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A Proof of an Inequality Related to Variable Zagreb Indices for Simple Connected Graphs

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Abstract

The paper is concerned with the inequality ${}^{\lambda}M_1(G)/n \leq {}^{\lambda}M_2(G)/m$ for the first and second generalized Zagreb indices of $(m\ n)$ - graphs. It was conjectured that this inequality holds for general graphs, whenever λ is in the interval $[0\ \sqrt{2}/2]$, but the same was only partially proved. We prove that the inequality holds for every $(m\ n)$ - graph and for every λ from the interval $(1/2\ \sqrt{2}/2]$.

1 Introduction

1.1 Definitions

In this paper *graph* has the meaning of a simple connected graph. Let $G = (V, E)$ be a graph with $n = |V|$ vertices and $m = |E|$ edges, and let denote the degree of a vertex $v \in V$ by $d(v)$. The *variable first and second Zagreb indices* are defined as

$${}^{\lambda}M_1(G) = \sum_{v \in V} (d(v))^{2\lambda}, \quad \text{and} \quad {}^{\lambda}M_2(G) = \sum_{uv \in E} (d(u)d(v))^{\lambda}$$

respectively, where $\lambda \in \mathbb{R}$. A question that remains to be answered is ***whether and when the following Inequality (1) holds?***

$$\frac{{}^{\lambda}M_1(G)}{n} \leq \frac{{}^{\lambda}M_2(G)}{m}\,. \tag{1}$$

Inequality (1) is known as *generalized Zagreb indices inequality.* From the definitions above, we can observe that the order of magnitude for ${}^{\lambda}M_1(G)$ is $O(n^3)$, while the order of magnitude for ${}^{\lambda}M_2(G)$ is $O(mn^2)$. Hence, ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ will have magnitudes of same order and, therefore, evaluation of (1) is more reasonable approach than comparing ${}^{\lambda}M_1(G)$ and ${}^{\lambda}M_2(G)$ directly. Assuming $\lambda = 1$, we take a step backward from this generalization to the originally stated definitions for the *first and second Zagreb indices* (introduced by I. Gutman and N. Trinajstić [1] in 1972) and the *Zagreb indices inequality.* In this case, we simply write $M_1(G)$ and $M_2(G)$ instead of ${}^{1}M_1(G)$ and ${}^{1}M_2(G)$. Now, let's introduce a definition which refers to the term of a chemical graph: A *chemical graph* is any simple connected graph G with maximum vertex degree $\Delta(G) \leq 4$.

In order to obtain a clear notion about the herein analyzed problem, we are going to expose several related results to this topic.

1.2 Known results and an open problem

In this section, we aim to give a brief idea about some of the already resolved issues arising from Inequality (1). In sequel, if in a specified case some inequality does not hold, but the reverse one (with reversed inequality sign) does, we may emphasize it by stating *opposite inequality*, or *opposite strict inequality* in case of omitted equality sign. Namely, the opposite of Inequality (1) is given by

$$\frac{{}^{\lambda}M_1(G)}{n} \geq \frac{{}^{\lambda}M_2(G)}{m} \,. \tag{2}$$

At first, the AutoGraphiX system [2] proposed the following conjecture:

Conjecture 1. *For all simple connected graphs G and $\lambda = 1$, Inequality* (1) *holds, and the bound is tight for complete graphs.*

In 2007, P. Hansen and D. Vukičević [3] disproved this claim for general graphs. Actually, the above conjecture does not refer to the variable Zagreb indices because the generalization happened later, but however, it was underlying motivation for many further results. Remember that when we say graph, we mean a simple connected graph. Notations and all other terms used in the below listed theorems, are in the sense of the classical Graph Theory.

Theorem 1. [3] *For all chemical graphs G and $\lambda = 1$, Inequality* (1) *holds.*

Theorem 2. [4] *For all trees T and $\lambda = 1$, Inequality* (1) *holds.*

Theorem 3. [5] *For all unicyclic graphs G and $\lambda = 1$, Inequality* (1) *holds.*

Theorem 4. [6] *For all graphs G such that*
$\Delta(G) - \delta(G) \leq 2$ *and* $\lambda = 1$, *Inequality* (1) *holds.*

Theorem 5. [6] *For all graphs G such that*
$\Delta(G) - \delta(G) \leq 3$ *and* $\delta(G) \neq 2$ *and* $\lambda = 1$, *Inequality* (1) *holds.*

Theorem 6. [7] *For all graphs $G = (V, E)$ such that*
$\forall v \in V,\ d(v) \in \{[s, s + \lceil \sqrt{s}\ \rceil] : s \in \mathbb{N}\}$ *and* $\lambda = 1$, *Inequality* (1) *holds.*

Theorem 7. [8] *For all graphs $G = (V, E)$ such that*
$\forall v \in V,\ d(v) \in \{[a, a + s] : a \geq \frac{s(s-1)}{2} : s \in \mathbb{N}\}$ *and* $\lambda = 1$, *Inequality* (1) *holds.*

The last two theorems slightly differ from their original, and they are adapted here for the purpose of this section. Now, let's move to the generalized variant.

Theorem 8. [9] *For all chemical graphs G and $\lambda \in [0, 1]$, Inequality* (1) *holds.*

Theorem 9. [10] *For all trees T and $\lambda \in [0, 1]$, Inequality* (1) *holds.*

Theorem 10. [11] *For all unicyclic graphs G and $\lambda \in [0, 1]$, Inequality* (1) *holds.*

Theorem 11. [12] *For all unicyclic graphs G and $\lambda \in (-\infty, 0)$, Opposite inequality* (2) *holds.*

It has also been proved that for the distinct classes of graphs mentioned in the above theorems, for $\lambda \in (1, +\infty)$ neither (1) nor (2) holds, i.e., the relationship of numerical value between ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ is not defined [12].

Theorem 12. [13] *For all graphs G such that $\Delta(G) - \delta(G) \leq 2$ and $\lambda \in [0, 1]$, Inequality* (1) *holds.*

Theorem 13. [13] *For all graphs G such that $\Delta(G) - \delta(G) \leq 2$ and $\lambda \in (-\infty, 0)$, Opposite inequality* (2) *holds.*

Theorem 14. [13] *For all graphs G such that $\Delta(G) - \delta(G) \leq 3$ and $\delta(G) \neq 2$ and $\lambda \in [0, 1]$, Inequality* (1) *holds.*

Theorem 15. [13] *For all graphs G such that $\Delta(G) - \delta(G) \leq 3$ and $\delta(G) \neq 2$ and $\lambda \in (-\infty, 0)$, Opposite inequality* (2) *holds.*

Theorem 16. [9] *For all complete unbalanced bipartite graphs G and $\lambda \in \mathbb{R}\backslash[0,1]$, Opposite strict inequality* (2) *holds.*

Theorem 17. [15] *For all graphs G and $\lambda \in [0, \frac{1}{2}]$, Inequality* (1) *holds.*

Theorem 18. [9] *For all $\lambda \in (\frac{\sqrt{2}}{2}, 1]$, there is a graph G such that Opposite strict inequality* (2) *holds.*

From the last two theorems, the next question for general graphs makes sense: ***What happens with Inequality* (1) *when* $\lambda \in (\frac{1}{2}, \frac{\sqrt{2}}{2}]$*?*** In other words, the same question can be restated as in [9]:

Open problem 1. [9] *Identify λ from the interval $(\frac{1}{2}, \frac{\sqrt{2}}{2}]$ such that Opposite strict inequality* (2) *holds.*

2 Preliminaries

2.1 Karamata Inequality

Definition 1. *Let $(a,b) \subseteq \mathbb{R}$. A function $f : (a,b) \to \mathbb{R}$ is said to be* ***convex*** *if for each two points $x, y \in (a,b)$ and each two nonnegative real numbers λ_x, λ_y satisfying $\lambda_x + \lambda_y = 1$, the following inequality holds:*

$$f(\lambda_x x + \lambda_y y) \leq \lambda_x f(x) + \lambda_y f(y)\,. \tag{3}$$

The function f is ***concave*** *if the function $-f$ is convex. If in the previous inequality (assuming $x \neq y$), the equality takes place only in the case when $\lambda_x = 0$ or $\lambda_y = 0$, then the function f is said to be* ***strictly convex****. The function f is* ***strictly concave*** *if the function $-f$ is strictly convex. For concave functions, the next inequality is valid:*

$$f(\lambda_x x + \lambda_y y) \geq \lambda_x f(x) + \lambda_y f(y)\,.$$

Now, let's define a *majorization* relation for finite sequences of real numbers.

