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E–L Equienergetic Graphs

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Abstract

Let G be a graph on n vertices. It was found that the ordinary energy $E(G)$ and the Laplacian energy $LE(G)$ have a number of analogous properties. In particular, if G is regular, then $E(G) = LE(G)$, and there are non-regular graphs with the same property. In this paper we consider the non-regular non-isomorphic connected graphs of the same order with the same energy and Laplacian energy, called as E-L equienergetic graphs. We construct a pair of E-L equienergetic graphs on n vertices for $n \equiv 0\,(mod\,7)$. Thus it is shown that there exist infinitely many pairs of E-L equienergetic graphs.

1 Introduction

Let G be a simple undirected graph possessing n vertices and m edges. Let A be the symmetric $(0,1)-$adjacency matrix of G and $D = diag(d_1, d_2, \dots d_n)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L = D - A$. Let λ_1 , λ_2 ,. . . λ_n be the adjacency spectrum of G , and let μ_1 , μ_2 ,. . . μ_n be the Laplacian spectrum of G .

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The energy $E(G)$ of a graph G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$ [4,6,8]. This quantity has a clear connection to chemical problems [5, 7] and has recently been much investigated (see [12, 15, 19, 21–23] and the references cited therein).

The Laplacian energy $LE(G)$ of a graph G has been defined [9] as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$. For recent investigations of this quantity see [13, 17, 18, 24, 25].

Two non-isomorphic graphs G_1 and G_2 of the same order are said to be equienergetic [1] if $E(G_1) = E(G_2)$. Similarly as in the case of graph energy, two non-isomorphic graphs G_1 and G_2 of the same order are said to be LE-equienergetic if $LE(G_1) = LE(G_2)$ [14].

The quantities $E(G)$ and $LE(G)$ were found to have a number of analogous properties [9, 13]. It is easy to see that if the graph G is regular, then $E(G) = LE(G)$, and there are non-regular graphs with the same property [10, 11]. So the equienergetic regular graphs are also the LE-equienergetic graphs. Besides, the two energies are equal. For instance, the regular graphs with 6 vertices of degree $2, 3, 4$ respectively, are both equienergetic graphs and LE-equienergetic graphs with the energies 8. Such case is of no interest. In this paper we are concerned with the non-regular non-isomorphic connected graphs of the same order with the same energy and Laplacian energy, which we call E-L equienergetic graphs. We construct a pair of E-L equienergetic graphs of order n, for all $n \equiv 0\,(mod\,7)$, and show that there exist infinitely many pairs of E-L equienergetic graphs.

2 Lemmas and results

Lemma 2.1 *There exists a pair of E-L equienergetic graphs of order* 7.

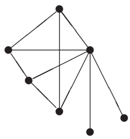
Proof. We consider the following two connected graphs of order 7. (G_{710}, G_{711} are graphs $10 - 333$, $11 - 362$ in [2] respectively). Clearly, G_{710} and G_{711} are non-regular. It was shown in [2] that the adjacency spectra of G_{710} and G_{711} are

$$\{3.41421, 0.58579, 0, 0, 0, -2, -2\} \qquad \text{and} \qquad \{3.41421, 0.58579, 0, 0, 0, -1, -3\}$$

respectively. Thus $E(G_{710}) = E(G_{711})$.

By direct computing, the Laplacian spectra of G_{710} and G_{711} are $\{7, 5, 3, 3, 1, 1, 0\}$ and $\{7, 5, 4, 2, 2, 2, 0\}$ respectively. It is then immediate to verify that $LE(G_{710}) = LE(G_{711})$. $\square$

In fact, the graphs in Lemma 2.1 are the minimal E-L equienergetic graphs.



G_{710}

G_{711}

Now we introduce an operation on graphs [16] and consider how it affects the adjacency spectrum and the Laplacian spectrum.

