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# On the Laplacian Estrada Index of Graphs

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**Abstract**

Assume that $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the Laplacian matrix of a graph $G$. The Laplacian Estrada index of $G$, is defined as $LEE(G) = \sum_{i=1}^{n} e^{\mu_i}$. In this note, we give upper bounds for $LEE(G)$ in terms of connectivity or matching number and characterize the corresponding extremal graphs.

## 1 Introduction

In this paper all graphs are finite and simple. The *join*, $G_1 \bigvee G_2$, of the graphs $G_1$ and $G_2$ is the graph obtained from the disjoint union $G_1 \bigcup G_2$ by adding new edges from each vertex in $G_1$ to every vertex in $G_2$. Let $G = (V(G), E(G))$ be a graph with $n$ vertices. By $G - U$ we mean the induced subgraph $G[V - U]$, if $U \subset V(G)$. The *adjacency matrix* of $G$ is $A(G) = ||a_{ij}||_{n\times n}$, where $a_{ij} = 1$ if two vertices $u_i$ and $u_j$ are adjacent in $G$ and $a_{ij} = 0$ otherwise. Since $A(G)$ is a real symmetric, it has only real eigenvalues, which can be ordered as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$.

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Let $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the the diagonal matrix of vertex degrees of $G$. We call the matrix $L(G) = D(G) - A(G)$ *Laplacian matrix* of $G$. It is well known that $L(G)$ is a real positive semidefinite matrix and its eigenvalues can be ordered as $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$. Denote the spectrum of $L(G)$ by $S(G) = (\mu_1, \mu_2, \ldots, \mu_n)$. It is well known that $\mu_i(G) = n - \mu_{n-i}(\overline{G})$ for $1 \leq i \leq n-1$. We refer the reader to [13, 18] for further information on the Laplacian matrix.

Recall that the Estrada index of a simple connected graph $G$, recently put forward by Estrada [6], is defined by

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i} .$$

The Estrada index has already found extensive applications, e. g., in chemistry [6, 7], in complex networks [8], in statistical thermodynamics [9, 10]. For recent research on $EE$ see [3, 4, 15, 19, 20].

Quite recently, in full analogy with the Estrada index, the Laplacian Estrada index of the graph $G$, $LEE$ for short, was introduced in [11] as

$$LEE(G) = \sum_{i=1}^{n} e^{\mu_i} . \tag{1}$$

Given a graph $G$ of order $n$ with $m$ edges, independently, in [16] the Laplacian Estrada index was defined as

$$LEE_{LSC}(G) = \sum_{i=1}^{n} e^{\mu_i - 2m/n} .$$

It is easy to see that two “Laplacian Estrada indices” are essentially equivalent in view of $LEE(G) = LEE_{LSC}(G) \times e^{2m/n}$. In the following we use the definition (1). Some properties of LEE have been reported in [1, 5, 11, 14, 16, 21, 22].

Let $k \geq 1$. We say that a graph $G$ is $k$-*connected* if either $G$ is the complete graph $K_{k+1}$, or $G$ has at least $k+2$ vertices and contains no $(k-1)$-vertex cut. Similarly, $G$ is $k$-*edge-connected* if it has at least two vertices and does not contain $(k-1)$-edge cut. The maximum value of $k$ for which a connected graph $G$ is $k$-connected is the connectivity of $G$, denoted by $\kappa(G)$. If $G$ is disconnected, we define $\kappa(G) = 0$. The edge-connectivity $\kappa'(G)$ is defined analogously. If $G$ is a graph of order $n$, then (1) $\kappa(G) \leq \kappa'(G) \leq n-1$, and (2) the three statements $\kappa(G) = n-1$, $\kappa'(G) = n-1$ and $G \cong K_n$ are equivalent.

Two distinct edges in a graph $G$ are *independent* if they are not incident with a common vertex in $G$. A set of pairwise independent edges in $G$ is called a *matching* in $G$. A matching of maximum cardinality is a *maximum matching* in $G$. The *matching number* $\beta(G)$ of $G$ is the cardinality of a maximum matching of $G$. It is well known that $\beta(G) \leq n/2$ with equality if and only if $G$ has a perfect matching. For the other graph theoretical terms used but not defined, we follow the book [2].

In this note, we determine the extremal graphs with given connectivity $k$ maximizing the Laplacian Estrada index. Moreover, we also characterize the $n$-vertex graphs with given matching number having the maximal Laplacian Estrada index.

