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Ranking Objects Using Fuzzy Orders, with an Application to Refrigerants

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Abstract

Ranking objects means pairwise comparing their features, mostly using certain parameters, for example real–valued functions, the values of which allow to put the objects in a certain order. However, due to the ordinal character of ranking not all slight numerical differences in the values of features should be considered as relevant. The possible appearance of data noise may support the idea to robustify the order relations. For this reason, elements of fuzzy order theory are used in the present paper that allow stepwise to relax the requirements on the influence of numerical differences of feature values. The methods described are applied to the ranking of refrigerants based upon their ozone depletion potential, global warming potential and atmospheric lifetime. It is found that trifluoroiodomethane, dimethyl ether and ammonia are the least environmentally problematic substances for a wide range of relaxation in their environmental properties. Pentafluorodimethyl ether, mooted as replacement of problematic refrigerants, turns out to be a problematic substance for different levels of data relaxation, which raises questions regarding its environmental impact.

Introduction

Ranking a set X of objects x according to their features is mostly based on a set of parameter values, usually real numbers. In chemical and environmental rankings, objects usually are chemical compounds or polluted sites, whose features are toxicity and concentration levels, to name but a few (Bruggemann and Münzer, 1993, [1]). For example,

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a refrigerant x can be evaluated according to its *ozone depletion potential* (ODP), its *global warming potential* (GWP) and its *atmospheric lifetime* (ALT). In this case we can associate to x the triple

$$(ODP(x), GWP(x), ALT(x))$$

of real numbers, and these triples define a canonical order on the set X of refrigerants if these numbers are available for every x. The following items explain the procedure.

— To begin with, we note that we cannot distinguish the quality of two refrigerants that have the same parameter values. Therefore, refrigerants with the same values are considered *equivalent*. This means that instead of x we in fact consider its *equivalence class* $\mathbf{x}$, consisting, possibly, of several objects, all with the same parameter values. Note that we can work with any of the elements in an equivalence class, for example with the element x of $\mathbf{x}$. The set of equivalence classes will be indicated as

$$\mathbf{X} = \{\mathbf{x} \mid x \in X\}.$$

— In order to prepare the interpretation of parameter values as fuzzy subsets, we should like to have the parameter values contained in the interval $[0, 1]$, consisting of the real numbers between 0 and 1. Moreover smaller parameter values should be interpreted as the better ones. For this reason we use the following transformation of data, if necessary.

Let q be one of the parameters. We distinguish two cases: Either the smaller values are the better ones, in which case we leave q as it is and say that q is *properly oriented*. In the other case we need a *reorientation* of the parameter, multiplying by -1. Moreover, we may need a *shift*, so that the values of the shifted parameter are nonnegative. Finally, dividing by the maximal value of the possibly reoriented and possibly shifted parameter, we arrange that the new values are contained in the interval $[0, 1]$. The following transformations, including in one step reorientation, shift and normalization, guarantees that the new parameter q' has its values in $[0, 1]$ and is properly oriented.

– In the first case, when q was properly oriented already, we put

$$q'(x) = \frac{q(x) - \min(q)}{\max(q) - \min(q)},$$

where $\min(q) = \min\{q(y) \mid y \in X\}$ and $\max(q) = \max\{q(y) \mid y \in X\}$, the minimal and the maximal values of q that occur.

– In the second case, when q is not yet properly oriented, we use the transformation

$$q'(x) = \frac{\max(q) - q(x)}{\max(q) - \min(q)}.$$

We call this transformation the *normalization* of data.

— After the normalization of data, we consider a refrigerant x as *at least as good* as y if the following is true:

$$ODP(x) \leq ODP(y) \text{ and } GWP(x) \leq GWP(y) \text{ and } ALT(x) \leq ALT(y).$$

We indicate this by writing $x \leq y$, and we note that in this case also $x' \leq y'$, for any other representatives of $x' \in \mathbf{x}$ and $y' \in \mathbf{y}$. Hence, the relation $\leq$ on X yields the relation

$$\{(\mathbf{x}, \mathbf{y}) \mid x \leq y\}$$

on the set $\mathbf{X}$, consisting of the pairs $(\mathbf{x}, \mathbf{y})$ with $x' \leq y'$, for every $x' \in \mathbf{x}$ and $y' \in \mathbf{y}$. It is obvious that this *relation* $\leq$ has the following properties:

– For each class $\mathbf{x}$ we have that $\mathbf{x} \leq \mathbf{x}$, i.e. the relation is *reflexive*.

– Moreover, if $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \leq \mathbf{z}$ hold, then also $\mathbf{x} \leq \mathbf{z}$ is true, the relation is *transitive*.

– Finally, if $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \leq \mathbf{x}$, then $\mathbf{x} = \mathbf{y}$, therefore the relation is also *antisymmetric*.

These three properties mean that the relation $\leq$ on $\mathbf{X}$ is a *partial order*, we call it the *canonical partial order* (CPO), obtaining the partially ordered set (poset, for short) $(\mathbf{X}, \leq)$, with its partial order

$$CPO = \{(\mathbf{x}, \mathbf{y}) \mid x \leq y\}.$$

This partial order is basic for the following argumentation and most sensitive with respect to numerical differences and hence eventually to data noise. We should carefully note that $\leq$ is not always a partial order on X, because it is not necessarily antisymmetric. For this reason we introduced equivalence classes.

Objects $x, y \in X$, for which neither $x \leq y$ nor $y \leq x$ hold, are said to be *incomparable*, denoted by $x \| y$, the same with classes $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, for which neither $\mathbf{x} \leq \mathbf{y}$ nor $\mathbf{y} \leq \mathbf{x}$ is true.

— It is obvious that the aforementioned concepts can easily be generalized to the ranking of general sets X of objects, where we have, say, a set of n real–valued parameters $p_1, \ldots, p_n$, that we can assume to be normalized. The objects form equivalence classes $\mathbf{x}$, and to the object x as well as to the other elements in that class, there corresponds the n–tuple

$$p(x) = (p_1(x), \ldots, p_n(x))$$

of real numbers $p_i(x)$, the parameter or *feature values*, and we assume that these tuples of parameter values are at hand. They are collected in the table

$$IB = \begin{array}{c|ccccc} & 1 & \ldots & i & \ldots & n \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{x} & p_1(x) & \ldots & p_i(x) & \ldots & p_n(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

that forms our *information basis* IB and *which is all what we are going to use in our evaluation.*

— The rows of the information basis give the canonical partial order CPO of equivalence classes, defined by

$$\mathbf{x} \leq \mathbf{y} \iff p_1(x) \leq p_1(y) \text{ and } p_2(x) \leq p_2(y) \text{ and } \ldots \text{ and } p_n(x) \leq p_n(y),$$

with $x \in \mathbf{x}, y \in \mathbf{y}$. But we ought to make clear that this partial order has serious drawbacks. It is quite sensitive concerning numerical differences in data:

- It is quite *sensitive concerning numerical differences in data*: Slight deviations of a value $p_i(x)$ may change the equivalence class, and therefore the order relation, too. We have to keep this lack of robustness under control!
- The canonical partial order is *not necessarily a total order*, as objects may occur that are incomparable as far as their features are concerned. But if decisions have to be made according to an evaluation, people prefer a total (or linear) order in which every object can be compared with every other one, as even the existence of partial orders is not very well known or at least far from being popular.

