values of M, i.e., the sum of the square roots of the eigenvalues of MM^t, where M^t is the transpose of M.

In line with Nikiforov's idea, Jooyandeh et al. [4–6] introduced the incidence energy $IE(G)$ of a graph G, $IE(G)$ was defined as the energy of its incidence matrix $I(G)$. They also found the relation between the energy and the incidence energy of graphs and some similar upper and lower bounds of energy for incidence energy. It was shown in [7] that the

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incidence energy $IE(G)$ coincides with the Laplacian-like energy $LEL(G)$ for a bipartite graph G. The Laplacian-like energy $LEL(G)$ of a graph G, introduced in [10], is the sum of the square roots of eigenvalues of its Laplacian matrix.

Lemma 1.1 ([7]) If G is a bipartite graph, then $IE(G) = LEL(G)$.

Since trees are bipartite, any result on LEL on trees is automatically applicable for IE.

Denote by $\psi(G,\lambda)$ the characteristic polynomial of the Laplacian matrix of a graph G. It is known that

$$\psi(G,\lambda) = \sum_{k\geq 0}(-1)^k c_k(G)\lambda^{n-k}$$

where $c_k(G) \geq 0$. By using the Coulson integral formula, Gutman et al. [7] obtained the next formula:

$$LEL(G) = \frac{1}{\pi}\int_0^{+\infty} \ln\left[\sum_{k\geq 0} c_k(G)x^{2k}\right]\frac{dx}{x^2}\ . \tag{1}$$

This shows that $LEL(G)$ is a monotonically increasing function of each of the coefficients $c_k(G)$. And the coefficients $c_k(G)$ of a tree G are related with the numbers $m_k(s(G))$ of k-matching of its subdivision $s(G)$:

Lemma 1.2 ([8]) Let G be a tree on n vertices. Then $c_k(G) = m_k(s(G))$ for $0 \leq k \leq n$.

Among the trees, it has been long known [11] that the path P_n has maximum energy and that the star S_n has minimal energy, and there are many results on graphs with extremal energy [12,12–16]. Gutman et al. also characterized the trees with the minimal and maximal incidence energy among all trees on n vertices.

Lemma 1.3 ([7]) Let T be any tree on n vertices. Then

$$IE(S_n) \leq IE(T) \leq IE(P_n)$$

with equality if and only if $T \cong S_n$ and $T \cong P_n$, where S_n and P_n are the star and path on n vertices, respectively.

In this work, we characterize the trees with the second smallest, the third smallest, the second greatest and third greatest incidence energy among all trees on n vertices.

2 The main result

First, we find the trees with the second and the third smallest incidence energy among all trees on n vertices.

Denote by S_{n_1,n_2} be the double star on n vertices ($n = n_1 + n_2 + 2, n_1 \geq n_2 \geq 1$) obtained by adding an edge between the centers of S_{n_1+1} and S_{n_2+1} and T_{n_1,n_2,n_3} be a tree on n vertices obtained from the path $P_3 = u_1u_2u_3$ by adding n_1, n_2, n_3 pendant edges on u_1, u_2, u_3, respectively, where $n_1 + n_2 + n_3 + 3 = n$, and $n_1, n_3 \geq 1$.

The following σ-transformation can transform every tree which is not a star into a double star.

σ-transformation: Let u_0 be a vertex of a tree T of degree $p + 1$. Suppose that $u_0u_1, u_0u_2, \cdots, u_0u_p$ are pendant edges incident with u_0, and that v_0 is the neighbor of u_0 distinct from $u_1, \cdots, u_p$. Then we form a tree $T^* = \sigma(T, u_0)$ by removing the edges $u_0u_1, \cdots, u_0u_p$ from T and adding p new pendant edges $v_0u_1, \cdots, v_0u_p$ incident with v_0. We say that T^* is a **σ-transformation** of T.

