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An Algebraic Approach for Solving Time-Dependent Potentials

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Abstract

Recently we introduced a nonlinear partial differential equation (nPDE) of the third order for the first time. This new model equation allows the extension of the Debye-Hückel Theory (DHT) in the sense of considering time dependence explicitly. This leads to a new formulation in the meaning of the known nonlinear Poisson-Boltzmann Equation (nPBE).
The purpose of the present paper follows two intensions: Firstly we show how we can solve the new model equation by an alternative algebraic method without using any approximations and numerical methods. This covers the mathematical standpoint of view. Secondly we show how we can interpret classes of solutions in the sense of describing electrochemical situations. Here the functions of interest will be directly connected with the time-dependent potential. Using a special algebraic approach involving elliptic functions, classes of solutions result naturally especially those of periodic behaviour.

1. Introduction – General remarks

Many problems of physical interest are described by PDEs with appropriate side conditions. These can be suitable chosen initial and/or boundary conditions. If the equation is linear, a widely used method for solving nPDEs is known as the superposition principle if certain convergence requirements are satisfied.
For nonlinear PDEs, however, the linear superposition principle cannot be applied to generate new classes of solutions.

Note: We stress the existence of a nonlinear superposition principle. This is sometimes known as the Bäcklund Transformation which means a special kind of contact transformation [1].
Later we show that the nPDE under consideration is not of Painlevé type and therefore a suitable Bäcklund system cannot be associated. Thus this fact justifies the usage of algebraic methods deriving analytical solutions if one is not interested in numerical approaches.

So, because most of the of solution methods for linear equations cannot be applied to nonlinear equations, there is no general method of finding analytical classes of solutions of nPDEs and numerical techniques are usually required for their solution. Sometimes a transformation of variables can be found that transforms nPDEs into linear PDEs, or some other ad hoc methods (and/or assumptions) can be used to find classes of solutions of a particular nonlinear equation.

Note: We arrange that we suppress the item ‘classes of solutions’, so we will simply understood ‘solutions’ instead of classes of solutions. Since the time occurs in the third-order derivation explicitly we will call such types of nPDEs evolution equations (EVE) since they allow the study of time-dependent procedures in a closed form.
Any nPDE may not have the outer form $u_t = K\left[u, u_x, u_{xx}, ...\right]$ necessarily being an EV where $K\left[u, u_x, u_{xx}, ...\right]$ is a nonlinear operator in general. Equations containing e.g. $u_{xt} = K[u]$ or $u_{xxt} = K[u]$ are sometimes also called EVEs.

Methods of solution for nonlinear equations represent only one aspect of the theory of nPDEs. Like linear equations, questions of existence, uniqueness, and stability of solutions are of fundamental importance.

2. The procedure under consideration

In the following note we stress the basic steps of the algorithm. Like many other ‘special function’ methods the benefit of the present computational algorithm, however, is the simplicity to handle difficulties appearing in dealing with nonlinear EVEs.
The crucial step is the assumption that the unknown solution functions of any nPDE under consideration are also solutions of some ordinary differential equations (ODE) which can be solved explicitly. Here in our approach we use a new formulated nODE of the first order which allows the determination of the solution manifold explicitly [2].
It is known that solutions of nonlinear EVEs can be expressed as finite series in terms of special functions, e.g. hyperbolic functions, Weierstrass and Lambert functions [3-13].
Further attention is given to new algebraic approaches derived by the author, e.g. [14-16].

Note: Consider a given nPDE in its two independent variables x and t

$$P\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}, ..., \frac{\partial^{n-1} u}{\partial x^{n-1}}, \frac{\partial^n u}{\partial x^n}\right) = 0 . \tag{a}$$

Firstly the nPDE is converted into a nonlinear ordinary differential equation (nODE) by using a frame of reference $u(x,t) = f(\xi)$, $\xi = x - \lambda\, t$ and λ is a constant to be determined (the transformation is closely related to group theory).