Therefore, in a most primitive — albeit very popular — approach we may be tempted to use a *weighted combination* of parameters, i.e. instead of the sequence $p(x) = (p_1(x), p_2(x), \ldots, p_n(x))$ consider a 'suitable' linear combination

$$P(x) = w_1 \cdot p_1(x) + w_2 \cdot p_2(x) + \ldots + w_n \cdot p_n(x),$$

using reasonable *weights* w_i, and claiming that $\mathbf{x}$ is at least as good as $\mathbf{y}$ if $P(x) \leq P(y)$. Since the *n–tuple* $p(x)$, consisting of n real numbers, is replaced in this way by a *single* real number $P(x)$, there are no longer incomparabilities, and so a linear or total order of sets of equivalence classes is obtained, where there is only one set of best classes of refrigerants, comprising the classes with smallest parameter value P (or a set of best refrigerants, the elements x of that best classes). Correspondingly, if the best classes do not contain each refrigerant, there is also a set of second best classes, and so on. But the approach of a weighted sum of features is obviously a manipulation which might be dangerous, a prejudice (Munda, 2008, [11]). The main difficulty is the selection of the weights w_i which is often highly subjective (for methods to obtain weights directly from data see OECD, 2008, [13]).

An approach that allows to take into account the sensitivity of partial order relations concerning numerical differences in data, without performing questionable feature combinations is to use the concept of *fuzzy relations, fuzzy preorders and fuzzy partial orders*. Fuzzy relations contain the given canonical order and further relations that form a sequence of orders, starting from the CPO and becoming coarser and coarser — since equivalence classes are collected successively into bigger classes — until finally all objects are equivalent and have trivially the same rank. The advantage is that this method is based on concepts from fuzzy order theory in the following manner.

Fuzzy relations, preorders and partial orders

The equivalence classes $\mathbf{x}$ of the elements in $X = \{x, y, z, \ldots\}$ are to be ranked according to their feature values, expressed as values of certain parameters $p_1, \ldots, p_n$, real numbers, collected in the information basis IB, shifted (so that they are nonnegative), normalized (so that they are contained in the interval $[0, 1]$) and reoriented (so that they are 'the smaller, the better'), if necessary.

In order to keep the sensitivity of the partial order relations with respect to slight numerical differences of the data under control, we are going to interpret the measurements as fuzzy subsets and introduce helpful concepts from fuzzy order theory in the following way:

Measurements as fuzzy subsets

We replace the equivalence classes $\mathbf{x}$ by *fuzzy subsets*[4] $\widetilde{\mathbf{x}}$ of $\{1, \ldots, n\}$, expressed in terms of the parameter values,

$$\widetilde{\mathbf{x}}\colon \{1, \ldots, n\} \to [0, 1]\colon i \mapsto p_i(x).$$

Less formally, the degree of the number i being an element of the fuzzy set $\widetilde{\mathbf{x}}$ is the feature value $p_i(x)$ of x. These fuzzy subsets of the set $\{1, \ldots, n\}$ form the rows of the information basis IB.

Now, we take from the theory of fuzzy sets that $\widetilde{\mathbf{x}}$ is *contained* in $\widetilde{\mathbf{y}}$ if and only if the degree of i, as a member of $\widetilde{\mathbf{x}}$, is less than or equal to the degree of i, being a member of $\widetilde{\mathbf{y}}$. In formal terms, and using the symbol $\subseteq$ also in the fuzzy case:

$$\widetilde{\mathbf{x}} \subseteq \widetilde{\mathbf{y}} \iff \text{for each } i \text{ we have that } p_i(x) \leq p_i(y).$$

This fits very well to our interest in evaluation: The objects in the class $\mathbf{x}$ are at least as good as the objects in the class $\mathbf{y}$ if and only if the corresponding fuzzy sets contain each other:

$$\mathbf{x} \leq \mathbf{y} \iff \widetilde{\mathbf{x}} \subseteq \widetilde{\mathbf{y}}.$$

This means that *the concept of fuzzy subset corresponds to the canonical partial order on the set of classes.* But there is much more available in the theory of fuzzy subsets that we can use. We follow the very clear description of these methods given by Naessens, De Baets and De Meyer in [12].

The degree of subsethood

Containment of sets, as well as of fuzzy sets, is a 'binary' concept, *either* a fuzzy set $\widetilde{\mathbf{x}}$ is contained in $\widetilde{\mathbf{y}}$, *or* not. This concept can easily be softened (or 'fuzzyfied') by introducing a *degree of subsethood*, which can be found in the standard literature on fuzzy sets, e.g. in

[4] A standard fuzzy subset of a set M is supposed to be a mapping from M to the interval $[0, 1]$

the book by Klir and Yuan (1995, [9]). It was used by Van de Walle et al. (1995, [17]) and De Baets and De Meyer (2003, [4]). The *degree* $SH(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ *of subsethood* of $\tilde{\mathbf{x}}$ in $\tilde{\mathbf{y}}$ can be defined[5] in terms of fuzzy cardinalities[6] by

$$\begin{aligned} SH(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) &= \begin{cases} \frac{|\tilde{\mathbf{x}} \cap \tilde{\mathbf{y}}|}{|\tilde{\mathbf{x}}|} & \text{if } |\tilde{\mathbf{x}}| \neq 0 \\ 1 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \frac{\sum_i \min\{p_i(x), p_i(y)\}}{\sum_i p_i(x)} & \text{if } \sum_i p_i(x) \neq 0 \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

(This construction of the fuzzy subsethood was motivated by its analogy with conditional probability (Kosko, 1991, [10]).)