Definition 2.1. *Let $a = (a_i)_{i=1}^n$ and $b = (b_i)_{i=1}^n$ be two (finite) sequences of real numbers. We say that the sequence a* ***majorizes*** *the sequence b, and write*

$$a \succ b \quad or \quad b \prec a \tag{4}$$

if there exist permutations $i_1 i_2 \dots i_n$ and $j_1 j_2 \dots j_n$ of all indices of the elements of a and b respectively, such that the following three conditions are simultaneously satisfied:

i. $a_{i_1} \geq a_{i_2} \geq \ldots \geq a_{i_n}$, *and* $b_{j_1} \geq b_{j_2} \geq \ldots \geq b_{j_n}$;

ii. $\sum_{k=1}^{m} a_{i_k} \geq \sum_{k=1}^{m} b_{j_k}$, $\forall m,\ 1 \leq m \leq n-1$;

iii. $\sum_{k=1}^{n} a_{i_k} = \sum_{k=1}^{n} b_{j_k}$.

Next, we give a theorem which plays an essential role in the rest of this paper.

Theorem 2.1. [Karamata Inequality] *Let $(\alpha, \beta) \subset \mathbb{R}$. Let $a = (a_i)_{i=1}^n$ and $b = (b_i)_{i=1}^n$ be two (finite) sequences of real numbers such that $a_i, b_i \in (\alpha, \beta)$ for all $i = 1, 2, \ldots, n$. If $f : (\alpha, \beta) \to \mathbb{R}$ is a convex function and $a \succ b$ (in other words* (3) *and* (4) *hold), then the following inequality is valid:*

$$\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i)\,. \tag{5}$$

2.2 Redefining Open problem 1

Now, we return to Open problem 1, confining to matters that are related to our approach to this problem. Namely, it has been shown (see [9, 14]) that the difference

$$\frac{{}^{\lambda}M_2(G)}{m} - \frac{{}^{\lambda}M_1(G)}{n} \tag{6}$$

depends on a function in which the degrees of the vertices of a graph play the role of variables. This function also depends on the parameter λ, and our interest is how the function behaves with respect to the mentioned parameter. What is important here is the sign of the expression (6) and, to this end, we are going to present some connections to this difference in the following paragraph.

Let $G = (V, E)$ be a (simple connected) graph and $D_G = \{d(v) : v \in V\}$ be the set of the degrees of the all vertices of G. Let $m_{i,j}$ be the number of those edges from E that connect vertices of degrees i and j. It is clear that $i, j \in \mathbb{N}$, and $m_{i,j} = m_{j,i}$ for all pairs (i, j). As shown (see [9, 14]), the sign of the difference (6) can be expressed by the following equivalence:

$$\mathrm{Sign}\left[\frac{{}^{\lambda}M_2(G)}{m} - \frac{{}^{\lambda}M_1(G)}{n}\right] = \mathrm{Sign}\left[\sum_{\substack{i\leq j\\ k\leq l}} {}^{\lambda}f(i,j,k,l)\, m_{i,j} m_{k,l}\right], \quad (i,j),(k,l) \in \mathbb{N}^2\,.$$

In order to single out the parameter λ from the other variables, we write ${}^{\lambda}f(i,j,k,l)$ instead of $f(i,j,k,l,\lambda)$. Since $m_{i,j}=0$ for all $(i,j)\notin D_G^2$ (the same applies to $m_{k,l}$), the sum on the right side of the previous equality can be taken over all pairs (i,j) and (k,l) from $\mathbb{N}^2$, such that $i\leq j$ and $k\leq l$. Clearly, $m_{i,j}$ and $m_{k,l}$ are positive integers or 0, thus, it turns out that the function ${}^{\lambda}f$ is a key factor for the sign of the difference (6). This function is defined as

$$ {}^{\lambda}f(\alpha)=i^{\lambda}j^{\lambda}\left(\frac{1}{k}+\frac{1}{l}\right)+k^{\lambda}l^{\lambda}\left(\frac{1}{i}+\frac{1}{j}\right)-i^{2\lambda-1}-j^{2\lambda-1}-k^{2\lambda-1}-l^{2\lambda-1}, \qquad (7)$$

where $\alpha=(i,j,k,l)\in\mathbb{N}^4$ and $\lambda\in\mathbb{R}$. In Section 1.2, we listed a number of theorems that actually refer to some restrictions on the variables i, j, k, l, and the parameter λ, under which the function ${}^{\lambda}f$ is nonnegative. Also, it was shown [8] that in general case, whenever ${}^{\lambda}f$ is negative for some fixed values i_g, j_g, k_g, l_g and λ_g, we have a way to construct a graph G_g with $D_{G_g}=\{i_g,j_g,k_g,l_g\}$, such that for $\lambda=\lambda_g$, Inequality (1) does not hold. From the previous fact and the list of theorems in Section 1.2, we may conclude that for all i, j, k, l such that $\max\{i,j,k,l\}\leq 4$, or $\max\{i,j,k,l\}-\min\{i,j,k,l\}\leq 2$, or $\max\{i,j,k,l\}-\min\{i,j,k,l\}\leq 3$ where $2\neq\min\{i,j,k,l\}$, we have ${}^{1}f\geq 0$. What is more important to us, is that we have ${}^{\lambda}f\geq 0$ for all i, j, k, l when $\lambda\in[0,\frac{1}{2}]$. On the other hand, the last theorem from the list says us that for $\lambda\in(\frac{\sqrt{2}}{2},1]$, we can always find i, j, k, l such that ${}^{\lambda}f<0$. Hence, the interval $[0,\frac{1}{2}]$ could be at most extended to the interval $[0,\frac{\sqrt{2}}{2}]$, on which ${}^{\lambda}f\geq 0$ would hold for any four positive integers i, j, k, l, and that is the assumption to be proved. Thus, Open problem 1 is reduced to

Problem 2. *Identify λ from the interval $(\frac{1}{2},\frac{\sqrt{2}}{2}]$ such that ${}^{\lambda}f$ is negative for some positive integers i, j, k, l.*

3 Solution of Problem 2

Let X be the set $(\mathbb{R}^+)^4$, $\omega=(x,y,p,q)$ be a point from X, and λ be a real number. Define a function $\phi: X\times\mathbb{R}\to\mathbb{R}$ as

$$\phi(\omega,\lambda)=x^{\lambda}y^{\lambda}\left(\frac{1}{p}+\frac{1}{q}\right)+p^{\lambda}q^{\lambda}\left(\frac{1}{x}+\frac{1}{y}\right)-x^{2\lambda-1}-y^{2\lambda-1}-p^{2\lambda-1}-q^{2\lambda-1}, \qquad (8)$$

and for the sake of formal similarity with the notation in (7), denote ${}^{\lambda}\phi(\omega) = \phi(\omega, \lambda)$. Now, from (7) and (8) it is clear that ${}^{\lambda}f$ is a restriction of ${}^{\lambda}\phi$ on $\mathbb{N}^4$, i.e.,

$$ {}^{\lambda}f = {}^{\lambda}\phi|_{\mathbb{N}^4} \,. \tag{9} $$

Some characteristics of the function ${}^{\lambda}\phi$ can be immediately perceived:

a. ${}^{0}\phi(\omega) = 0\,, \quad \forall\omega \in X\,;$
${}^{\lambda}\phi$ is continuous in both variables ω and λ.

Further, the function ${}^{\lambda}\phi$ has an useful symmetry property with respect to the position of x, y, p, q, i.e., for a fixed value of λ and an arbitrary point $\omega = (x, y, p, q) \in X$, obviously it is valid

b. ${}^{\lambda}\phi(x, y, p, q) = {}^{\lambda}\phi(y, x, p, q)\,;$
${}^{\lambda}\phi(x, y, p, q) = {}^{\lambda}\phi(p, q, x, y)\,.$

In accordance with Problem 2, our interest is limited in terms of the range of values of λ. The interval which we consider is

$$ \mathbf{I} = \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right]. $$

Let's define the sets:

$$ X_r = \{\omega \in X : x \le y,\ p \le q,\ x = \min\{x, y, p, q\}\}\,; \quad X_\omega = \{\omega \in X_r : q < y\}\,; $$
$$ Y_r = \{\omega \in X : x \le y,\ p \le q,\ y = \max\{x, y, p, q\}\}\,; \quad Y_\omega = \{\omega \in Y_r : x < p\}\,. $$