# 2 The Laplacian Estrada index and connectivity

**Lemma 1.** [12, p. 291] *Let $G$ be a simple non-complete graph with $n$ vertices. If $G+e$ is obtained from $G$ by adding an edge $e$ to $G$, then $0 = \mu_n(G) \leq \mu_n(G+e) \leq \cdots \leq \mu_2(G) \leq \mu_2(G+e) \leq \mu_1(G) \leq \mu_1(G+e)$.*

Noting that $\sum_{i=1}^{n} \mu_i(G+e) - \sum_{i=1}^{n} \mu_i(G) = 2$, we immediately obtain the following result by Lemma 1.

**Lemma 2.** *Let $G$ be a simple non-complete graph with $n$ vertices. Then $LEE(G) < LEE(G+e)$.*

In the following, we will present one main result of this paper.

**Theorem 1.** *Let $G$ be a graph of order $n$ with vertex connectivity $k$. Then $LEE(G) \leq k\,e^n + (n-k-2)\,e^{n-1} + e^k + 1$ with equality if and only if $G \cong K_k \bigvee (K_1 \bigcup K_{n-k-1})$.*

*Proof.* Let $G$ be a graph of order $n$ with given vertex connectivity $k$. It is clear for $k = n-1$. Thus, we can suppose that $1 \leq k \leq n-2$ in the following. Assume that $G$ has the maximal Laplacian Estrada index among all all connected graphs with $n$ vertices and vertex connectivity $k$. By hypothesis there exists a vertex cut set $U$ of order $k$ such that $G-U$ is disconnected. Let $G_1, G_2, \ldots, G_r$ be connected components of $G-U$. If $r > 2$, then adding an edge between $G_1$ and $G_2$ will preserve the connectivity of $G$ but increase the Laplacian Estrada index by Lemma 2, a contradiction. Thus, $r = 2$. Similarly, we also have all of $G[U]$, $G_1$ and $G_2$ are cliques and every vertex in $U$ is adjacent to all vertices in $G_1$ and $G_2$ in view of Lemma 2. Consequently, $G$ can be written as $K_k \bigvee (K_i \bigcup K_{n-k-i})$ for $1 \le i \le (n-k)/2$.

It is well known that $S(K_p) = (\underbrace{p, p, \ldots, p}_{p-1}, 0)$. Noting that

$$\begin{aligned} S(\overline{G}) &= S(K_{i,n-k-i} \bigcup \overline{K_k}) \\ &= \left( n-k, \underbrace{n-k-i, \ldots, n-k-i}_{i-1}, \underbrace{i, \ldots, i}_{n-k-i-1}, \underbrace{0, \ldots, 0}_{k+1} \right) \end{aligned}$$

we have

$$S(G) = (\underbrace{n, \ldots, n}_{k}, \underbrace{n-i, \ldots, n-i}_{n-k-i-1}, \underbrace{i+k, \ldots, i+k}_{i-1}, k, 0)\ .$$

Consequently, we obtain that

$$LEE(G) = ke^n + e^k + (i-1)e^{i+k} + (n-k-i-1)e^{n-i} + 1\ .$$

Assume that $g(x) = (x-1)\, e^{x+k} + (n-k-x-1)\, e^{n-x}$. It is clear that

$$g'(x) = x\, e^{x+k} - (n-k-x)\, e^{n-x} \le 0$$

for $1 \le x \le (n-k)/2$ with equality if and only if $x = (n-k)/2$. Hence, we have

$$LEE(G) \le k\, e^n + e^k + (n-k-2)\, e^{n-1} + 1$$

where equality holds if and only if $G \cong K_k \bigvee (K_1 \bigcup K_{n-k-1})$, as required. □

As well known, $\kappa(G) \le \kappa'(G) \le \delta(G)$. Note that $K_k \bigvee (K_1 \bigcup K_{n-k-1})$ has minimum degree $k$ and edge connectivity $k$ and the function $k\, e^n + e^k + (n-k-2)\, e^{n-1} + 1$ is increasing with respect to $k$. Thus the following corollaries immediately follows from Theorem 1.