A fuzzy preorder

We consider SH as a function on the cartesian square $\widetilde{\mathbf{X}}^2 = \{(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \mid \mathbf{x}, \mathbf{y} \in \mathbf{X}\}$ of the set of fuzzy subsets, namely as the mapping

$$SH\colon \widetilde{\mathbf{X}}^2 \to [0,1]\colon (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \mapsto \frac{|\tilde{\mathbf{x}} \cap \tilde{\mathbf{y}}|}{|\tilde{\mathbf{x}}|}\,.$$

Thus, SH can be understood as a fuzzy subset of $\widetilde{\mathbf{X}}^2$. $SH(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is the degree of subsethood of $\tilde{\mathbf{x}}$ in $\tilde{\mathbf{y}}$. However, the main point is that *this value is the degree, by which* $\mathbf{x}$ *is better than* $\mathbf{y}$, and this is what we really want, *a degree and not a crisp either/or.*

We may also interpret SH as a fuzzy (binary) *relation* on $\widetilde{\mathbf{X}}$ with the following properties:

- The pair $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ belongs to the relation SH with the degree $SH(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$.
- It is *reflexive*, since $SH(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}) = 1$, for all $\tilde{\mathbf{x}} \in \widetilde{\mathbf{X}}$.
- But it is not necessarily *transitive*, since this needs that a generalized transitivity condition suitable for fuzzy order is to be fulfilled,

$$\min\{SH(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}), SH(\tilde{\mathbf{y}}, \tilde{\mathbf{z}}) \mid \tilde{\mathbf{y}} \in \widetilde{\mathbf{X}}\} \leq SH(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}).$$

For this reason, we consider the *transitive closure* SHT of SH that fulfills this condition (see Klir and Yuan, 1995, [9], for applications see Haven, 1998, [7], also De Baets and De Meyer, 2003, [4]). It is defined as follows:

$$SHT(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) = \max\{\min\{SH(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}), SH(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})\} \mid \tilde{\mathbf{y}} \in \widetilde{\mathbf{X}}\}.$$

[5]There are various other ways of defining subsethood, an interesting axiomatic approach is described, for example, in [8].

[6]The fuzzy cardinality $|\tilde{\mathbf{x}}|$ of the fuzzy set $\tilde{\mathbf{x}}$ is defined to be the sum of the values of that mapping: $|\tilde{\mathbf{x}}| = \sum_i p_i(x)$ while the (standard) fuzzy intersection $\tilde{\mathbf{x}} \cap \tilde{\mathbf{y}}$ is the fuzzy set with values $\min\{p_i(x), p_i(y)\}$ and, correspondingly, it is of order $|\tilde{\mathbf{x}} \cap \tilde{\mathbf{y}}| = \sum_i \min\{p_i(x), p_i(y)\}$.

Relations that are reflexive and transitive are called *preorders*, and so we have obtained a *fuzzy preorder* SHT on $\widetilde{\mathbf{X}}$, the transitive closure of the subsethood relation SH.

The cutsets of SHT

Having constructed the transitive closure SHT we arrive at the crucial point of our discussion: We know that each fuzzy subset can be reconstructed from its *cutsets*, for example SHT from its cutsets

$$SHT_{\geq\alpha} = \{(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in \widetilde{\mathbf{X}}^2 \mid SHT(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \geq \alpha\}.$$

These cutsets are crisp subsets of $\widetilde{\mathbf{X}}^2$, i.e. subsets in the classical sense, and, moreover, they form a chain of subsets, since obviously

$$\alpha \leq \beta \Longrightarrow SHT_{\geq\alpha} \supseteq SHT_{\geq\beta}$$

is true. For example: $SHT_{\geq 0} = \widetilde{\mathbf{X}}^2$. Less formally, two pairs $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ and $(\tilde{\mathbf{x}}', \tilde{\mathbf{y}}')$ that are contained in $SHT_{\geq\beta}$, for a $\beta \geq \alpha$, are also contained in $SHT_{\geq\alpha}$.

It is easily seen that these cutsets are also preorders, and that each of them defines an equivalence relation $SHTE_{\geq\alpha}$, obtained by collecting pairs in order to obtain a *symmetric* relation:

$$(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in SHTE_{\geq\alpha} \iff (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in SHT_{\geq\alpha} \text{ and } (\tilde{\mathbf{y}}, \tilde{\mathbf{x}}) \in SHT_{\geq\alpha}.$$

These equivalence relations form a chain, since also

$$\alpha \leq \beta \Longrightarrow SHTE_{\geq\alpha} \supseteq SHTE_{\geq\beta}$$

is true. In words, $\alpha \leq \beta$ implies that $SHE_{\geq\beta}$ *refines* $SHE_{\geq\alpha}$, classes $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ that are equivalent in $SHE_{\geq\beta}$ are also equivalent with respect to $SHE_{\geq\alpha}$. For example: $SHTE_{\geq 0} = \{\widetilde{\mathbf{X}}^2\}$, the equivalence relation consisting of a single class, since all the fuzzy subsets $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are pairwise equivalent.

The α-cuts are important since each one of them permits to assess the order relationships between the objects to rank at certain levels of data relaxation linked to α (Haven, 1998, [7]). Thus, the higher the α-value, the lower the data relaxation. De Baets used the following helpful formulation for this interesting fact:

> 'Incomparability disappears at the cost of increasing indifference'

The maximum α-value yields the canonical partial order, whereas the minimum α-value leads to a data relaxation such that all objects become equivalent.

The partial orders corresponding to the cutsets

The final step is the evaluation of the partial orders obtained from the cutsets.

The equivalence relations form a chain under refinement, and each of these equivalence relations $SHTE_{\geq\alpha}$ yields a partial order $\leq_\alpha$, defined as follows: Denoting the equivalence class of $\tilde{\mathbf{x}}$ in $SHTE_{\geq\alpha}$ by $[\tilde{\mathbf{x}}]_\alpha$ we have that

$$[\tilde{\mathbf{x}}]_\alpha \leq_\alpha [\tilde{\mathbf{y}}]_\alpha \iff (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in SHT_{\geq\alpha} \text{ while } (\tilde{\mathbf{y}}, \tilde{\mathbf{x}}) \notin SHT_{\geq\alpha}.$$

For example, as $SHT_{\geq 0}$ consists of a single class, its Hasse diagram consists of a single vertex.

As a relevant environmental example of application of this procedure, let us consider the data matrix of 18 refrigerants.

An application: Ranking of refrigerants

In a recent study 40 refrigerants were ranked with respect to their ozone depletion potential, global warming potential, and atmospheric lifetime (Restrepo et al., 2008, [15, 16]). In those publications the focus was on the application of the concept of stability fields (stability fields are of interest when composite indicators are to be discussed, see Bruggemann et al., 2008, [3]).

Here, for demonstration, 18 most hazardous refrigerants (Table 1) were studied according to the mentioned properties. For sake of simplicity of notation (and the size of the tables) we replace the objects $x_i \in X$, their equialence classes $\mathbf{x}_i \in \mathbf{X}$ as well as the corresponding fuzzy subsets $\tilde{\mathbf{x}}_i \in \widetilde{\mathbf{X}}$ by the labels i that they have in a set of altogether 40 refrigerants considered in [15, 16]. This is possible since the equivalence classes are singletons, $\mathbf{x}_i = \{x_i\}$.