Thus we have

$$Q\left(f(\xi), f'(\xi), f''(\xi), \ldots\right) = 0 . \tag{b}$$

The nODE (b) is integrated as long as all terms contain derivatives. Further the associated integration constants can taken to be zero in view of the localized solutions one is looking for. This is a necessary (but not sufficient) condition that $f(\xi)$ tends to zero as $\xi \to \pm\infty$. This covers an important fact in the study of solitons.
The next step is that the solution can be expressed in terms of the following series representation by using an auxiliary variable $\omega = \omega(\xi)$ such that

$$f(\xi) = f(\omega(\xi)) = a_0 + \sum_{i=1}^{n} sn^{i-1}\{\omega(\xi)\}\left[a_i sn(\omega(\xi)) + b_i cn(\omega(\xi))\right] . \tag{c}$$

where $sn(.)$ and $cn(.)$ have the meaning of the Jacobian elliptic functions. Next one assumes that the function $\omega(\xi)$ satisfies a nonlinear ODE involving the delta amplitude such that we have

$$\frac{d\omega}{d\xi} = \omega' = b\, dn(\xi), \quad \omega : R \times R \to R \;, \; \omega \neq 0 \,, \; \omega' \neq 0 \,, \; \xi \in R \,, \; \omega \in R . \tag{d}$$

and b means an arbitrary constant. We find it useful to restrict the domain of the delta amplitude to be $-k' < dn\xi < 1$ where k' means the complementary modulus defined by $k' = \sqrt{1-k^2}$.
The parameter n in Eq. (c) is found by balancing the highest derivative with the nonlinear terms in the reduced nODE, Eq. (b). This parameter must be a positive integer since it represents the number of terms in the series (c).

In the case of fractions one can take some suitable transformations as shown below. Substituting (c) and (d) into the relevant nODE yields a system of nonlinear algebraic polynomial equations with respect to $a_0, a_1, \ldots, a$ and λ.
Solutions of the nODE Eq. (d) can be expressed explicitly to give

$$\omega = C_1 + b\, am\, \xi \,, -1 \leq \xi \leq 1 , \tag{e}$$

where $am(.)$ means the Jacobian amplitude which is closely related with the elliptic integral of the first kind by the transformation $u = am^{-1}\varphi = F(\varphi)$ and C_1 means a further arbitrary constant.
The success of all algebraic methods strongly depend upon the solubility of the nonlinear polynomial system and this fact cannot be predicted in general (trivial solutions have to be rejected).

The experience shows that most of these systems are over-determined and ways to influence the solutions are restricted. However increasing the number of unknowns as shown in [17] is a suitable starting point in handling such problems efficiently. Introducing Eq. (e) into the polynomial ansatz Eq. (c), solutions of any nPDE are readily obtained.

We assume some basic notes: In the first instance we look for real-valued solutions assuming that solutions exist and are unique in some domain D so that $D \subseteq R^{n+1}$ open, $D \neq 0$ and $\omega : D \to R$ continuous.
Then we seek for solutions for which $\omega = F(\xi)$, where $F \in R^3$ and $D \subset R^2$ is an open set and further we exclude $D := \left\{(\omega, \xi) \in \widetilde{D} : \omega(\xi) = 0, \omega'(\xi) = 0\right\}$. Suitable solutions are $\omega \in I$, I an interval, so that $I \subseteq D$ and $\omega : I \to R^2$.

Since the r.h.s of the nODE, Eq. (d) is a continuous function we ensure at least the existence locally and due to the lemmas both from Peano and Picard-Lindelöf we assume uniqueness (also at least locally) in a considered domain.

We also note that it may necessary to expand the domain so that we admit complex-valued solutions. For the nODE, Eq. (d) we require therefore:
Let $D(C)$ be a complex domain and $D(C) \subseteq C \times C$ for all holomorphic functions and further let $\xi : C^\infty \times C^\infty \to C^\infty$, such that $\left\{\omega, \omega', \omega'', \ldots, \omega^{(n)}\right\} \neq 0$ where the prime means $d/d\xi$.