From $X_r \subset X$, for the images of X_r and X under the function ${}^{\lambda}\phi$, we conclude that ${}^{\lambda}\phi(X_r) \subseteq {}^{\lambda}\phi(X)$, and by assertion **b**, for any point $\omega \in X$, there is a point $\omega_r \in X_r$ (just a rearrangement of the values x, y, p, q), such that ${}^{\lambda}\phi(\omega) = {}^{\lambda}\phi(\omega_r)$, which means ${}^{\lambda}\phi(X_r) \supseteq {}^{\lambda}\phi(X)$. Further, from $X_\omega \subset X_r$, for the images of X_ω and X_r under the function ${}^{\lambda}\phi$, we have ${}^{\lambda}\phi(X_\omega) \subseteq {}^{\lambda}\phi(X_r)$. It is easy to see that an analogous inclusion is valid for the sets Y_ω, Y_r, X, and for their images ${}^{\lambda}\phi(Y_\omega)$, ${}^{\lambda}\phi(Y_r)$, ${}^{\lambda}\phi(X)$. We obtain:

$$ {}^{\lambda}\phi(X_\omega) \subseteq {}^{\lambda}\phi(X_r) = {}^{\lambda}\phi(X)\,; \quad {}^{\lambda}\phi(Y_\omega) \subseteq {}^{\lambda}\phi(Y_r) = {}^{\lambda}\phi(X)\,. \tag{10} $$

From this point, we consider the restrictions of the function ${}^{\lambda}\phi$ on the sets X_ω, X_r, and Y_ω, Y_r, and according to (10), it is enough to show that for every $\lambda \in \mathbf{I}$,

$$ {}^{\lambda}\phi|_{X_r}(\omega) \ge 0\,, \quad \text{or} \quad {}^{\lambda}\phi|_{Y_r}(\omega) \ge 0 \tag{11} $$

holds for all ω. The definitions of the sets allow three possible situations for each of the cases $\omega \in X_r$, $\omega \in Y_r$, in terms of the relationship between x, y, p, q. Those are:

$$\begin{array}{l|l}
\omega = (x, y, p, q) \in X_r : & \omega = (x, y, p, q) \in Y_r : \\
\\
\mathbf{1}^{'}\ x \le y \le p \le q & \mathbf{1}^{''}\ p \le q \le x \le y \\
\mathbf{2}^{'}\ x \le p \le y \le q & \mathbf{2}^{''}\ p \le x \le q \le y \\
\mathbf{3}^{'}\ x \le p \le q < y & \mathbf{3}^{''}\ x < p \le q \le y
\end{array}$$

Note that the set of all points ω from X_r which coordinates x, y, p, q satisfy $\mathbf{1}^{'}$ or $\mathbf{2}^{'}$, and do not satisfy $\mathbf{3}^{'}$, is the set $X_r \backslash X_\omega$. Analogously, the set of all points ω from Y_r which coordinates x, y, p, q satisfy $\mathbf{1}^{''}$ or $\mathbf{2}^{''}$, and do not satisfy $\mathbf{3}^{''}$, is the set $Y_r \backslash Y_\omega$ (to see this, it is enough to recall the definitions of these sets). Furthermore, there is an equivalence between the points from $X_r \backslash X_\omega$ and those from $Y_r \backslash Y_\omega$ in the next sense: The point $\omega_x = (x, y, p, q) \in X_r \backslash X_\omega$ if and only if the point $\omega_y = (p, q, x, y) \in Y_r \backslash Y_\omega$. Even more, the coordinates of ω_x satisfy $\mathbf{1}^{'}$ if and only if the coordinates of ω_y satisfy $\mathbf{1}^{''}$ and, the coordinates of ω_x satisfy $\mathbf{2}^{'}$ if and only if the coordinates of ω_y satisfy $\mathbf{2}^{''}$. For the points ω_x and ω_y, the property **b** implies that ${}^{\lambda}\phi(\omega_x) = {}^{\lambda}\phi(\omega_y)$ and, hence, the following two claims are equivalent. Here, we set them out in form of lemmas:

Lemma 1. *Let $\omega = (x, y, p, q) \in X_r \backslash X_\omega$ (x, y, p, q satisfy $\mathbf{1}^{'}$ or $\mathbf{2}^{'}$) and $\lambda \in [0, 1]$. Then ${}^{\lambda}\phi(\omega) \ge 0$ holds.*

Lemma 2. *Let $\omega = (x, y, p, q) \in Y_r \backslash Y_\omega$ (x, y, p, q satisfy $\mathbf{1}^{''}$ or $\mathbf{2}^{''}$) and $\lambda \in [0, 1]$. Then ${}^{\lambda}\phi(\omega) \ge 0$ holds.*

A proof of Lemma 1 will be given at the end of this paper. So, for now, we assume that in cases $\mathbf{1}^{'}$ and $\mathbf{2}^{'}$ ($\mathbf{1}^{''}$ and $\mathbf{2}^{''}$), the inequality ${}^{\lambda}\phi \ge 0$ is valid for $\lambda \in [0, 1]$. Starting with case $\mathbf{3}^{'}$, or with case $\mathbf{3}^{''}$, we have the next two choices respectively: To consider ${}^{\lambda}\phi|_{X_\omega}$, or to consider ${}^{\lambda}\phi|_{Y_\omega}$, which is the same in sense of our problem. If we take the first of these two possibilities, the points which coordinates satisfy $x = p \le q < y$ are covered by $\mathbf{2}^{''}$, and by Lemma 2, we have ${}^{\lambda}\phi \ge 0$ at these points. Otherwise, if we decide to take the second possibility, the points which coordinates satisfy $x < p \le q = y$ are covered by $\mathbf{2}^{'}$, and by Lemma 1, we have ${}^{\lambda}\phi \ge 0$. Ignoring these two sets of points, cases $\mathbf{3}^{'}$ and $\mathbf{3}^{''}$ become identical. In this way, we also avoid some further undefined expressions. Therefore, without loss of generality, we assume

$$x < p \le q < y\,. \tag{12}$$

Case 3′. Let $\omega = (x, y, p, q) \in X_\omega$ and $\lambda \in \mathbb{R}\backslash\{0\}$. Let's fix λ, and make change of the coordinates of ω by introducing a continuous bijective map $g_\lambda : X_\omega \to \mathbb{R}^4$ such that $\omega \mapsto \omega'$, which is given by $g_\lambda(\omega) = (g_x(\omega), g_y(\omega), g_p(\omega), g_q(\omega))$, where the components $g_x,\ g_y,\ g_p,\ g_q$ are real functions on X_ω defined as $g_x(\omega) = \lambda \ln x,\ g_y(\omega) = \lambda \ln y$, $g_p(\omega) = \lambda \ln p,\ g_q(\omega) = \lambda \ln q$, and for simpler further writing let denote $x' = g_x(\omega)$, $y' = g_y(\omega)$, $p' = g_p(\omega)$, $q' = g_q(\omega)$. Remember that in this moment λ does not vary. Now, with an analogous notation as above, we define a function ${}^\lambda\psi : g_\lambda(X_\omega) \to \mathbb{R}$ as

$$
\begin{aligned}
{}^\lambda\psi(\omega') = e^{p'+q'-\frac{x'}{\lambda}} + e^{p'+q'-\frac{y'}{\lambda}} + e^{x'+y'-\frac{p'}{\lambda}} + e^{x'+y'-\frac{q'}{\lambda}} - \\
-e^{2y'-\frac{y'}{\lambda}} - e^{2q'-\frac{q'}{\lambda}} - e^{2p'-\frac{p'}{\lambda}} - e^{2x'-\frac{x'}{\lambda}}\,. \qquad (13)
\end{aligned}
$$

The Jacobian of the map g_λ is positive at every point ($\mathbf{J}_{g_\lambda}(\omega) = \frac{\lambda^4}{xypq} > 0$ for all $\omega \in X_\omega$ and $\lambda \neq 0$), which means that we can turn back to the original coordinates at any moment and, hence, by elementary transformation of (13) we may check that

$$
{}^\lambda\phi(\omega) = {}^\lambda\psi(g_\lambda(\omega))\,, \quad \forall \omega \in X_\omega\,. \qquad (14)
$$