Table 1. Labels, molecular formulas and non–proprietory names of the refrigerants studied.

Label	Molecular formula	Non–proprietory name
1	CCl_3F	R11
2	CCl_2F_2	R12
6	$C_2H_3Cl_2F$	R141b
7	$C_2H_3ClF_2$	R142b
8	CHF_3	R23
16	C_3H_8	R290
21	CO_2	R744
22	$CBrClF_2$	R12B1
23	C_4F_8	RC318
29	C_2HF_5O	HFE–125
32	CH_3Cl	R40
33	$C_2Cl_3F_3$	R113
35	$C_2Cl_2F_4$	R114
36	CF_3I	R13I1
37	C_2H_6O	—
38	NH_3	R717
39	$C_2H_3F_3O$	HFE–143
40	$C_3H_3F_5O$	HFE–245

The labels correspond to those used in [16]. The following table is our information basis, containing the results of measurements:

Table 2. Parameter values for the chosen refrigerants, where ODP, GWP and ALT stand for ozone depletion potential, global warming potential and atmospheric lifetime, respectively.

$IB =$

Label	ODP	GWP	ALT
1	1	4680	45
2	0.82	10720	100
6	0.12	713	9.3
7	0.065	2270	17.9
8	0.0004	14310	270
16	0	20	0.041
21	0	1	120
22	5.1	1300	11
23	0	10000	3200
29	0	14800	165
32	0.02	16	1.3
33	0.9	6000	85
35	0.85	9800	300
36	0	1	0.1
37	0	1	0.015
38	0	0	0.25
39	0	656	5.7
40	0	697	4

The normalized properties of these 18 refrigerants are shown in Table 3.

Table 3. Normalized property values of the 18 refrigerants studied, the *data matrix*.

$S =$

Label	ODP	GWP	ALT
1	0.19608	0.31622	0.01406
2	0.16078	0.72432	0.03125
6	0.02353	0.04818	0.00290
7	0.01275	0.15338	0.00559
8	0.00008	0.96689	0.08437
16	0	0.00135	0.00001
21	0	0.00007	0.03750
22	1	0.08784	0.00343
23	0	0.67568	1
29	0	1	0.05156
32	0.00392	0.00108	0.00040
33	0.17647	0.40541	0.02656
35	0.16667	0.66216	0.09375
36	0	0.00007	0.00003
37	0	0.00007	0
38	0	0	0.00007
39	0	0.04432	0.00178
40	0	0.04709	0.00125

Here come the tables of the subsethood degrees (Table 4) and that of its transitive closure, SHT (Table 5).

Table 4. The matrix of subsethood degrees $SH(i, j)$.

	1	2	6	7	8	16	21	22	23	29	32	33	35	36	37	38	39	40
1	1.0	0.933	0.142	0.326	0.628	0.003	0.027	0.546	0.627	0.627	0.01	0.963	0.944	0.0	0.0	0.0	0.088	0.092
2	0.536	1.0	0.081	0.187	0.825	0.001	0.034	0.275	0.771	0.825	0.006	0.647	0.932	0.0	0.0	0.0	0.05	0.053
6	1.0	1.0	1.0	0.856	0.686	0.018	0.04	1.0	0.685	0.685	0.072	1.0	1.0	0.001	0.001	0.001	0.618	0.648
7	1.0	1.0	0.372	1.0	0.926	0.008	0.033	0.606	0.926	0.926	0.031	1.0	1.0	0.001	0.0	0.0	0.268	0.282
8	0.314	0.719	0.049	0.151	1.0	0.001	0.036	0.087	0.723	0.969	0.001	0.411	0.71	0.0	0.0	0.0	0.044	0.046
16	1.0	1.0	1.0	1.0	1.0	1.0	0.059	1.0	1.0	1.0	0.801	1.0	1.0	0.059	0.051	0.007	1.0	1.0
21	0.376	0.834	0.079	0.151	1.0	0.002	1.0	0.093	1.0	1.0	0.013	0.709	1.0	0.003	0.002	0.002	0.049	0.035
22	0.263	0.231	0.068	0.095	0.084	0.001	0.003	1.0	0.084	0.084	0.005	0.245	0.236	0.0	0.0	0.0	0.042	0.044
23	0.197	0.422	0.03	0.095	0.454	0.001	0.022	0.054	1.0	0.434	0.001	0.258	0.451	0.0	0.0	0.0	0.028	0.029
29	0.314	0.719	0.049	0.151	0.969	0.001	0.036	0.087	0.692	1.0	0.001	0.411	0.679	0.0	0.0	0.0	0.044	0.046
32	1.0	1.0	1.0	1.0	0.289	0.202	0.087	1.0	0.274	0.274	1.0	1.0	1.0	0.019	0.013	0.013	0.274	0.274
33	0.833	0.974	0.123	0.282	0.71	0.002	0.044	0.44	0.71	0.71	0.009	1.0	0.984	0.0	0.0	0.0	0.076	0.079
35	0.539	0.926	0.081	0.186	0.809	0.001	0.041	0.28	0.819	0.774	0.006	0.649	1.0	0.0	0.0	0.0	0.05	0.052
36	1.0	1.0	1.0	1.0	1.0	0.8	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.7	0.3	1.0	1.0
37	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0	1.0	1.0
38	1.0	1.0	1.0	1.0	1.0	0.143	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.429	0.0	1.0	1.0	1.0
39	1.0	1.0	1.0	1.0	1.0	0.03	0.04	1.0	1.0	1.0	0.032	1.0	1.0	0.002	0.002	0.002	1.0	0.989
40	1.0	1.0	1.0	1.0	1.0	0.028	0.027	1.0	1.0	1.0	0.031	1.0	1.0	0.002	0.001	0.001	0.943	1.0

Table 5. The matrix of transitive closure SHT of the subsethood relation SH, containing the values $SHT(i, j)$.