Simultaneously as in the real-valued case we further require that the nODE, Eq. (d) has at least one solution and let this solution be unique. The solution develops completely in an interval I for which $\left\{\xi, w(\xi) \middle| \xi \in I\right\} \subset D(C)\ \forall \xi \in I$ holds and so we ensure complex-valued solutions.

3. The model equation

The fundamental equation describing electrical interfaces and/or the potential distribution around the central ion is a combination of the Poisson-Equation (from electrostatic view) and the Boltzmann law of distribution, [18-20].

However the theory remains time-independent explicitly. Solutions of the nPBE have thus found applicability in explaining the physics of a wide variety of phenomena, some of which are: **(i)** the estimation of ionic radii in solutions, [18], **(ii)** the theory of the disjoining pressure due to the overlapping of diffuse double layers and its application to instability phenomena in thin liquid films, [19], **(iii)** a wide variety of electrokinetic phenomena that include streaming potentials, electrophoresis, electroviscous effects, e.g. [20] to mention some examples.

Another point is the fact that the PBE, due to the nonlinearity can only be solved by numerical standard methods, e.g. [21-23]. A practicable approach representing linearization describing successfully chemical quantities (e.g. the activity coefficient) was done in the past [24-27].

The purpose of our recent paper [28] was to introduce time-dependence in the DHT explicitly. Therefore we used the electro quasistatic approach (EQS) for the first time [28]. The crucial step is the fact that the time-dependent electric field may derived from a scalar potential which is a solution of a certain nPDE of the third order [29]

$$\frac{\partial^2 u}{\partial x^2} + \tau \frac{\partial^3 u}{\partial x^2 \partial t} - \mu^2 \frac{\partial u}{\partial t} e^{-\eta u} = 0, \quad u = u(x,t), \quad -\infty < x < \infty. \tag{1}$$

Let us formally impose boundary conditions so that $\lim_{x\to\infty} u_0 = u_L$ and $\lim_{x\to\infty} \frac{du}{dx} = 0$ holds; they are necessary conditions in electrochemical applications for the function $u = u(x,t)$. It is convenient to introduce the following abbreviations

$$\frac{\varepsilon_0 \varepsilon}{\sigma} = \tau, \quad \frac{e_0 z_i}{kT} = \eta, \quad \frac{1}{\sigma k T} \sum_{i=1}^{N} \left(z_i^2 e_0^2 N_i^0 \right) = \mu^2 , \tag{2}$$

where τ means a characteristic system time in the range of $\tau \approx 10^{-7} s$.

The quantity μ has the dimension $\dim[\mu] = s.m^{-2}$ referring to a reciprocal diffusion constant if we compare with the expression obtained in the DHT. μ^2 depends reciprocal upon the conductivity and temperature.

We firstly convert the Eq. (1) by introducing a frame of reference $u(x,t) = f(\xi)$, $\xi = x - \lambda t$ to derive the nODE of the third order

$$\lambda \tau \frac{d^3 f}{d\xi^3} - \frac{d^2 f}{d\xi^2} - \lambda \mu^2 \frac{df}{d\xi} \exp[-\eta f] = 0, \; f = f(\xi), \quad -\infty < \xi < \infty. \tag{3}$$

Note: The quantity λ is seen as a pure quantity of calculation. Without any loss of generality we set $\lambda = 1$ for other considerations (especially in wave theory λ represents the velocity of a wave or a soliton).

After integration and application of the transformation $f(\xi) = \frac{1}{\eta} \ln[w(\xi)]$ we derive

$$\tau w \frac{d^2 w}{d\xi^2} - \tau \left(\frac{dw}{d\xi} \right)^2 - w \frac{dw}{d\xi} - \mu^2 w = 0, \; w = w(\xi). \tag{4}$$

This nODE is the starting point to apply algebraic methods. First of all we need the parameter n of the series, Eq. (c). Balancing the highest-order nonlinear term and the highest-order linear term results in $n = -2$. This is impossible since the number must be $n \in Z^+$.