**Corollary 1.** *Let $G$ be a graph of order $n$ with given edge connectivity $k$. Then $LEE(G) \le k\, e^n + e^k + (n-k-2)\, e^{n-1} + 1$, where equality holds if and only if $G \cong K_k \bigvee (K_1 \bigcup K_{n-k-1})$.*

**Corollary 2.** *Let $G$ be a graph with $n$ vertices and minimum degree $k$. Then $LEE(G) \le k\, e^n + e^k + (n-k-2)\, e^{n-1} + 1$, where equality holds if and only if $G \cong K_k \bigvee (K_1 \bigcup K_{n-k-1})$.*

# 3 The Laplacian Estrada index and chromatic number

Obviously, $\beta(G) = 0$ if and only if $G$ is an empty graph. A component of a graph is said to be *even* (respectively *odd* ) if it has an even (respectively odd) number of vertices. The following Tutte–Berge formula is very important.

**Lemma 3.** [17] *Let $G$ be a graph of order $n$ with matching number $\beta$. Let $o(G)$ be the number of odd components of $G$. Then*

$$n - 2\beta = \max\{o(G - S) - |S| : S \subset V(G)\}.$$

In what follows we present a result for Laplacian Estrada index related to matching number.

**Theorem 2.** *Let $G$ be a connected graph with $n$ vertices and matching number $\beta$.*

(i) *If $\beta = \lfloor n/2 \rfloor$, then $LEE(G) \leq (n-1)\, e^n + 1$ with equality if and only if $G \cong K_n$.*

(ii) *If $1 \leq \beta < \lfloor n/2 \rfloor$, then $LEE(G) \leq 1 + \beta\, e^n + (n - \beta - 1)\, e^\beta$ with equality if and only if $G \cong K_\beta \bigvee \overline{K_{n-\beta}}$.*

*Proof.* Let $G$ be a graph with the maximum $LEE$ among all connected graphs with $n$ vertices and matching number $\beta$. Then by Lemma 3 there exists a subset $S$ of $V(G)$ on $s$ vertices such that $G - S$ has $n + s - 2\beta$ odd components.

Assume that $s = 0$. It follows that $G - S = G$ and $n - 2\beta \leq 1$. If $n - 2\beta = 0$, then $\beta = n/2$, and if $n - 2\beta = 1$, then $\beta = (n-1)/2$. In either case, by Lemma 2 we have $G = K_n$ and $LEE(G) = (n-1)\, e^n + 1$.

In what follows, we suppose that $s \geq 1$. So $n - 2\beta + s \geq 1$. Let $G_1, G_2, \ldots, G_{n-2\beta+s}$ be the odd components of $G - S$. If $G - S$ has an even component, then by adding an edge to $G$ joining a vertex of an even component and a vertex of an odd component of $G - S$, we obtain a graph $G^*$, for which $n - 2\beta(G^*) \geq o(G^* - S) - s = o(G - S) - s$. This implies that $\beta(G^*) = \beta$. However $LEE(G^*) > LEE(G)$ by Lemma 2, a contradiction. Hence $G - S$ does not have an even component. Similarly, $G_1, G_2, \ldots, G_{n-2\beta+s}$ and $G[S]$ are all cliques, and every vertex in $S$ is adjacent to all vertices in $G_1, G_2, \ldots, G_{n-2\beta+s}$. Consequently, we can write $G$ as

$K_s \bigvee (K_{n_1} \bigcup K_{n_2} \bigcup \cdots \bigcup K_{n_{n-2\beta+s}})$ , where $n_i = |V(G_i)|$ for $1 \le i \le n-2\beta+s$ . Without loss of generality, we can suppose that $n_1 \le n_2 \le \ldots \le n_{n-2\beta+s}$ . Note that $S(K_p) = (\underbrace{p, p, \ldots, p}_{p-1}, 0)$ . It is easy to obtain that

$$\begin{aligned} S\left(K_{n_1} \bigcup K_{n_2} \bigcup \cdots \bigcup K_{n_{n-2\beta+s}}\right) &= (\underbrace{n_{n-2\beta+s}, \ldots, n_{n-2\beta+s}}_{n_{n-2\beta+s}-1}, \ldots \\ &\quad \ldots, \underbrace{n_1, \ldots, n_1}_{n_1-1}, \underbrace{0, \ldots, 0}_{n-2\beta+s}) \end{aligned}$$

$$S\left(\overline{K_{n_1} \bigcup K_{n_2} \bigcup \cdots \bigcup K_{n_{n-2\beta+s}}}\right) = (\underbrace{n-s, \ldots, n-s}_{n-2\beta+s-1}, \ldots$$

$$\ldots \underbrace{n-s-n_1, \ldots, n-s-n_1}_{n_1-1}, \ldots, \underbrace{n-s-n_{n-2\beta+s}, \ldots, n-s-n_{n-2\beta+s}}_{n_{n-2\beta+s}-1}, 0)$$