Lemma 2.1 ([9]) Let $T^* = \sigma(T, u_0)$ be a σ-transformation of a tree T of order n. Then $c_k(T) > c_k(T^*)$ for $2 \leq k \leq n-2$, and $c_k(T) = c_k(T^*)$ for $k = 0, 1, n-1, n$.

Lemma 2.2 If $n_1 \geq n_2 > 1$, then $c_k(S_{n_1,n_2}) > c_k(S_{n_1+1,n_2-1})$ for $2 \leq k \leq n-2$, and $c_k(S_{n_1,n_2}) = c_k(S_{n_1+1,n_2-1})$ for $k = 0, 1, n-1, n$.

Proof Let $u, v, w \in S_{n_1,n_2}$ and $d(u) = n_1 + 1$, $d(v) = n_2 + 1$, $vw \in E(S_{n_1,n_2})$. S_{n_1+1,n_2-1} can be formed by removing the edges uv from S_{n_1,n_2} and adding a new pendant edges uw.

From Lemma 1.2, we have

$$\begin{aligned}
& c_k(S_{n_1,n_2}) = m_k(s(S_{n_1,n_2})) \\
& = m_k(s(S_{n_1,n_2} - w)) + m_{k-1}(s(S_{n_1,n_2} - w)) + m_{k-1}(s(S_{n_1,n_2} - w) - v)
\end{aligned}$$

and

$$\begin{aligned}
& c_k(S_{n_1+1,n_2-1}) = m_k(s(S_{n_1+1,n_2-1})) \\
& = m_k(s(S_{n_1+1,n_2-1} - w)) + m_{k-1}(s(S_{n_1+1,n_2-1} - w)) + m_{k-1}(s(S_{n_1+1,n_2-1} - w) - u) \\
& = m_k(s(S_{n_1,n_2} - w)) + m_{k-1}(s(S_{n_1,n_2} - w)) + m_{k-1}(s(S_{n_1,n_2} - w) - u)\ .
\end{aligned}$$

By some computations, we have

$$m_{k-1}(s(S_{n_1,n_2} - w) - v) = \binom{n_1 + n_2}{k-1} + n_1 \binom{n_1 + n_2 - 2}{k-2}$$

and

$$m_{k-1}(s(S_{n_1,n_2} - w) - u) = \binom{n_1 + n_2}{k-1} + (n_2 - 1) \binom{n_1 + n_2 - 2}{k-2}.$$

$$
\begin{aligned}
c_k(S_{n_1,n_2}) - c_k(S_{n_1+1,n_2-1}) &= m_{k-1}(s(S_{n_1,n_2} - w) - v) - m_{k-1}(s(S_{n_1,n_2} - w) - u) \\
&= (n_1 - n_2 + 1)\begin{pmatrix} n_1 + n_2 - 2 \\ k-2 \end{pmatrix}.
\end{aligned}
$$

So, $c_k(S_{n_1,n_2}) > c_k(S_{n_1+1,n_2-1})$ for $2 \le k \le n-2$, and $c_k(S_{n_1,n_2}) = c_k(S_{n_1+1,n_2-1})$ for $k = 0, 1, n-1, n$.

Lemma 2.3. If $n \ge 6$, then $c_k(S_{n-4,2}) < c_k(T_{1,n-4,1})$ for $3 \le k \le n-2$, and $c_k(S_{n-4,2}) = c_k(T_{1,n-4,1})$ for $k = 0, 1, 2, n-1, n$.

Proof. Let $u, v, w \in S_{n-4,1}$ and $d(u) = n-3$, $d(v) = 2, d(w) = 1$, $uw \in E(S_{n-4,1})$.