Let G be a graph on n vertices and m edges, and $t > 0$ be an integer. Write $G^{(t)}$ for the graph obtained by replacing each vertex $u \in V(G)$ by a set V_u of t vertices and joining $x \in V_u$ to $y \in V_u$ if and only if $uv \in E(G)$. Notice that the order of $G^{(t)}$ is $v(G^{(t)}) = tn$, and the number of edges of $G^{(t)}$ is $e(G^{(t)}) = t^2 m$. The following result holds.

Lemma 2.2 [16] *The adjacency eigenvalues of* $G^{(t)}$ *are* $t\lambda_1(G)$, $\ldots$, $t\lambda_n(G)$ *together with* $n(t-1)$ *additional* 0*'s.*

From this we get the following result:

Theorem 2.1 *If* $E(G_1) = E(G_2)$, *then* $E(G_1^{(t)}) = E(G_2^{(t)})$, *for all integers* $t > 0$.

The following result on Laplacian spectrum was presented in [3].

Lemma 2.3 *Let* G *be a simple graph of order* n. *If we put two similar graphs* G *side by side, and any vertex of the first graph* G *is connected by edges with the vertices which are adjacent to the corresponding vertex of the second graph* G, *then the resulting graph has Laplacian eigenvalues* $2\mu_1(G)$, $\ldots$, $2\mu_n(G)$ *and the remaining eigenvalues are* $2d_1, 2d_2, \ldots, 2d_n$.

In fact, Lemma 2.3 gives the Laplacian spectrum of $G^{(2)}$. It can be extended to $G^{(t)}$ for all integers $t > 0$.

Lemma 2.4 *The Laplacian eigenvalues of* $G^{(t)}$ *are* $t\mu_1(G)$, $\ldots$, $t\mu_n(G)$ *and the remaining eigenvalues are* $td_1, td_2, \cdots, td_n$ *with multiplicities* $t-1$, *i. e.,*

$$\begin{pmatrix} t\mu_1(G) & t\mu_2(G) & \cdots & t\mu_n(G) & td_1 & td_2 & \cdots & td_n \\ 1 & 1 & \cdots & 1 & t-1 & t-1 & \cdots & t-1 \end{pmatrix}.$$

Remark 2.1 *By Lemma* 2.4 *, it is an easy computation to show that*

$$LE(G^{(t)}) = t\,LE(G) + t(t-1)\sum_{i=1}^{n}\left|d_i - \frac{2m}{n}\right| \ . \tag{1}$$

Notice that $LE(G_1) = LE(G_2)$ *does not imply that* $LE(G_1^{(t)}) = LE(G_2^{(t)})$ *.*

Finally, we show that there exist infinitely many pairs of E-L equienergetic graphs.

Theorem 2.2 *There exists a pair of E-L equienergetic graphs of order* n *, for all* $n \equiv 0\,(mod\,7)$ *.*

Proof. Let $G_1 = G_{710}^{(t)}$, $G_2 = G_{711}^{(t)}$, where G_{710} , G_{711} be the graphs in Lemma 2.1 , and $t > 0$ be an integer. Then by Theorem 2.1 , we have $E(G_1) = E(G_2)$.

Moreover, by equation (1),

$$LE(G_1) = t\,LE(G_{710}) + t(t-1)\sum_{i=1}^{7}\left|d_{i0} - \frac{20}{7}\right| = t\,LE(G_{710}) + \frac{52}{7}\,t(t-1)$$

and

$$LE(G_2) = t\,LE(G_{711}) + t(t-1)\sum_{i=1}^{7}\left|d_{i1} - \frac{22}{7}\right| = t\,LE(G_{711}) + \frac{52}{7}t(t-1)$$

where d_{i0}, d_{i1} , $i = 1, 2, \ldots, 7$, are the vertex degrees of G_{710} and G_{711} , respectively.

Hence, there exists a pair of E-L equienergetic graphs of order n , for all $n \equiv 0\,(mod\,7)$.

□

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