The last equality suggests to pay attention to the sign of the function ${}^\lambda\psi$, which would lead us to the same goal. Before we start, allow λ to vary over the whole range of values. Then, for each separate λ, in the above presented manner, we are able to construct the corresponding map g_λ, after what (14) still applies for all $(\omega, \lambda, g_\lambda(\omega))$. Further, simplify the writing in (13), (14) by introducing the following substitutions:

$$
\begin{aligned}
a_1 &= p' + q' - \frac{x'}{\lambda} = \ln\frac{(pq)^\lambda}{x}\,; \qquad & a_2 &= p' + q' - \frac{y'}{\lambda} = \ln\frac{(pq)^\lambda}{y}\,;\\
a_3 &= x' + y' - \frac{p'}{\lambda} = \ln\frac{(xy)^\lambda}{p}\,; \qquad & a_4 &= x' + y' - \frac{q'}{\lambda} = \ln\frac{(xy)^\lambda}{q}\,;\\
b_1 &= 2y' - \frac{y'}{\lambda} = \ln\frac{y^{2\lambda}}{y}\,; \qquad & b_2 &= 2q' - \frac{q'}{\lambda} = \ln\frac{q^{2\lambda}}{q}\,;\\
b_3 &= 2p' - \frac{p'}{\lambda} = \ln\frac{p^{2\lambda}}{p}\,; \qquad & b_4 &= 2x' - \frac{x'}{\lambda} = \ln\frac{x^{2\lambda}}{x}\,;
\end{aligned}
$$

$$
{}^\lambda\phi(\omega) = {}^\lambda\psi(g_\lambda(\omega)) = e^{a_1}+e^{a_2}+e^{a_3}+e^{a_4}-e^{b_1}-e^{b_2}-e^{b_3}-e^{b_4}\,, \quad \forall \omega \in X_\omega,\ \lambda \neq 0\,. \quad (15)
$$

Since ordering (12) holds for the points ω, due to the monotonicity of the logarithm function, $x' < p' \leq q' < y'$ remains to hold for the points ω'. Now, let's split our consideration on three cases by setting the following conditions:

- $pq > xy\,; \quad \lambda \in [\frac{1}{2}, +\infty)$
- $pq < xy\,; \quad \lambda \in [\frac{1}{2}, +\infty)$
- $pq = xy\,; \quad \lambda \in [\frac{1}{2}, +\infty)$

- Let $pq > xy$; $\lambda \in [\frac{1}{2}, +\infty)$.

First, make several direct estimates referring to exponents a_i, b_i $(i = 1, 2, 3, 4)$, which will be valid whenever is $pq > xy$ and $\lambda \geq \frac{1}{2}$. Since (12) holds, we easily observe that

$$a_1 = \max\{a_1, a_2, a_3, a_4\}\,; \quad a_3 \geq a_4\,; \quad b_1 > b_2 \geq b_3 > b_4\,.$$

Under these conditions, the next three orderings are possible:

1° $a_1 \geq a_2 \geq a_3 \geq a_4$

2° $a_1 \geq a_3 \geq a_2 \geq a_4$

3° $a_1 \geq a_3 \geq a_4 \geq a_2$

For ordering **1**° to be true, an additional constraint is needed on λ, under which $a_2 \geq a_3$ will hold. Similarly, for ordering **2**°, the missing constraints are those under which $a_3 \geq a_2$ and $a_2 \geq a_4$ will hold, and for ordering **3**° that refers to $a_4 \geq a_2$. All these restrictions on λ are given by

$$\mathbf{S}_1 : a_2 \geq a_3 \Leftrightarrow \lambda \geq \frac{\ln \frac{y}{p}}{\ln \frac{pq}{xy}}\,; \qquad \mathbf{S}_2 : a_3 \geq a_2 \Leftrightarrow \lambda \leq \frac{\ln \frac{y}{p}}{\ln \frac{pq}{xy}} \quad (= F_1)\,;$$

$$\mathbf{S}_3 : a_2 \geq a_4 \Leftrightarrow \lambda \geq \frac{\ln \frac{y}{q}}{\ln \frac{pq}{xy}}\,; \qquad \mathbf{S}_4 : a_4 \geq a_2 \Leftrightarrow \lambda \leq \frac{\ln \frac{y}{q}}{\ln \frac{pq}{xy}} \quad (= F_4)\,,$$

where F_1, F_4 (and F_2, F_3, F_5 below) are labels for simpler and shorter notation in the sequel. Obviously, the values F_1 and F_4 are always positive. Further, looking to provide the necessary conditions **i**, **ii** and **iii** of Definition 2.1, we are going to compare the corresponding partial sums for each of the sequences **1**°, **2**° and **3**° (which elements appear in the above-specified order), with the partial sums of the sequence $b = (b_i)_{i=1}^4$.

$$\mathbf{P}_1 : a_1 \geq b_1 \Leftrightarrow \ln \frac{(pq)^\lambda}{x} \geq \ln \frac{y^{2\lambda}}{y} \Leftrightarrow \frac{pq}{xy} \geq \left(\frac{y^2}{pq}\right)^{\lambda-1},$$

thus, we can see for certain that this applies when $\lambda \leq 1$;

$$\mathbf{P}_2 : a_1 + a_2 \geq b_1 + b_2 \Leftrightarrow \ln \frac{(pq)^{2\lambda}}{xy} \geq \ln \frac{(qy)^{2\lambda}}{qy} \Leftrightarrow \frac{p^{2\lambda}}{x} \geq \frac{y^{2\lambda}}{q} \Leftrightarrow \lambda \leq \frac{1}{2}\frac{\ln \frac{q}{x}}{\ln \frac{y}{p}} \quad (= F_2);$$

$$\mathbf{P}_3 : a_1 + a_3 \geq b_1 + b_2 \Leftrightarrow \ln \frac{(xpqy)^\lambda}{xp} \geq \ln \frac{(qy)^{2\lambda}}{qy} \Leftrightarrow \frac{qy}{xp} \geq \left(\frac{qy}{xp}\right)^\lambda \Leftrightarrow \lambda \leq 1;$$

$$\mathbf{P}_4 : a_1 + a_2 + a_3 \geq b_1 + b_2 + b_3 \Leftrightarrow$$
$$\Leftrightarrow \ln \frac{(xy)^\lambda (pq)^{2\lambda}}{xpy} \geq \ln \frac{(pqy)^{2\lambda}}{pqy} \Leftrightarrow \frac{x^\lambda}{x} \geq \frac{y^\lambda}{q} \Leftrightarrow \lambda \leq \frac{\ln \frac{q}{x}}{\ln \frac{y}{x}} \quad (= F_3);$$

$$\mathbf{P}_5 : a_1 + a_3 + a_4 \geq b_1 + b_2 + b_3 \Leftrightarrow$$
$$\Leftrightarrow \ln \frac{(xy)^{2\lambda}(pq)^\lambda}{xpq} \geq \ln \frac{(pqy)^{2\lambda}}{pqy} \Leftrightarrow \frac{x^{2\lambda}}{x} \geq \frac{(pq)^\lambda}{y} \Leftrightarrow \lambda \leq \frac{\ln \frac{y}{x}}{\ln \frac{pq}{x^2}} \quad (= F_5).$$

Values F_2, F_3, F_5 are also positive. Since $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$ is valid without any restriction on λ, it is not a limiting factor in sense of Definition 2.1. Now, combining the conditions $\mathbf{S}_1, \ldots, \mathbf{S}_4$ and $\mathbf{P}_1, \ldots, \mathbf{P}_5$, we can cover all three possibilities **1°**, **2°** and **3°** in a way that assumptions **i**, **ii** and **iii** of Definition 2.1 always remain satisfied. Namely, if we assume that the conditions $\mathbf{S}_1\mathbf{P}_1\mathbf{P}_2\mathbf{P}_4$ are simultaneously satisfied, then ordering **1°** becomes true, and also the majorization we want $(a_1, a_2, a_3, a_4) \succ (b_1, b_2, b_3, b_4)$ is achieved. Similarly, if the conditions $\mathbf{S}_2\mathbf{S}_3\mathbf{P}_1\mathbf{P}_3\mathbf{P}_4$ ($\mathbf{S}_4\mathbf{P}_1\mathbf{P}_3\mathbf{P}_5$) are simultaneously satisfied, then ordering **2°** (ordering **3°**) and the same majorization $a = (a_{i'})_{i=1}^4 \succ b = (b_i)_{i=1}^4$ are to be valid. The function e^r ($r \in \mathbb{R}$) is strictly convex. Thus, under the previous assumptions, we infer that Theorem 2.1 can be applied in (15), so by (5) it follows

$$\mathbf{S}_1\mathbf{P}_1\mathbf{P}_2\mathbf{P}_4 \vee \mathbf{S}_2\mathbf{S}_3\mathbf{P}_1\mathbf{P}_3\mathbf{P}_4 \vee \mathbf{S}_4\mathbf{P}_1\mathbf{P}_3\mathbf{P}_5 \Rightarrow \sum_{i=1}^{4}(e^{a_i} - e^{b_i}) \geq 0. \tag{16}$$