From Lemma 1.2, we have

$$
\begin{aligned}
&c_k(S_{n-4,2}) = m_k(s(S_{n-4,2})) \\
&= m_k(s(S_{n-4,1})) + m_{k-1}(s(S_{n-4,1})) + m_{k-1}(s(S_{n-4,1}) - v)
\end{aligned}
$$

and

$$
\begin{aligned}
&c_k(T_{1,n-4,1}) = m_k(s(T_{1,n-4,1})) \\
&= m_k(s(S_{n-4,1})) + m_{k-1}(s(S_{n-4,1})) + m_{k-1}(s(S_{n-4,1}) - w)\ .
\end{aligned}
$$

By some computations, we have

$$
m_{k-1}(s(S_{n-4,1}) - v) = \begin{pmatrix} n-3 \\ k-1 \end{pmatrix} + (n-4)\begin{pmatrix} n-4 \\ k-2 \end{pmatrix}
$$

and

$$
\begin{aligned}
m_{k-1}(s(S_{n-4,1)}) - w) &= \begin{pmatrix} n-3 \\ k-1 \end{pmatrix} + (n-4)\begin{pmatrix} n-4 \\ k-2 \end{pmatrix} + \begin{pmatrix} n-3 \\ k-2 \end{pmatrix} + \begin{pmatrix} n-5 \\ k-2 \end{pmatrix} \\
&\quad +(n-5)\begin{pmatrix} n-6 \\ k-3 \end{pmatrix} + 2\begin{pmatrix} n-5 \\ k-3 \end{pmatrix} + \begin{pmatrix} n-5 \\ k-4 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
c_k(T_{1,n-4,1}) - c_k(S_{n-4,2}) &= m_{k-1}(s(S_{n-4}) - w) - m_{k-1}(s(S_{n-4,1}) - v) \\
&= \begin{pmatrix} n-3 \\ k-2 \end{pmatrix} + \begin{pmatrix} n-5 \\ k-2 \end{pmatrix} + (n-5)\begin{pmatrix} n-6 \\ k-3 \end{pmatrix} \\
&\quad +2\begin{pmatrix} n-5 \\ k-3 \end{pmatrix} + \begin{pmatrix} n-5 \\ k-4 \end{pmatrix}.
\end{aligned}
$$

So $c_k(S_(n-4,2) < c_k(T_{1,n-4,1})$ for $3 \le k \le n-2$, and $c_k(S_(n-4,2) = c_k(T_{1,n-4,1})$ for $k = 0, 1, 2, n-1, n$.

Theorem 2.4 If T is a tree on $n \geq 6$ vertices and $T \neq S_n, S_{n-3,1}, S_{n-4,2}$, then $LEL(T) > LEL(S_{n-4,2}) > LEL(S_{n-3,1}) > LEL(S_n)$, i.e., $S_{n-3,1}$ and $S_{n-4,2}$ are the unique tree with the second and the third smallest Laplacian-like energy among all trees on n vertices, respectively.

Proof Since Laplacian-like energy is a monotonically increasing function of each of the coefficients. We only need to prove that $c_k(T) > c_k(S_{n-4,2})$ for $3 \leq k \leq n-2$ and $c_k(T) = c_k(S_{n-4,2})$ for $k = 0, 1, 2, n-1, n$.

Since $T \neq S_n, S_{n_1,n_2}$, T can be transformed into T_{n_1,n_2,n_3} by $\sigma-$transformation, where $n = n_1 + n_2 + n_3 + 3, n_1 \geq n_3 \geq 1$ and T_{n_1,n_2,n_3} can be transformed into S_{n_1,n_2+n_3+1}. By Lemmas 2.1, 2.2 and 2.3, we have $c_k(T) > c_k(T_{n_1,n_2,n_3}) > c_k(S_{n_1,n_2+n_3+1}) > c_k(S_{n-4,2}) > c_k(S_{n-3,1})$ for $2 \leq k \leq n-2$ and $c_k(T) = c_k(T_{n_1,n_2,n_3}) = c_k(S_{n_1,n_2+n_3+1}) = c_k(S_{n-4,2}) = c_k(S_{n-3,1})$ for $k = 0, 1, n-1, n$.

By Lemma 1.1, we have next corollary.