We employ the transformation $w = p(\xi)^{-2}$ to derive a further second-order nODE with C as an arbitrary constant of integration

$$2\tau p \frac{d^2 p}{d\xi^2} - 2\tau \left(\frac{dp}{d\xi} \right)^2 - 2p \frac{dp}{d\xi} - \mu^2 p^4 + C p^2 = 0, \quad p = p(\xi). \tag{5}$$

Now the balancing procedure results in the suitable form $n = 1$ and the polynomial ansatz of the first order for the function $p(\xi)$ is appropriate: $p(\xi) = a_0 + a_1 sn(\omega(\xi)) + b_1 cn(\omega(\xi))$ with $\omega(\xi)$ given from Eq. (e).

Note: We stress that $p = 0$ is the singular point of the Eq. (5).

Let us give an exact explanation for using algebraic approaches referring to the Painlevé property.
If we are able to show that the Eq. (1) is of P-type we can develop a solution near the singularity about the singular manifold and the Eq. (1) is integrable completely.
Thus let us show the following important theorem of long-ranged effects:
Theorem: The nPDE, Eq. (1) does not possess the Painlevé property.
Proof: Since the nODEs, Eq. (4) and Eq. (5) could derived from the nPDE Eq. (1) via the similarity transformation $u(x,t) = f(\xi)$, $\xi = x - \lambda t$, $\lambda \in R^{\pm}$, it is sufficient to show that the nODE Eq. (5) has the

Painlevé property. Also, without loss of generality we can set the constants τ and μ equal the identity in order to prove the P-property.

Further we assume the solution of the nODE Eq. (5) in the form $f \sim \alpha(z-z_0)^p$ where $\mathrm{Re}(p)<0$, z_0 arbitrary, and substitute it into Eq. (5).

Then α and p could determined so that different sets of terms may balance to provide the values of these coefficients. For each choice of p, the terms that balance in Eq. (5) are suitable for a possible expansion near the singularity. Here we find two possibilities: $p_1=-1$ and $p_2=-\frac{1}{2}$.

If any p is a fraction we have a branch point and the Eq. (5) does not possess the P-property [33]

At this stage of the analysis one cannot decide between algebraic and/or logarithmic branch points as the singularities. So, to represent the solution in case of occurrence of branch points it is necessary to introduce logarithmic terms in the solution more and more. Alternatively, psi-series can be used [32].

4. Algebraic classes of solutions

Let us now proceed further to derive solutions of the nPDE, Eq. (1). At the end of the chapter 3 we saw that the balancing number is given by $n=1$and therefore a linear ansatz is suitable.

We substitute Eq. (c) and Eq. (d) into the nODE Eq. (5) and change it into polynomial identities for the variable $\omega(\xi)$ by using the standard relations for the elliptic functions (e.g. $cn^2u+sn^2u=1$).

Then all terms with the same power in $\mathrm{cn}^k\omega\,\mathrm{sn}^l\omega,\ \mathrm{k,l}=0,1,....$ are collected and set to zero their coefficients to get a nonlinear algebraic system of polynomial equations which has to be solved for the unknowns a, a_0, a_1, b_1, b and C_1 in a consistent way whereby it may happen that solutions are only of trivial form and therefore useless for the solution manifold of Eq. (5) and hence for the nPDE Eq. (1), respectively.

In total we find twelve nonlinear polynomial equations for five unknowns which is solved by real and complex parameter values

Case (i): real-valued: $\quad a_0=0,\ a_1=\pm b_1,\ C=4b,\ b\neq 0,\ b_1\neq 0\,,$ (6)

Case(ii): complex-valued: $\quad a_0=0,\ a_1=\pm i b_1,\ C=\pm 2ib,\ a_1\neq 0\,.$ (6a)

Applying all transformations we find a solution of the potential for the real case in form of

$$u(x,t)=f(\xi)=\frac{1}{\eta}\ln\left[\frac{1}{1+2cn[\Phi]\times sn[\Phi]}\right]=-\frac{1}{\eta}\{\ln 2+\ln[cn[\Phi]]+\ln[sn[\Phi]]\}\,, \tag{7}$$

where we have made use of the simplifications $C_1 = b_1 = b = 1$. Further note that the argument of the elliptic functions is given by $\Phi = (1 + am\,\xi)$. We stress that the first expression is not defined at the points where the denominator vanishes, e.g. at the point $\xi = 0$ as $sn(0) = 0$. Moreover the expression may have singularities at some points.