The conditions from the left side of the previous implication can be overwritten as:

$$\mathbf{Q}_1 : \mathbf{S}_1\mathbf{P}_1\mathbf{P}_2\mathbf{P}_4 \quad \Longleftrightarrow \quad \lambda \in A \cap [\tfrac{1}{2}, 1] \quad \text{where} \quad A = [F_1, \min\{F_2, F_3\}]\,;$$

$$\mathbf{Q}_2 : \mathbf{S}_2\mathbf{S}_3\mathbf{P}_1\mathbf{P}_3\mathbf{P}_4 \quad \Longleftrightarrow \quad \lambda \in B \cap [\tfrac{1}{2}, 1] \quad \text{where} \quad B = [F_4, \min\{F_1, F_3\}]\,;$$

$$\mathbf{Q}_3 : \mathbf{S}_4\mathbf{P}_1\mathbf{P}_3\mathbf{P}_5 \quad \Longleftrightarrow \quad \lambda \in C \cap [\tfrac{1}{2}, 1] \quad \text{where} \quad C = [\tfrac{1}{2}, \min\{F_4, F_5\}]\,.$$

The proof of this case shall be successfully accomplished if we can show the following:

$$\mathbf{I} = \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right] \subseteq \bigcap_{\substack{pq>xy \\ \omega\in X_\omega}} \left((A \cup B \cup C) \cap \left[\frac{1}{2}, 1\right]\right)\,. \tag{17}$$

It means that whenever $\lambda \in \mathbf{I}$, at least one of the assumptions $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$ becomes true, which implies (16) also to be true. Since in (17) the interval $[\frac{1}{2}, 1]$ has no other impact except discarding the numbers greater than 1, we have only to show the next:

$$\mathbf{I} \subseteq A \cup B \cup C\,, \quad \{\omega \in X_\omega : pq > xy\}\,. \tag{18}$$

Now, we analyze the set $(A \cup B \cup C) \cap [\frac{1}{2}, 1]$. Suppose that $F_2 < F_3$. Then it follows

$$\frac{1}{2}\frac{\ln\frac{q}{x}}{\ln\frac{y}{p}} < \frac{\ln\frac{q}{x}}{\ln\frac{y}{x}} \Rightarrow p^2 < xy\,,$$

hence, combining with (12) we have

$$1 < \frac{\ln\frac{y}{p}}{\ln\frac{q}{p}} = \frac{\ln\frac{y}{p}}{\ln\frac{pq}{p^2}} < \frac{\ln\frac{y}{p}}{\ln\frac{pq}{xy}} = F_1\,.$$

On the other hand, $F_3 < 1$ always holds, thus under the previous assumption we get $F_2 < F_3 < F_1$, which means $A = \varnothing$, i.e., $B \cup A = [F_4, F_3] \cup \varnothing$. Conversely, if $F_3 \le F_2$, then the upper endpoint of A is F_3. The assumption $p \le q$ implies that we always have $F_4 \le F_1$, so if we assume $F_1 \le F_3$, then $B \cup A = [F_4, F_1] \cup [F_1, F_3] = [F_4, F_3]$. Otherwise, if we now assume $F_3 < F_1$ to hold, then again $B \cup A = [F_4, F_3] \cup \varnothing$. In this way, we show that $B \cup A = [F_4, F_3]$ in all cases. Next, it will be proved that neither $F_5 \le F_4 < F_3$ ($F_5 < F_4 \le F_3$) nor $F_3 \le F_4 < F_5$ ($F_3 < F_4 \le F_5$) is possible. In that order, let's take the case $F_5 \le F_4 < F_3$. Then we have

$$F_5 \le F_4 \Leftrightarrow \ln\frac{y}{x}\ln\frac{pq}{xy} \le \ln\frac{y}{q}\ln\frac{pq}{x^2}\,;$$

$F_4 < F_3 \Leftrightarrow \ln\frac{y}{q}\ln\frac{y}{x} < \ln\frac{q}{x}\ln\frac{pq}{xy}$. Now, sum the left and the right sides:

$$\ln\frac{pq}{xy}\left(\ln\frac{y}{x} - \ln\frac{q}{x}\right) < \ln\frac{y}{q}\left(\ln\frac{p}{x} + \ln\frac{q}{x} - \ln\frac{y}{x}\right) \Rightarrow \ln\frac{pq}{xy}\ln\frac{y}{q} < \ln\frac{y}{q}\ln\frac{pq}{xy} \Rightarrow 0 < 0\,,$$

which is impossible. In the same manner we disprove the other cases, so it remains to have only the next two conjunctions to be true:

$$F_4 \le F_3 \wedge F_4 \le F_5\,;$$
$$F_4 > F_3 \wedge F_4 > F_5\,.$$

Remark. From the preceding contradiction, we observe $F_4 = F_3$ *if and only if* $F_4 = F_5$. Further, from the definitions of F_3 and F_5, we note that both are always less than 1. Thus, when the first of these two cases occurs, i.e., when we have the situation $F_4 \le F_3$ and $F_4 \le F_5$, it follows that $(C \cup B \cup A) \cap [\frac{1}{2}, 1] = ([\frac{1}{2}, F_4] \cup [F_4, F_3]) \cap [\frac{1}{2}, 1] = [\frac{1}{2}, F_3]$. In this case, the important value is F_3. When the second case occurs, i.e., when $F_4 > F_3$ and $F_4 > F_5$, it turns out that $(C \cup B \cup A) \cap [\frac{1}{2}, 1] = ([\frac{1}{2}, F_5] \cup \varnothing) \cap [\frac{1}{2}, 1] = [\frac{1}{2}, F_5]$. Up to this point, it is still possible that the result is the empty set, but it does not affect our conclusions. What is important here is that the set $(C \cup B \cup A) \cap [\frac{1}{2}, 1]$ depends solely on the values F_3 and F_5. Thus, we will end up this case by proving the following two implications:

$$\text{a}^*\ F_4 \le F_3 \wedge F_4 \le F_5 \Rightarrow F_3 \ge \frac{\sqrt{2}}{2}$$
$$\text{b}^*\ F_4 > F_3 \wedge F_4 > F_5 \Rightarrow F_5 \ge \frac{\sqrt{2}}{2}$$

a* Let $F_4 \le F_3 \wedge F_4 \le F_5$.

As we showed, in this case $(A \cup B \cup C) \cap [\frac{1}{2}, 1] = [\frac{1}{2}, F_3]$. Suppose that there exist a point $(x, y, p, q) \in X_\omega$ and $\alpha \in (0, \frac{\sqrt{2}}{2}]$, such that $\alpha > F_3$.

$$\alpha > F_3 \Leftrightarrow \ln\frac{y}{x} > \frac{1}{\alpha}\ln\frac{q}{x}\,;\quad \ln\frac{y}{q} + \ln\frac{q}{x} > \frac{1}{\alpha}\ln\frac{q}{x}\,;\quad \ln\frac{y}{q} > \left(\frac{1}{\alpha} - 1\right)\ln\frac{q}{x}\,.$$

$$\alpha > F_4 \Leftrightarrow \ln\frac{pq}{xy} > \frac{1}{\alpha}\ln\frac{y}{q}\,;\quad \ln\frac{p}{x} + \ln\frac{q}{y} > \frac{1}{\alpha}\ln\frac{y}{q}\,;\quad \ln\frac{p}{x} > \left(\frac{1}{\alpha} + 1\right)\ln\frac{y}{q}\,.$$

Hence, $\alpha > F_3$ together with $\alpha > F_4$ imply the following:

$$\ln\frac{y}{q} > \left(\frac{1}{\alpha} - 1\right)\ln\frac{q}{x} \ge \left(\frac{1}{\alpha} - 1\right)\ln\frac{p}{x} > \left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} + 1\right)\ln\frac{y}{q}\,.$$

Since $\ln\frac{y}{q} > 0$, finally we obtain: $1 > \frac{1}{\alpha^2} - 1\,;\quad \alpha > \frac{\sqrt{2}}{2}\,.$

This contradicts the above assumption and, therefore, we conclude that whenever $F_4 \le F_3$ holds, then $\frac{\sqrt{2}}{2} \le F_3$ must also be valid. Hence, the inclusion which we want

$$\mathbf{I} \subseteq A \cup B \cup C = \left[\frac{1}{2}, F_3\right] \tag{19}$$

is obtained, and that is what we intended to show. As it can be noted, the condition $F_4 \le F_5$ is not used anywhere above, so the same proof can be applied for the next case when $F_4 \ge F_3$, to bound F_4 from below. Therefore, we infer that the following two implications must always be correct:

$$F_3 \ge F_4 \Rightarrow F_3 \ge \frac{\sqrt{2}}{2}, \quad \text{and} \quad F_4 \ge F_3 \Rightarrow F_4 \ge \frac{\sqrt{2}}{2}. \tag{20}$$

b* Let $F_4 > F_3 \wedge F_4 > F_5$.