	1	2	6	7	8	16	21	22	23	29	32	33	35	36	37	38	39	40
1	1.0	0.963	0.326	0.326	0.825	0.072	0.072	0.546	0.819	0.825	0.072	0.963	0.963	0.059	0.059	0.059	0.326	0.326
2	0.649	1.0	0.326	0.326	0.825	0.072	0.072	0.546	0.819	0.825	0.072	0.649	0.932	0.059	0.059	0.059	0.326	0.326
6	1.0	1.0	1.0	0.856	0.856	0.072	0.072	1.0	0.856	0.856	0.072	1.0	1.0	0.059	0.059	0.059	0.648	0.648
7	1.0	1.0	0.372	1.0	0.926	0.072	0.072	0.606	0.926	0.926	0.072	1.0	1.0	0.059	0.059	0.059	0.372	0.372
8	0.649	0.719	0.326	0.326	1.0	0.072	0.072	0.546	0.723	0.969	0.072	0.649	0.719	0.059	0.059	0.059	0.326	0.326
16	1.0	1.0	1.0	1.0	1.0	1.0	0.087	1.0	1.0	1.0	0.801	1.0	1.0	0.059	0.059	0.059	1.0	1.0
21	0.709	0.926	0.326	0.326	1.0	0.072	1.0	0.546	1.0	1.0	0.072	0.709	1.0	0.059	0.059	0.059	0.326	0.326
22	0.263	0.263	0.263	0.263	0.263	0.072	0.072	1.0	0.263	0.263	0.072	0.263	0.263	0.059	0.059	0.059	0.263	0.263
23	0.454	0.454	0.326	0.326	0.454	0.072	0.072	0.454	1.0	0.454	0.072	0.454	0.454	0.059	0.059	0.059	0.326	0.326
29	0.649	0.719	0.326	0.326	0.969	0.072	0.072	0.546	0.723	1.0	0.072	0.649	0.719	0.059	0.059	0.059	0.326	0.326
32	1.0	1.0	1.0	1.0	0.926	0.202	0.087	1.0	0.926	0.926	1.0	1.0	1.0	0.059	0.059	0.059	0.648	0.648
33	0.833	0.974	0.326	0.326	0.825	0.072	0.072	0.546	0.819	0.825	0.072	1.0	0.984	0.059	0.059	0.059	0.326	0.326
35	0.649	0.926	0.326	0.326	0.825	0.072	0.072	0.546	0.819	0.825	0.072	0.649	1.0	0.059	0.059	0.059	0.326	0.326
36	1.0	1.0	1.0	1.0	1.0	0.8	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.7	0.3	1.0	1.0
37	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.3	1.0	1.0
38	1.0	1.0	1.0	1.0	1.0	0.429	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.429	0.429	1.0	1.0	1.0
39	1.0	1.0	1.0	1.0	1.0	0.072	0.072	1.0	1.0	1.0	0.072	1.0	1.0	0.059	0.059	0.059	1.0	0.989
40	1.0	1.0	1.0	1.0	1.0	0.072	0.072	1.0	1.0	1.0	0.072	1.0	1.0	0.059	0.059	0.059	0.943	1.0

The following 33 α-cuts resulted: 0.059; 0.072; 0.087; 0.202; 0.263; 0.3; 0.326; 0.372; 0.429; 0.454; 0.546; 0.606; 0.648; 0.649; 0.7; 0.709; 0.719; 0.723; 0.8; 0.801; 0.819; 0.825; 0.833; 0.856; 0.926; 0.932; 0.943; 0.963; 0.969; 0.974; 0.984; 0.989; 1.

Each α-cut induces an equivalence relation and gives a partial order as described above. In order to assess the effect of data relaxation, we selected eleven values $\alpha \in [0, 1]$, namely $1, 0.9, 0.8, \ldots, 0.1$ and 0. The cutsets $SHT_{\geq\alpha}$ can be obtained from the following matrix (Table 6), obtained from the matrix with its entries $SHT(i, j)$ given above by restricting attention to one decimal only:

Table 6. The matrix obtained from the above matrix $SHT(i, j)$ by restricting the entries to the first decimal place.

	1	2	6	7	8	16	21	22	23	29	32	33	35	36	37	38	39	40
1	1.0	0.9	0.3	0.3	0.8	0.0	0.0	0.5	0.8	0.8	0.0	0.9	0.9	0.0	0.0	0.0	0.3	0.3
2	0.6	1.0	0.3	0.3	0.8	0.0	0.0	0.5	0.8	0.8	0.0	0.6	0.9	0.0	0.0	0.0	0.3	0.3
6	1.0	1.0	1.0	0.8	0.8	0.0	0.0	1.0	0.8	0.8	0.0	1.0	1.0	0.0	0.0	0.0	0.6	0.6
7	1.0	1.0	0.3	1.0	0.9	0.0	0.0	0.6	0.9	0.9	0.0	1.0	1.0	0.0	0.0	0.0	0.3	0.3
8	0.6	0.7	0.3	0.3	1.0	0.0	0.0	0.5	0.7	0.9	0.0	0.6	0.7	0.0	0.0	0.0	0.3	0.3
16	1.0	1.0	1.0	1.0	1.0	1.0	0.0	1.0	1.0	1.0	0.8	1.0	1.0	0.0	0.0	0.0	1.0	1.0
21	0.7	0.9	0.3	0.3	1.0	0.0	1.0	0.5	1.0	1.0	0.0	0.7	1.0	0.0	0.0	0.0	0.3	0.3
22	0.2	0.2	0.2	0.2	0.2	0.0	0.0	1.0	0.2	0.2	0.0	0.2	0.2	0.0	0.0	0.0	0.2	0.2
23	0.4	0.4	0.3	0.3	0.4	0.0	0.0	0.4	1.0	0.4	0.0	0.4	0.4	0.0	0.0	0.0	0.3	0.3
29	0.6	0.7	0.3	0.3	0.9	0.0	0.0	0.5	0.7	1.0	0.0	0.6	0.7	0.0	0.0	0.0	0.3	0.3
32	1.0	1.0	1.0	1.0	0.9	0.2	0.0	1.0	0.9	0.9	1.0	1.0	1.0	0.0	0.0	0.0	0.6	0.6
33	0.8	0.9	0.3	0.3	0.8	0.0	0.0	0.5	0.8	0.8	0.0	1.0	0.9	0.0	0.0	0.0	0.3	0.3
35	0.6	0.9	0.3	0.3	0.8	0.0	0.0	0.5	0.8	0.8	0.0	0.6	1.0	0.0	0.0	0.0	0.3	0.3
36	1.0	1.0	1.0	1.0	1.0	0.8	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.7	0.3	1.0	1.0
37	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.3	1.0	1.0
38	1.0	1.0	1.0	1.0	1.0	0.4	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.4	0.4	1.0	1.0	1.0
39	1.0	1.0	1.0	1.0	1.0	0.0	0.0	1.0	1.0	1.0	0.0	1.0	1.0	0.0	0.0	0.0	1.0	0.9
40	1.0	1.0	1.0	1.0	1.0	0.0	0.0	1.0	1.0	1.0	0.0	1.0	1.0	0.0	0.0	0.0	0.9	1.0

Their corresponding equivalence relations and partial orders will be given. To begin with, we show the equivalence relations, they are contained in the next table:

Table 7. The (tree of) equivalence relations $SHTE_{\geq\alpha}$ turns out to be the following.