Note: This problem is closely related with the solution of the equation $2\,cn[\Phi]\,sn[\Phi] = 1$ if we assume $C_1 = 0$. Then, depending upon the modulus the equation has the solution $\xi = -1 \pm 3\pi/4$ for the case $k = 0$ and for the case $k = 1$ we have $\xi = \ln\left[\tan\left[\frac{1}{4}\left[2\,\mathrm{sech}^{-1}\left[\frac{1}{\sqrt{2}}\right] - 2\right]\right]\right]$.

In Fig.1 we show graphically the periodic solution Eq. (7).

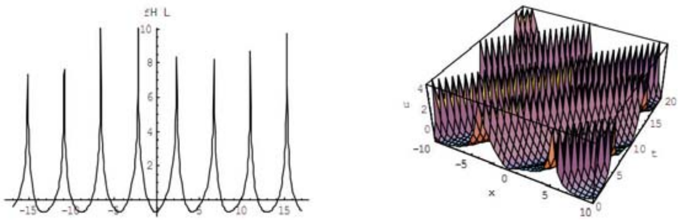

Fig.1 The planar periodic solution Eq. (7) left and a cut-out of the three-dimensional domain right. The periodic behaviour comes from the fact that the superposition of periodic elliptic functions also leads to a periodic movement. For the time-dependent solution right we assume $\lambda = 1$ and the modulus is $k = 0{,}9$. It is also shown that the function is stable.

Let us further deal with some special cases depending upon the values of the modulus. Thus we have to treat two sub cases: Without loss of generality we can set $C_1 = 0$ and $k = 1$ for the modulus we derive the special solution

$$f_1(\xi) = \frac{1}{\eta}\ln\left[\left(2\,\mathrm{sech}\left[1 - \tfrac{\pi}{2} + 2\arctan\left[e^{\xi}\right]\right] \times \tanh\left[1 - \tfrac{\pi}{2} + 2\arctan\left[e^{\xi}\right]\right]\right)^{-1}\right] =$$
$$= -\frac{1}{\eta}\ln\left[2\,\mathrm{sech}\left[1 + gd(\xi)\right] \times \tanh\left[1 + gd(\xi)\right]\right], \tag{7a}$$

where we introduced the Gudermannian function (sometimes called as the hyperbolic amplitude).

Note: Generally, the Gudermannian function can be treated as a parameter integral so that one defines by using the substitution $u = \sinh t$

$$gd(x) = \int_0^x sech\,dt = \int_0^x \frac{dt}{\cosh(t)} = \int_{u=0}^{\sinh x} \frac{du}{1+u} = \arctan(u)\Big|_{u=0}^{\sinh x} = \arctan(\sinh x)\,. \tag{f}$$

Alternatively one can write

$$gd(x) = 2\arctan\left(e^{x}\right) - \pi/2\,. \tag{g}$$

To prove this one has to show that both sides vanishes as $x = 0$. Left side: $gd(0) = \arctan(\sinh(0)) = \arctan(0) = 0$, right side: $2\arctan(e^0) - \pi/2 = 1 - \pi/2 = 2\frac{\pi}{4} - \frac{\pi}{2} = 0$.

Otherwise, for the choice $k = 0$ we find the following solution

$$f_2(\xi) = -\frac{1}{\eta}\ln\left[\left(1+\sin\left[2\,(1+\xi)\right]\right)^{-1}\right] = -\frac{1}{\eta}\ln\left[1+\sin\left[2(1+\xi)\right]\right] . \tag{7b}$$

The Fig.2 shows the graphical behaviour for both of the cases considering a special initial condition as $t = 0$ so that the dependence occurs frame-like. We remark that the solution is not defined at the point $\xi_1 = -\frac{1}{4}(\pi+4)$ therefore the domain of definition is $D := \{\xi \in R^+ : R^+ \backslash (-\frac{1}{4}(\pi+4))\}$. Since $\ln(1) = 0$ the solution vanishes at the point $\xi_2 = -1$.