Then, we have $(A \cup B \cup C) \cap [\frac{1}{2}, 1] = [\frac{1}{2}, F_5]$, and we now need to show that under these conditions, $F_5 \ge \frac{\sqrt{2}}{2}$ is valid. Suppose in contrary that there exist a point $(x, y, p, q) \in X_\omega$ and $\alpha \in (0, \frac{\sqrt{2}}{2}]$, such that $\alpha > F_5$. Since $F_4 > F_3$, from (20) it is $F_4 \ge \frac{\sqrt{2}}{2}$ and, therefore, we have $F_4 \ge \alpha$.

$$F_4 \ge \alpha \Leftrightarrow \ln\frac{y}{q} \ge \alpha \ln\frac{pq}{xy} = \alpha\left(\ln\frac{p}{x} + \ln\frac{q}{y}\right); \quad \left(\frac{1}{\alpha} + 1\right)\ln\frac{y}{q} \ge \ln\frac{p}{x}.$$

$$\alpha > F_5 \Leftrightarrow \ln\frac{p}{x} + \ln\frac{q}{x} > \frac{1}{\alpha}\ln\frac{y}{x} = \frac{1}{\alpha}\left(\ln\frac{y}{q} + \ln\frac{q}{x}\right). \text{ Now, replace } \ln\frac{p}{x}:$$

$$\left(\frac{1}{\alpha} + 1\right)\ln\frac{y}{q} + \ln\frac{q}{x} > \frac{1}{\alpha}\left(\ln\frac{y}{q} + \ln\frac{q}{x}\right);$$

$$\ln\frac{y}{x} > \frac{1}{\alpha}\ln\frac{q}{x}; \quad \alpha > \frac{\ln\frac{q}{x}}{\ln\frac{y}{x}} = F_3.$$

It turns out that the condition $F_4 \ge \alpha > F_5$ implies $\alpha > F_3$. If we assume $F_3 > F_5$, then there must be β such that $F_3 > \beta > F_5$, and from $F_4 > \beta > F_5$, as we showed, it follows that $\beta > F_3$, which is impossible. Hence, it has to be true that $F_5 \ge F_3$, and together with the assumption $\alpha > F_5$, again we arrive at the following contradiction:

$$\frac{\sqrt{2}}{2} > F_5 \Leftrightarrow \frac{\sqrt{2}}{2} \ln \frac{pq}{x^2} > \ln \frac{y}{x}\,;$$

$$\frac{\sqrt{2}}{2} > F_3 \Leftrightarrow \ln \frac{y}{x} > \sqrt{2} \ln \frac{q}{x}\,.$$ If we rewrite the same in one raw, we get

$$\sqrt{2} \ln \frac{\sqrt{pq}}{x} > \ln \frac{y}{x} > \sqrt{2} \ln \frac{q}{x}\,,$$ which is clearly impossible, since $p \leq q$.

Thus, under the above conditions, we infer that our starting assumption was wrong and, therefore, $F_5 \geq \frac{\sqrt{2}}{2}$ must be valid. From this, we see that the inclusion we want

$$\mathbf{I} \subseteq A \cup B \cup C = \left[\frac{1}{2}, F_5\right]$$

holds, and together with (19), implies that we also have (18), (17) and (16) as valid. Before we move to the next case which will be proved in a quite analogous way, let's mention (to avoid confusion), we will keep a part of the previous labeling unchanged.

- Let $pq < xy$; $\lambda \in [\frac{1}{2}, +\infty)$.

Again, make several direct estimates referring to exponents a_i, b_i $(i = 1, 2, 3, 4)$, which will be valid whenever is $pq < xy$ and $\lambda \geq \frac{1}{2}$. Since (12) holds, we easily observe that

$$a_2 = \min\{a_1, a_2, a_3, a_4\}\,; \quad a_3 \geq a_4\,; \quad b_1 > b_2 \geq b_3 > b_4\,.$$

Under these conditions, the next three orderings are possible:

$$\mathbf{1}^\circ \ a_3 \geq a_4 \geq a_1 \geq a_2$$

$$\mathbf{2}^\circ \ a_3 \geq a_1 \geq a_4 \geq a_2$$

$$\mathbf{3}^\circ \ a_1 \geq a_3 \geq a_4 \geq a_2$$

For ordering $\mathbf{1}^\circ$ to be true, an additional constraint is needed on λ, under which $a_4 \geq a_1$ will hold. Similarly, for ordering $\mathbf{2}^\circ$, the missing constraints are those under which $a_3 \geq a_1$ and $a_1 \geq a_4$ will hold, and for ordering $\mathbf{3}^\circ$ that refers to $a_1 \geq a_3$. All these restrictions on λ are given by

$$\mathbf{S}_1 : a_4 \geq a_1 \Leftrightarrow \lambda \geq \frac{\ln \frac{q}{x}}{\ln \frac{xy}{pq}}\,; \qquad \mathbf{S}_2 : a_1 \geq a_4 \Leftrightarrow \lambda \leq \frac{\ln \frac{q}{x}}{\ln \frac{xy}{pq}} \quad (= G_1)\,;$$

$$\mathbf{S}_3 : a_3 \geq a_1 \Leftrightarrow \lambda \geq \frac{\ln \frac{p}{x}}{\ln \frac{xy}{pq}}\,; \qquad \mathbf{S}_4 : a_1 \geq a_3 \Leftrightarrow \lambda \leq \frac{\ln \frac{p}{x}}{\ln \frac{xy}{pq}} \quad (= G_4)\,,$$

where G_1, G_4 (and G_3, G_5, G_2, G_6 below) are labels for simpler and shorter notation in the sequel. Clearly, the values G_1 and G_4 are always positive. Further, looking to provide the necessary conditions **i**, **ii** and **iii** of Definition 2.1, we are going to compare the corresponding partial sums for each of the sequences $\mathbf{1}^\circ$, $\mathbf{2}^\circ$ and $\mathbf{3}^\circ$ (which elements appear in the above-specified order), with the partial sums of the sequence $b = (b_i)_{i=1}^4$.

$$\mathbf{P}_1 : a_3 \geq b_1 \Leftrightarrow \ln \frac{(xy)^\lambda}{p} \geq \ln \frac{y^{2\lambda}}{y} \Leftrightarrow \frac{y}{p} \geq \left(\frac{y}{x}\right)^\lambda \Leftrightarrow \lambda \leq \frac{\ln \frac{y}{p}}{\ln \frac{y}{x}} \quad (= G_3)\,;$$

$$\mathbf{P}_2 : a_1 \geq b_1 \Leftrightarrow \ln \frac{(pq)^\lambda}{x} \geq \ln \frac{y^{2\lambda}}{y} \Leftrightarrow \frac{y}{x} \geq \left(\frac{y^2}{pq}\right)^\lambda \Leftrightarrow \lambda \leq \frac{\ln \frac{y}{x}}{\ln \frac{y^2}{pq}} \quad (= G_5)\,;$$

$$\mathbf{P}_3 : a_3 + a_4 \geq b_1 + b_2 \Leftrightarrow \ln \frac{(xy)^{2\lambda}}{pq} \geq \ln \frac{(qy)^{2\lambda}}{qy} \Leftrightarrow \frac{y}{p} \geq \left(\frac{q}{x}\right)^{2\lambda} \Leftrightarrow \lambda \leq \frac{1}{2}\frac{\ln \frac{y}{p}}{\ln \frac{q}{x}} \quad (= G_2)\,;$$

$$\mathbf{P}_4 : a_1 + a_3 \geq b_1 + b_2 \Leftrightarrow \ln \frac{(xpqy)^\lambda}{xp} \geq \ln \frac{(qy)^{2\lambda}}{qy} \Leftrightarrow \frac{qy}{xp} \geq \left(\frac{qy}{xp}\right)^\lambda \Leftrightarrow \lambda \leq 1\,;$$

$$\mathbf{P}_5 : a_1 + a_3 + a_4 \geq b_1 + b_2 + b_3 \Leftrightarrow$$
$$\Leftrightarrow \ln \frac{(xy)^{2\lambda}(pq)^\lambda}{xpq} \geq \ln \frac{(pqy)^{2\lambda}}{pqy} \Leftrightarrow \frac{x^{2\lambda}}{x} \geq \frac{(pq)^\lambda}{y} \Leftrightarrow \lambda \leq \frac{\ln \frac{y}{x}}{\ln \frac{pq}{x^2}} \quad (= G_6)\,.$$