$$\begin{aligned} S(\overline{G}) &= S\left(\overline{K_s} \bigcup \overline{K_{n_1} \bigcup K_{n_2} \bigcup \cdots \bigcup K_{n_{n-2\beta+s}}}\right) \\ &= (\underbrace{n-s, \ldots, n-s}_{n-2\beta+s-1}, \underbrace{n-s-n_1, \ldots, n-s-n_1}_{n_1-1}, \ldots, \\ &\quad \ldots \underbrace{n-s-n_{n-2\beta+s}, \ldots, n-s-n_{n-2\beta+s}}_{n_{n-2\beta+s}-1}, \underbrace{0, \ldots, 0}_{s+1}) \ . \end{aligned}$$

Thus, we have

$$\begin{aligned} S(G) &= (\underbrace{n, \ldots, n}_{s}, \underbrace{s+n_{n-2\beta+s}, \ldots, s+n_{n-2\beta+s}}_{n_{n-2\beta+s}-1}, \ldots, \\ &\quad \ldots \underbrace{s+n_1, \ldots, s+n_1}_{n_1-1}, \underbrace{s, \ldots, s}_{n-2\beta+s-1}, 0) \ . \end{aligned}$$

Therefore, we obtain that

$$LEE(G) = 1 + s\, e^n + (n-2\beta+s-1)\, e^s + \sum_{i=1}^{n-2\beta+s} (n_i - 1)\, e^{s+n_i} \ .$$

Assume that $g(x) = (x-1)\, e^{s+x} + (m-x-1)\, e^{s+m-x}$ . It is easy to show that

$$g'(x) = xe^{s+x} - (m-x)\, e^{s+m-x} \le 0$$

for $1 \le x \le m/2$ , where equality holds if and only if $x = m/2$ . This implies that

$$(n_i - 1)\, e^{n-n_i} + (n_j - 1)\, e^{n-n_j} < (n_i - 2)\, e^{n-n_i+1} + n_j\, e^{n-n_j-1}$$

for $n_i \leq n_j$. Thus, by replacing any pair $(n_i, n_j)$ with $n_i \leq n_j$ by $(n_i - 1, n_j + 1)$ in the sum $\sum_{i=1}^{n-2\beta+s}(n_i - 1)\, e^{s+n_i}$, we increase the sum. By repeating this process, we attain the maximum of $1 + s\, e^n + (n - 2\beta + s - 1)\, e^s + \sum_{i=1}^{n-2\beta+s}(n_i - 1)\, e^{s+n_i}$ if and only if $n_1 = n_2 = \ldots = n_{n-2\beta+s-1} = 1$ and $n_{n-2\beta+s} = 2\beta - 2s + 1$. It follows that $G$ can be written as

$$K_s \bigvee (\overline{K_{n-2\beta+s-1}} \bigcup K_{2\beta-2s+1})$$

and

$$LEE(G) = 1 + s\, e^n + (n - 2\beta + s - 1)\, e^s + (2\beta - 2s)\, e^{2\beta-s+1} \ .$$

Since $n - s \geq n - 2\beta + s$, we we have $s \leq \beta$. Suppose that $h(s) = 1 + s\, e^n + (n - 2\beta + s - 1)\, e^s + (2\beta - 2s)\, e^{2\beta-s+1}$. By taking derivatives, we have

$$h''(s) = (n - 2\beta + s + 1)\, e^s + 2(\beta - s + 2)\, e^{2\beta-s+1} > 0 \ .$$

It follows that $h(s)$ is a strictly convex function for $1 \leq s \leq \beta$, and the maximum is achieved for $s = 1$ or $s = \beta$. It is easy to show that

$$\begin{aligned} h(\beta) - h(1) &= (\beta - 1)\, e^n - 2(\beta - 1)\, e^{2\beta} + (n - \beta - 1)\, e^\beta - (n - 2\beta)\, e \\ &\geq (\beta - 1)\, e^{2\beta+1} - 2(\beta - 1)\, e^{2\beta} + (n - \beta - 1)\, e^\beta - (n - 2\beta)\, e \\ &\geq 0 \end{aligned}$$

where equalities hold if and only if $\beta = 1$. Thus, $LEE(G) \leq 1 + \beta\, e^n + (n - \beta - 1)\, e^\beta$ with equality if and only if $G \cong K_\beta \bigvee \overline{K_{n-\beta}}$. This completes the proof. $\square$

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