$\alpha = 1$	{36}	{37}	{38}	{21}	{16}	{32}	{22}	{7}	{6}	{39}	{40}	{23}	{1}	{33}	{8}	{29}	{2}	{35}
$\alpha = .9$	{36}	{37}	{38}	{21}	{16}	{32}	{22}	{7}	{6}	{39	40}	{23}	{1}	{33}	{8	29}	{2	35}
$\alpha = .8$	{36}	{37}	{38}	{21}	{16}	{32}	{22}	{7}	{6}	{39	40}	{23}	{1	33}	{8	29}	{2	35}
$\alpha = .7$	{36	37}	{38}	{21}	{16}	{32}	{22}	{7}	{6}	{39	40}	{23}	{1	33}	{8	29	2	35}
$\alpha = .6$	{36	37}	{38}	{21}	{16}	{32}	{22}	{7}	{6	39	40}	{23}	{1	33	8	29	2	35}
$\alpha = .5$	{36	37}	{38}	{21}	{16}	{32}	{22}	{7}	{6	39	40}	{23}	{1	33	8	29	2	35}
$\alpha = .4$	{36	37}	{38}	{21}	{16}	{32}	{22}	{7}	{6	39	40}	{23	1	33	8	29	2	35}
$\alpha = .3$	{36	37	38}	{21}	{16}	{32}	{22}	{7	6	39	40	23	1	33	8	29	2	35}
$\alpha = .2$	{36	37	38}	{21}	{16}	{32}	{22	7	6	39	40	23	1	33	8	29	2	35}
$\alpha = .1$	{36	37	38}	{21}	{16}	{32}	{22	7	6	39	40	23	1	33	8	29	2	35}
$\alpha = .0$	{36	37	38	21	16	32	22	7	6	39	40	23	1	33	8	29	2	35}

It is worth mentioning the transitions from $\alpha = 0.5$ to $\alpha = 0.6$ as well as from $\alpha = 0.1$ to $\alpha = 0.2$, where the number and sort of equivalence classes remain. Whereas the transition $\alpha = 0.1$ to $\alpha = 0.2$ does not change anything, the transition from $\alpha = 0.5$ to $\alpha = 0.6$ keeps the equivalence classes unchanged, but changes the order relations: Specifically the comparabilities of the refrigerant no 22, i.e. of $CBrClF_2$ (refrigerant R12B1) are changed.

In order to prepare the corresponding partial orders, we have to check Table 6. For example, the entry $(1,36)$ is 0, i.e. $(1,36) \notin SHT_{\geq\alpha}$, for $\alpha > 0$, while the entry $(36,1)$ is 1, i.e. $(1,36) \in SHT_{\geq\alpha}$, for each α. This proves

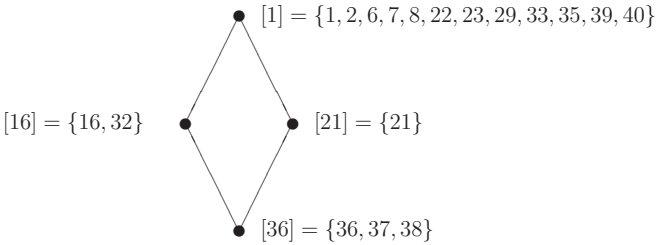
$$[36]_\alpha <_\alpha [1]_\alpha, \text{ for all } \alpha > 0.$$

Here are a few examples:

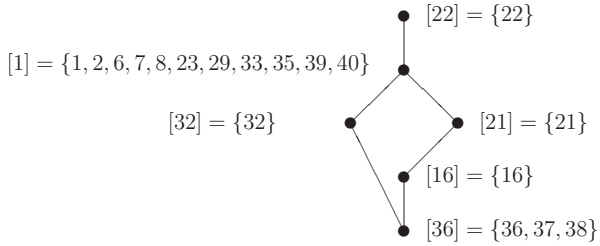
$\alpha = 0$ yields a partial order consisting of a single element, since there is a single equivalence class in $SHTE_{\geq 0}$ only:

- $[1] = \{1, 2, 6, 7, 8, \ldots, 39, 40\}$

$SHTE_{\geq 0.1}$ as well as $SHTE_{\geq 0.2}$ consist of four classes, the Hasse diagram looks in both cases as follows:

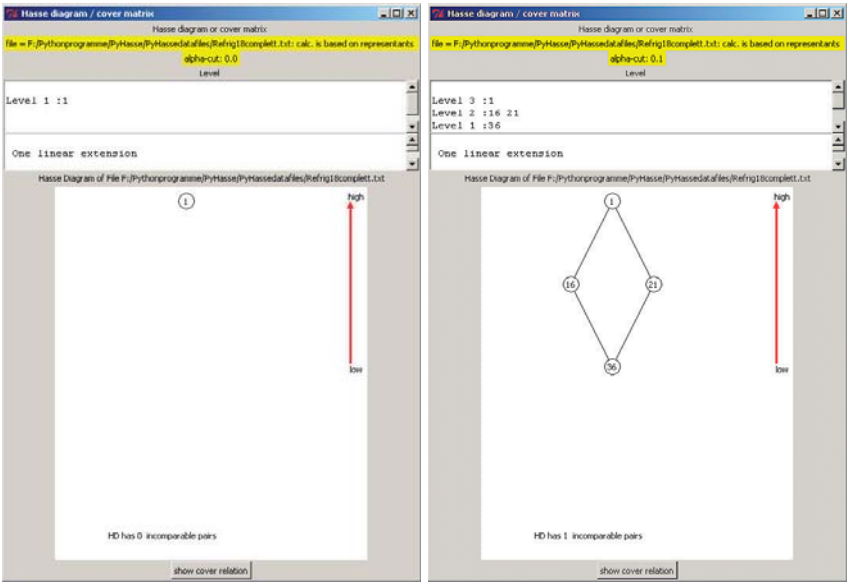


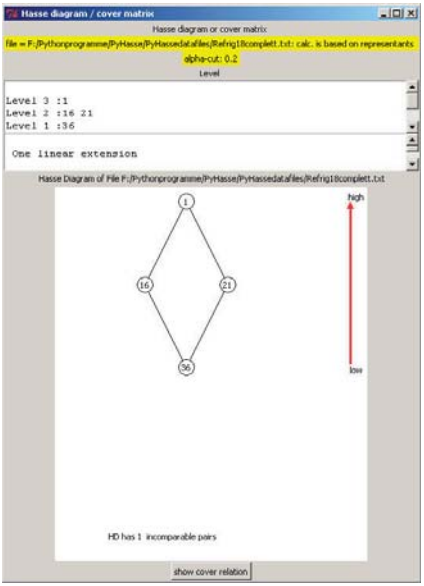
$SHTE_{\geq 0.3}$ yields the following Hasse diagram:

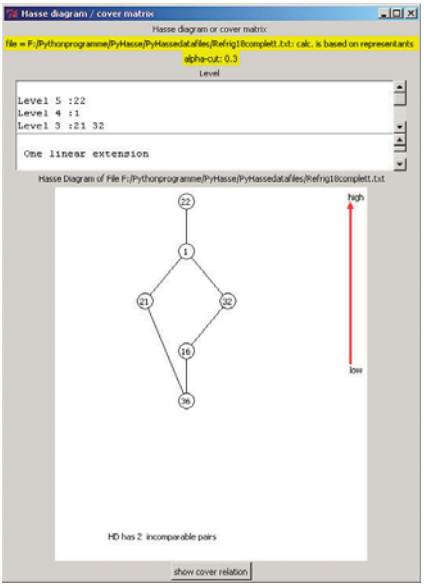


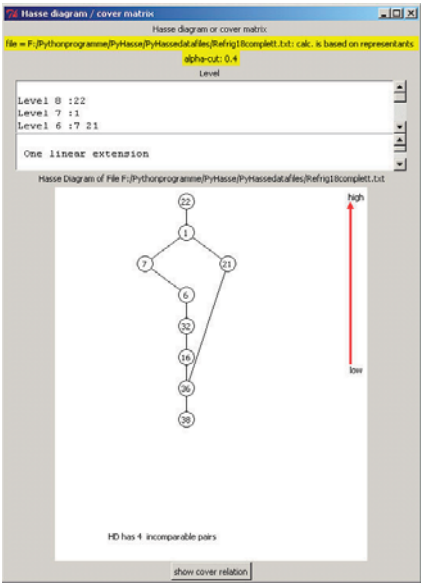
All the needed calculations, starting with the correctly oriented data matrix until the final Table 7, can be performed applying the software package PyHasse. We obtain the various partial orders corresponding to the α-cuts. PyHasse is a software package and freely available on request. It is written in the language Python. Additional information on PyHasse, which is steadily under development, can be found in Voigt et al, 2010, [18] and in Bruggemann and Voigt, 2009, [2].

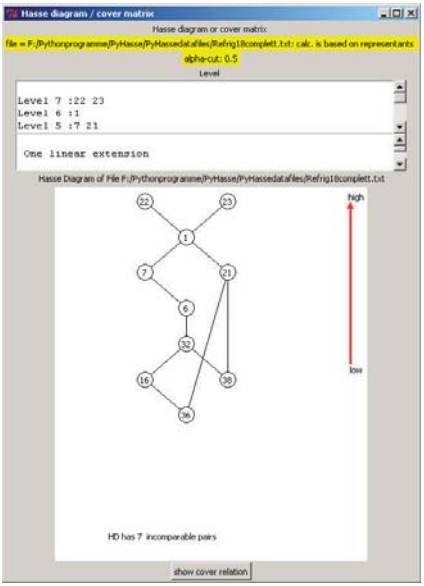
In the following there are screenshots, also obtained by an application of PyHasse. They show the Hasse diagrams of the partial orders. The nodes of the Hasse diagrams are labeled by the numbers of representatives of the classes, i.e. by numbers of the refrigerants that represent the various classes. Relatively good chemicals are at the bottom, relatively bad chemicals are at the top of the diagrams. Hence the minimal and maximal elements of the poset are of special concern.

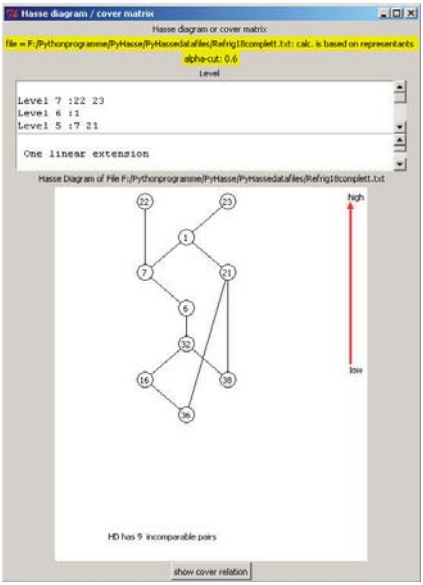


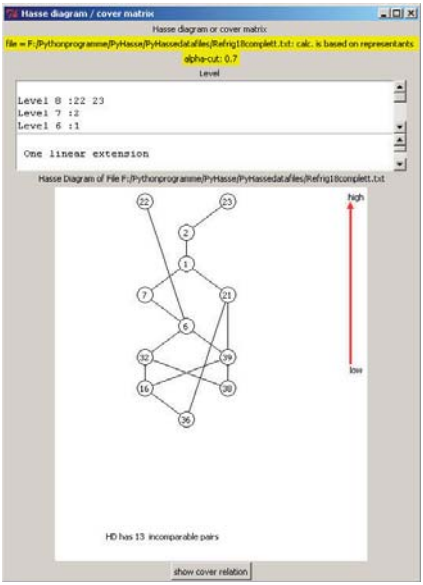

Hasse diagram / cover matrix
Hasse diagram or cover matrix
file = F:/Pythonprogramme/PyHasse/PyHassedataFiles/Refrig18complett.txt: calc. is based on representants
alpha-cut: 0.2
Level
Level 3 :1
Level 2 :16 21
Level 1 :36
One linear extension
Hasse Diagram of File F:/Pythonprogramme/PyHasse/PyHassedataFiles/Refrig18complett.txt
high
low
1
16
21
36
HD has 1 incomparable pairs
show cover relation

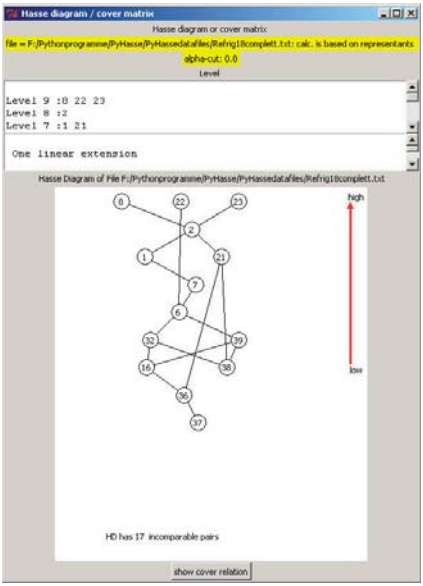

Hasse diagram / cover matrix
Hasse diagram or cover matrix
file = F:/Pythonprogramme/PyHasse/PyHassedataFiles/Refrig18complett.txt: calc. is based on representants
alpha-cut: 0.3
Level
Level 5 :22
Level 4 :1
Level 3 :21 32
One linear extension
Hasse Diagram of File F:/Pythonprogramme/PyHasse/PyHassedataFiles/Refrig18complett.txt
high
low
22
1
21
32
16
36
HD has 2 incomparable pairs
show cover relation