Here, G_3, G_5, G_2, G_6 are also positive. Since $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$ holds without any restriction on λ, it is not a limiting factor in sense of Definition 2.1. Now, combining the conditions $\mathbf{S}_1, \ldots, \mathbf{S}_4$ and $\mathbf{P}_1, \ldots, \mathbf{P}_5$, we can cover all three possibilities $\mathbf{1}^\circ$, $\mathbf{2}^\circ$ and $\mathbf{3}^\circ$ in a way that assumptions **i**, **ii** and **iii** of Definition 2.1 always remain satisfied. Namely, if we assume that the conditions $\mathbf{S}_1\mathbf{P}_1\mathbf{P}_3\mathbf{P}_5$ are simultaneously satisfied, then ordering $\mathbf{1}^\circ$ becomes true, and also the majorization we want $(a_3, a_4, a_1, a_2) \succ (b_1, b_2, b_3, b_4)$ is achieved. Similarly, if the conditions $\mathbf{S}_2\mathbf{S}_3\mathbf{P}_1\mathbf{P}_4\mathbf{P}_5$ ($\mathbf{S}_4\mathbf{P}_2\mathbf{P}_4\mathbf{P}_5$) are simultaneously satisfied, then ordering $\mathbf{2}^\circ$ (ordering $\mathbf{3}^\circ$) and the same majorization $a = (a_{i'})_{i=1}^4 \succ b = (b_i)_{i=1}^4$ are to be valid. The function e^r $(r \in \mathbb{R})$ is strictly convex. Thus, under the previous assumptions, we infer that Theorem 2.1 can be applied in (15), so by (5) it follows

$$\mathbf{S}_1\mathbf{P}_1\mathbf{P}_3\mathbf{P}_5 \vee \mathbf{S}_2\mathbf{S}_3\mathbf{P}_1\mathbf{P}_4\mathbf{P}_5 \vee \mathbf{S}_4\mathbf{P}_2\mathbf{P}_4\mathbf{P}_5 \Rightarrow \sum_{i=1}^{4}(e^{a_i} - e^{b_i}) \geq 0\,. \tag{21}$$

Directly, we can observe that whenever $\mathbf{P}_1$ is true or $\mathbf{P}_2$ is true, then $\mathbf{P}_5$ must also be true. Namely, since $pq < xy$, we have

$$\ln\frac{pq}{x^2} < \ln\frac{y}{x}\,,\quad \ln\frac{pq}{x^2} < \ln\frac{y^2}{pq}\,,$$

which means that it is always $G_3 < G_6$ and $G_5 < G_6$. Therefore, $\mathbf{P}_5$ does not have meaning when it is in conjunction with $\mathbf{P}_1$ or $\mathbf{P}_2$, and will not be taken into account. The conditions from the left side of the implication (21) can be overwritten as follows:

$$\mathbf{Q}_1 : \mathbf{S}_1\mathbf{P}_1\mathbf{P}_3 \iff \lambda\in A\cap[\tfrac{1}{2},1] \quad \text{where} \quad A=[G_1,\min\{G_2,G_3\}]\,;$$

$$\mathbf{Q}_2 : \mathbf{S}_2\mathbf{S}_3\mathbf{P}_1\mathbf{P}_4 \iff \lambda\in B\cap[\tfrac{1}{2},1] \quad \text{where} \quad B=[G_4,\min\{G_1,G_3\}]\,;$$

$$\mathbf{Q}_3 : \mathbf{S}_4\mathbf{P}_2\mathbf{P}_4 \iff \lambda\in C\cap[\tfrac{1}{2},1] \quad \text{where} \quad C=[\tfrac{1}{2},\min\{G_4,G_5\}]\,.$$

The proof of this case shall be successfully accomplished if we can show the following:

$$\mathbf{I}=\left[\frac{1}{2},\frac{\sqrt{2}}{2}\right]\subseteq\bigcap_{\substack{pq<xy\\ \omega\in X_\omega}}\left((A\cup B\cup C)\cap\left[\frac{1}{2},1\right]\right)\,. \tag{22}$$

It means that whenever $\lambda\in\mathbf{I}$, at least one of the assumptions $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$ becomes true, which implies (21) also to be true. Since in (22) the interval $[\frac{1}{2},1]$ has no other impact except discarding the numbers greater than 1, we have only to show the next:

$$\mathbf{I}\subseteq A\cup B\cup C\,,\quad \{\omega\in X_\omega : pq<xy\}\,. \tag{23}$$

Let's consider the set $(A\cup B\cup C)\cap[\frac{1}{2},1]$. First, suppose $G_2 < G_3$. Then it follows

$$\frac{1}{2}\frac{\ln\frac{y}{p}}{\ln\frac{q}{x}} < \frac{\ln\frac{y}{p}}{\ln\frac{y}{x}} \Rightarrow xy < q^2\,,$$

hence, combining with (12) we have

$$1<\frac{\ln\frac{q}{x}}{\ln\frac{q}{p}}=\frac{\ln\frac{q}{x}}{\ln\frac{q^2}{pq}}<\frac{\ln\frac{q}{x}}{\ln\frac{xy}{pq}}=G_1\,.$$

On the other hand, $G_3 < 1$ always holds, thus under the previous assumption we get $G_2 < G_3 < G_1$, which means $A=\varnothing$, i.e., $B\cup A=[G_4,G_3]\cup\varnothing$. Conversely, if $G_3\le G_2$, then the upper endpoint of A is G_3. The assumption $p\le q$ implies that we always have $G_4\le G_1$, so if we assume $G_1\le G_3$, then $B\cup A=[G_4,G_1]\cup[G_1,G_3]=[G_4,G_3]$. Otherwise, if we now assume $G_3 < G_1$ to hold, then again $B\cup A=[G_4,G_3]\cup\varnothing$. In this way, we show that $B\cup A=[G_4,G_3]$ in all cases. Next, it will be proved that neither $G_5 \le G_4 < G_3$ ($G_5 < G_4 \le G_3$) nor $G_3 \le G_4 < G_5$ ($G_3 < G_4 \le G_5$) is possible. In that order, let's take the case $G_5 \le G_4 < G_3$. Then we have

$G_5 \le G_4 \Leftrightarrow \ln\dfrac{y}{x}\ln\dfrac{xy}{pq} \le \ln\dfrac{p}{x}\ln\dfrac{y^2}{pq}$;

$G_4 < G_3 \Leftrightarrow \ln\dfrac{p}{x}\ln\dfrac{y}{x} < \ln\dfrac{y}{p}\ln\dfrac{xy}{pq}$. Now, sum the left and the right sides:

$$\ln\frac{xy}{pq}\left(\ln\frac{y}{x}-\ln\frac{y}{p}\right) < \ln\frac{p}{x}\left(\ln\frac{y}{p}+\ln\frac{y}{q}-\ln\frac{y}{x}\right) \Rightarrow \ln\frac{xy}{pq}\ln\frac{p}{x} < \ln\frac{p}{x}\ln\frac{xy}{pq} \Rightarrow 0<0\,,$$

which is impossible. In the same manner we disprove the other cases, so it remains to have only the next two conjunctions to be true:

$$G_4 \le G_3 \wedge G_4 \le G_5\,;$$
$$G_4 > G_3 \wedge G_4 > G_5\,.$$

Remark. From the preceding contradiction, we see $G_4 = G_3$ *if and only if* $G_4 = G_5$. Further, from the definitions of F_3 and F_5, we note that both are always less than 1. Thus, when the first of these two cases occurs, i.e., when we have the situation $F_4 \le F_3$ and $F_4 \le F_5$, it follows that $(C\cup B\cup A)\cap[\frac{1}{2},1] = ([\frac{1}{2},F_4]\cup[F_4,F_3])\cap[\frac{1}{2},1] = [\frac{1}{2},F_3]$. In this case, the important value is F_3. When the second case occurs, i.e., when $F_4 > F_3$ and $F_4 > F_5$, it turns out that $(C\cup B\cup A)\cap[\frac{1}{2},1] = ([\frac{1}{2},F_5]\cup\varnothing)\cap[\frac{1}{2},1] = [\frac{1}{2},F_5]$. Up to this point, it is still possible that the result is the empty set, but it does not affect our conclusions. What is important here is that the set $(C\cup B\cup A)\cap[\frac{1}{2},1]$ depends solely on the values F_3 and F_5. Thus, we will end up this case by proving the following two implications:

$$\text{c}^*\ G_4 \le G_3 \wedge G_4 \le G_5 \Rightarrow G_3 \ge \frac{\sqrt{2}}{2}$$
$$\text{d}^*\ G_4 > G_3 \wedge G_4 > G_5 \Rightarrow G_5 \ge \frac{\sqrt{2}}{2}$$

c* Let $G_4 \le G_3 \wedge G_4 \le G_5$.