Corollary 2.5. If T is a tree on $n \geq 6$ vertices, and $T \neq S_n, S_{n-3,1}, S_{n-4,2}$, then $IE(T) > IE(S_{n-4,2}) > IE(S_{n-3,1}) > IE(S_n)$, i. e., $S_{n-3,1}$ and $S_{n-4,2}$ are the unique tree with the second and the third smallest incidence energy among all trees on n vertices, respectively.

In the following, we find the trees with the second and the third greatest incidence energy among all trees on n vertices. For any graph, the incidence energy is a half of the energy of its subdivision graph.

Lemma 2.6 ([7]) For any graph G,

$$IE(G) = \frac{1}{2}E(s(G))$$

where $E(s(G))$ is the energy of its subdivision graph $s(G)$.

In [12], the trees with the maximal, second maximal and the third maximal energy were determined.

Lemma 2.7 ([12]) Among all trees on n vertices, (i) P_n is the unique tree with the maximal energy; (ii) for $n \leq 3$, there is no tree with the second maximal energy; for $n = 4$, $n = 5$ and $n \geq 6$, the trees with the second maximal energy are S_4, T_1 and $T_n^{3,2}$, respectively, depicted in Figure 1.(iii) for $n \leq 5$, there is no tree with the third maximal energy; for $n = 5$,$n = 6$, $n = 7$ and $n = 9$ the trees with the third maximal energy are S_5, T_2,T_3,T_4,respectively, depicted in Figure 1. For $n = 8$ and $n \geq 10$ the trees with the third maximal energy is $T_n^{5,2}$, respectively, depicted in Figure 1.

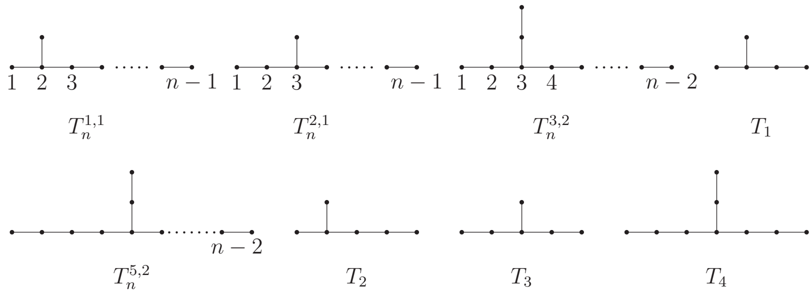

Figure 1.

It is easy to see that there is no tree with the second maximal incidence energy for $n \leq 3$ (There is only one tree on $n \leq 3$ vertices, respectively). There are only two trees S_4 and P_4 on 4 vertices and we have $IE(S_4) < IE(P_4)$. There are exactly three tree s S_5, $T_5^{1,1}$ and P_5 on 5 vertices and we have $IE(S_5) < IE(T_5^{1,1}) < IE(P_5)$.

Theorem 2.8 If T be a tree on $n \geq 6)$ vertices and $T \neq P_n, T_n^{2,1}, T_n^{1,1}$ (depicted in Figure 1), then

$$IE(T) < IE(T_n^{2,1}) < IE(T_n^{1,1}) < IE(P_n) \ .$$

Proof. Let $n \geq 6$. If T is any tree on n vertices and $T \neq P_n, T_n^{2,1}, T_n^{1,1}$, then their subdivisions $s(P_n) = P_{2n-1}$, $s(T_n^{1,1}) = T_{2n-1}^{3,2}$, $s(T_n^{2,1}) = T_{2n-1}^{5,2}$ and $s(T) \neq P_{2n-1}, T_{2n-1}^{3,2}, T_{2n-1}^{5,2}$.

By Lemma 2.6, we have

$$E(s(T)) < E(S(T_n^{2,1})) < E(s(T_n^{1,1})) < E(s(P_n))$$

and

$$IE(T) < IE(T_n^{2,1}) < IE(T_n^{1,1}) < IE(P_n) \ .$$

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