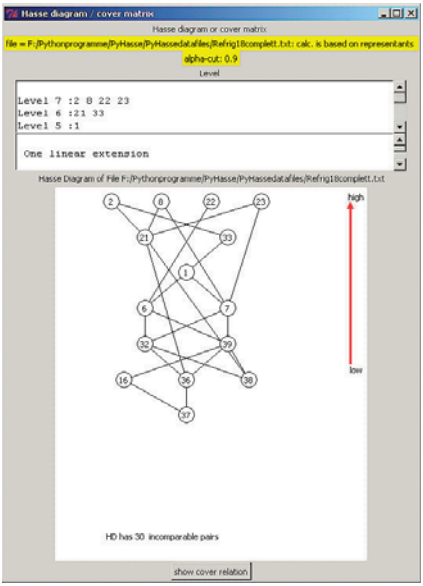

Hasse diagram / cover matrix
Hasse diagram or cover matrix
file = F:/Pythonprogramme/PyHasse/PyHassedataFiles/Refrig18complett.txt: calc. is based on representants
alpha-cut: 0.4
Level
Level 8 :22
Level 7 :1
Level 6 :7 21
One linear extension
Hasse Diagram of File F:/Pythonprogramme/PyHasse/PyHassedataFiles/Refrig18complett.txt
high
low
22
1
7
21
6
32
16
36
38
HD has 4 incomparable pairs
show cover relation


Hasse diagram / cover matrix
Hasse diagram or cover matrix
file = F:/Pythonprogramme/PyHasse/PyHassedataFiles/Refrig18complett.txt: calc. is based on representants
alpha-cut: 0.5
Level
Level 7 :22 23
Level 6 :1
Level 5 :7 21
One linear extension
Hasse Diagram of File F:/Pythonprogramme/PyHasse/PyHassedataFiles/Refrig18complett.txt
high
low
22
23
1
7
21
6
32
16
38
36
HD has 7 incomparable pairs
show cover relation

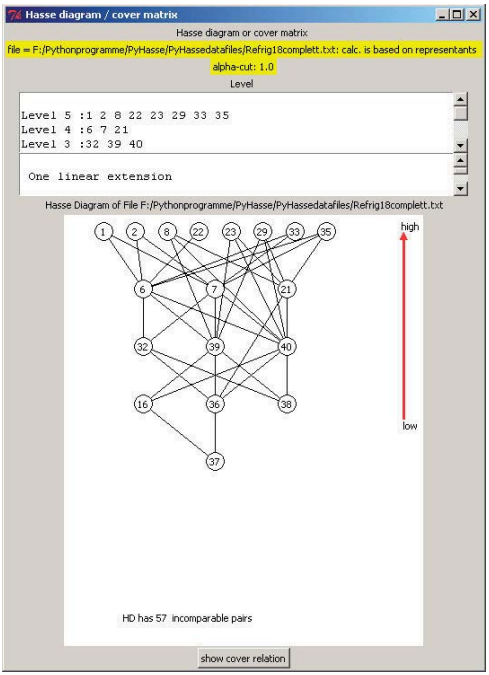








The final screenshot shows the canonical partial order CPO:



Summary

In this manuscript we have ranked refrigerants according to three environmental properties, however the method can easily be extended to more properties, not only to environmental ones, but to chemical, physical, thermodynamic and technical ones. By the partial order methodology together with fuzzy concepts and the software package PyHasse we compared 18 refrigerants and one of the main results is obtained by checking the Hasse diagrams and the tree of equivalence classes (Table 7) that the refrigerants with the labels 36, 37 and 38 are the recommended ones, when the environmental aspects ozone depletion potential, global warming potential and atmospheric lifetime are considered. The three refrigerants are trifluoroiodomethane, dimethylether and ammonia. The chemical pentafluorodimethyl ether which is suggested as a replacement of hazarduous refrigerants is strikingly often a maximal element (i.e. a maximal element for different levels of data relaxation). Therefore this chemical must be considered as a problematic one — at least with respect to environmental hazards. Besides the possibility to perform to a large ex-

tend comparisons among the 18 refrigerants without the need of subjective weightings, we note the following items:

— The generality of the procedure allows also the application to other fields seeking for rankings, for example in the selection of biodiverse ecosystems for governmental protection or in different areas such as the assessment of research institutions, scientists and publications.

— PyHasse provides corresponding software that allows to submit an information basis containing feature values of the objects that should be ranked. The feature values are assumed to be real numbers, parameter values of the objects. We suppose that the information basis is complete, i.e. that the feature values are known for each object, and completely. The output of the special module fuzzyHD12.py of PyHasse which is of relevance here, is a set of partial orders corresponding to the user defined cutsets.

— The approach considered here needs still further research. For example, the following questions arise:

  — How can we find a best cutset? Recently De Loof et al., suggested a method based on a 'linearity index', [5].

  — The normalization can be considered as an application of an operator f on the parameters p_i. The subsethood degree depends on f and hence the final SHT–matrix depends also on f. Even the number of α-cuts may change if f is modified. Furthermore, many other transformations are possible. For instance the normalization used here is sensitive to statistical outliers and is therefore statistically not a robust one. The α-cuts will depend on how the transformation is done and hence outliers may influence the fuzzy analysis. The transformations must be selected in dependence of the context and the distributional properties of each feature.

  — Partial orders have the great advantage that we do not need to find weights for the attributes. Here, however, the subsethood degree sums the minimal values of the p_i which implies some usually unwanted compensations among the attributes. Therefore other approaches are of interest to robustify the

partial order relations. In a paper of Fattore an elegant method is introduced, which is based on membership–functions derived from the linear extensions of the partial order (Fattore, 2008, [6]).

— Another application of fuzzy techniques in environmental sciences can be found in Pudenz et al. (2000, [14]), where fuzzy clustering was used to compare ecosystems. The authors characterize cluster membership by two parameters: Crucial membership and the number of clusters. In the future it will be of high interest to try to relate these ideas with the data relaxation treated in the current manuscript.

Anyway, this is a lucid, reasonable and general procedure for evaluation of objects of any kind according to (real) feature values, and the appropriate software (PyHasse) is easily available.

Acknowledgement

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