As we showed, in this case $(A\cup B\cup C)\cap[\frac{1}{2},1] = [\frac{1}{2},G_3]$. Suppose that there exist a point $(x, y, p, q) \in X_\omega$ and $\alpha \in (0, \frac{\sqrt{2}}{2}]$, such that $\alpha > G_3$.

$$\alpha > G_3 \Leftrightarrow \ln\frac{y}{x} > \frac{1}{\alpha}\ln\frac{y}{p}; \quad \ln\frac{y}{p} + \ln\frac{p}{x} > \frac{1}{\alpha}\ln\frac{y}{p}; \quad \ln\frac{p}{x} > \left(\frac{1}{\alpha} - 1\right)\ln\frac{y}{p}.$$

$$\alpha > G_4 \Leftrightarrow \ln\frac{xy}{pq} > \frac{1}{\alpha}\ln\frac{p}{x}; \quad \ln\frac{x}{p} + \ln\frac{y}{q} > \frac{1}{\alpha}\ln\frac{p}{x}; \quad \ln\frac{y}{q} > \left(\frac{1}{\alpha} + 1\right)\ln\frac{p}{x}.$$

Hence, $\alpha > G_3$ together with $\alpha > G_4$ imply the following:

$$\ln\frac{p}{x} > \left(\frac{1}{\alpha} - 1\right)\ln\frac{y}{p} \geq \left(\frac{1}{\alpha} - 1\right)\ln\frac{y}{q} > \left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} + 1\right)\ln\frac{p}{x}.$$

Since $\ln\frac{p}{x} > 0$, finally we obtain: $1 > \frac{1}{\alpha^2} - 1; \quad \alpha > \frac{\sqrt{2}}{2}$.

This contradicts the above assumption and, therefore, we conclude that whenever $G_4 \leq G_3$ holds, then $\frac{\sqrt{2}}{2} \leq G_3$ must also be valid. Hence, the inclusion which we want

$$\mathbf{I} \subseteq A \cup B \cup C = \left[\frac{1}{2}, G_3\right] \tag{24}$$

is obtained, and that is what we intended to show. As it can be noted, the condition $G_4 \leq G_5$ is not used anywhere above, so the same proof can be applied for the next case when $G_4 \geq G_3$, to bound G_4 from below. Therefore, we infer that the following two implications must always be correct:

$$G_3 \geq G_4 \Rightarrow G_3 \geq \frac{\sqrt{2}}{2}, \quad \text{and} \quad G_4 \geq G_3 \Rightarrow G_4 \geq \frac{\sqrt{2}}{2}. \tag{25}$$

d* Let $G_4 > G_3 \wedge G_4 > G_5$.

Then, we have $(A \cup B \cup C) \cap [\frac{1}{2}, 1] = [\frac{1}{2}, G_5]$, and we now need to show that under these conditions, $G_5 \geq \frac{\sqrt{2}}{2}$ is valid. Suppose in contrary that there exist a point $(x, y, p, q) \in X_\omega$ and $\alpha \in (0, \frac{\sqrt{2}}{2}]$, such that $\alpha > G_5$. Since $G_4 > G_3$, from (25) it is $G_4 \geq \frac{\sqrt{2}}{2}$ and, therefore, we have $G_4 \geq \alpha$.

$$G_4 \geq \alpha \Leftrightarrow \ln\frac{p}{x} \geq \alpha\ln\frac{xy}{pq} = \alpha\left(\ln\frac{x}{p} + \ln\frac{y}{q}\right); \quad \left(\frac{1}{\alpha} + 1\right)\ln\frac{p}{x} \geq \ln\frac{y}{q}.$$

$$\alpha > G_5 \Leftrightarrow \ln\frac{y}{p} + \ln\frac{y}{q} > \frac{1}{\alpha}\ln\frac{y}{x} = \frac{1}{\alpha}\left(\ln\frac{y}{p} + \ln\frac{p}{x}\right)$$. Now, replace $\ln\frac{y}{q}$:

$$\ln\frac{y}{p} + \left(\frac{1}{\alpha}+1\right)\ln\frac{p}{x} > \frac{1}{\alpha}\left(\ln\frac{y}{p} + \ln\frac{p}{x}\right)\,;$$

$$\ln\frac{y}{x} > \frac{1}{\alpha}\ln\frac{y}{p}\,; \quad \alpha > \frac{\ln\frac{y}{p}}{\ln\frac{y}{x}} = G_3\,.$$

It turns out that the condition $G_4 \geq \alpha > G_5$ implies $\alpha > G_3$. If we assume $G_3 > G_5$, then there must be β such that $G_3 > \beta > G_5$, and from $G_4 > \beta > G_5$, as we showed, it follows that $\beta > G_3$, which is impossible. Hence, it has to be true that $G_5 \geq G_3$, and together with the assumption $\alpha > G_5$, again we arrive at the following contradiction:

$$\frac{\sqrt{2}}{2} > G_5 \Leftrightarrow \frac{\sqrt{2}}{2}\ln\frac{y^2}{pq} > \ln\frac{y}{x}\,;$$

$$\frac{\sqrt{2}}{2} > G_3 \Leftrightarrow \ln\frac{y}{x} > \sqrt{2}\ln\frac{y}{p}$$. If we rewrite the same in one raw, we get

$$\sqrt{2}\ln\frac{y}{\sqrt{pq}} > \ln\frac{y}{x} > \sqrt{2}\ln\frac{y}{p}$$, which is clearly impossible, since $p \leq q$.

Thus, under the above conditions, we infer that our starting assumption was wrong and, therefore, $G_5 \geq \frac{\sqrt{2}}{2}$ must be valid. From this, we see that the inclusion we want

$$\mathbf{I} \subseteq A \cup B \cup C = \left[\frac{1}{2}, G_5\right]$$

holds, and together with (24), implies that we also have (23), (22) and (21) as valid.

- Let $pq = xy$; $\lambda \in [\frac{1}{2}, +\infty)$.

This case is much simpler. According to (12), the following orderings hold for all λ:

$$a_1 > a_3 \geq a_4 > a_2\,; \quad b_1 > b_2 \geq b_3 > b_4\,.$$

Again, looking for the necessary conditions **i**, **ii** and **iii** of Definition 2.1, we may use the known relations from the previous two cases, and having $pq = xy$, simplify them:

$a_1 \geq b_1$ holds for $\lambda \leq 1$;

$a_1 + a_3 \geq b_1 + b_2$ holds for all λ;

$a_1 + a_3 + a_4 \geq b_1 + b_2 + b_3$ holds for $\lambda \leq 1$;

$a_1 + a_3 + a_4 + a_2 = b_1 + b_2 + b_3 + b_4$ holds for all λ.

Therefore, the majorization $(a_1, a_3, a_4, a_2) \succ (b_1, b_2, b_3, b_4)$ is achieved when $\lambda \in [\frac{1}{2}, 1]$. Since the function $e^r\,(r \in \mathbb{R})$ is strictly convex, the conditions under which it is possible Theorem 2.1 to be applied in (15) are obtained for this interval.

From the last and the previous two cases, we conclude that ${}^{\lambda}\phi|_{X_\omega}(\omega) \geq 0$ is valid for all $\lambda \in \mathbf{I}$. This conclusion together with Lemma 1 give us also the validity of (11), which was to be proved. ■

Since in [13], the proof of Lemma 1 is short, we include it for the sake of completeness.

- *Proof of Lemma 1.* [13] Let $\omega = (x, y, p, q) \in X_r \backslash X_\omega$.

Thus, either ordering $\mathbf{1}'$ or ordering $\mathbf{2}'$ appears as valid. It is easy to verify that the function ${}^{\lambda}\phi$ defined by (8), can be represented in the following form:

$$
\begin{aligned}
{}^{\lambda}\phi(\omega) = (x^{\lambda-1} - y^{\lambda-1})(y^{\lambda} - x^{\lambda}) + (p^{\lambda-1} - q^{\lambda-1})(q^{\lambda} - p^{\lambda}) + \\
+ \tfrac{1}{xypq}[xp(q-y) + yq(p-x)](p^{\lambda}q^{\lambda} - x^{\lambda}y^{\lambda})\,.
\end{aligned}
$$

Obviously, when $\lambda \in [0, 1]$, all differences on the right side in the above equality are nonnegative, which proves the lemma. ■

After all, we are in position to answer the question from which we started: There is no value of λ within the interval $\mathbf{I}$, such that ${}^{\lambda}\phi$ would be negative at some point $(x, y, p, q) \in (\mathbb{R}^+)^4$. Since ${}^{\lambda}f$ is given by (9), we now establish the following theorem:

Theorem 19. *For all graphs G and $\lambda \in (\frac{1}{2}, \frac{\sqrt{2}}{2}]$, Inequality* (1) *holds.*

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