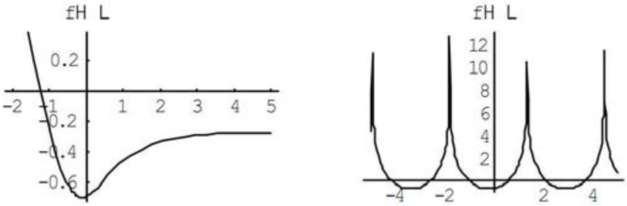

Fig.2 The solutions eq(7a) left and Eq. (7b) right. A marked minimum is seen and after crossing the ordinate the function tends to a finite value. The behaviour is similar to known potentials. The right curve represents also a periodic run.
Note that we chose special values for the modulus: Left the special case $k = 1$ and right $k = 0$.

4.a A short numerical statement

As mentioned above the highly nPDE Eq. (1) and all nODEs derived from it, especially the nODE Eq. (5) can only be solved under special circumstances. It is clear that by applying numerical standard procedures a closed-form solution is always obtained (considering suitable boundary and/or initial conditions). One of these standard procedures is the representation in ascending power series which is useful in numerical calculations. At the regular point $\xi = 0$ of the nODE Eq. (5) one can assume a series representation up to order four in the form (here we set the constants η and μ equal the identity since the do not influence the result necessarily)

$$\begin{aligned} p(\xi) = {} & a_0\xi + \frac{1}{4}\left(a_0 + a_0^3 + 2a_1\right)\xi^2 + \frac{1}{12}\left(a_0 + a_0^3 + 3a_1 + 3a_0^2 a_1\right)\xi^3 + \\ & + \frac{1}{96}\left(3a_0 + 6a_0^3 + 3a_0^5 + 8a_1 + 12a_0^2 a_1 + 12 a_0 a_1^2\right)\xi^4 + O[\xi]^5 , \end{aligned} \tag{8}$$

with arbitrary chosen coefficients a_0 and a_1. Unfortunately the algorithm (by Taylor) which was used does not provide any ready way to determine the radius of convergence of the series. One can say that by considering higher order terms the series converges relatively fast.

In Fig.3 we show some integral curves by choosing different values of the parameters a_i, $i=0{,}1$ and this is compared with a solution derived by a Runge-Kutta-like algorithm by assuming different initial values.

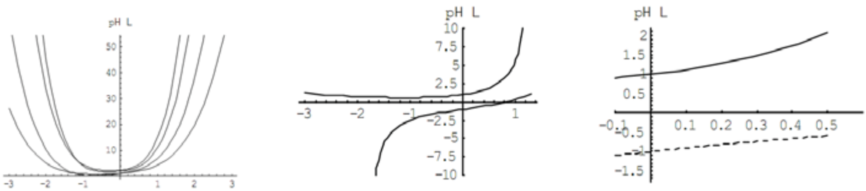

Fig.3 Some integral curves for the nODE Eq. (5). Left: The series solution Eq. (8) generated by different values of the parameters $a_i, 1 \le a_i \le 2, i = 1,2$ and a numerical simulation with different initial conditions: middle: $C = 1$, $p(0) = 1, p'(0) = 1$ for the top curve and $p(0) = -1, p'(0) = 1$ for the upper curve. Right: $C = -1$, $p(0) = 1, p'(0) = 1$ for the top curve and $p(0) = -1, p'(0) = 1$ for the upper graph.

4.b Complex-valued solutions

By using the parameter values of Eq. (6a) we find by assuming $C_1 = 0$ and $C = b = 1$

$$f(\xi) = \frac{1}{\eta}\ln\left[\left(1+2i\,cn\left[\frac{i}{2}am\xi\right]\times sn\left[\frac{i}{2}am\xi\right]\right)^{-1}\right] = -\frac{1}{\eta}\{\ln 2i + \ln[cn[\Phi]] + \ln[sn[\Phi]]\}, \qquad (9)$$

with $\Phi = \frac{i}{2}am\xi$ and $i^2 = -1$.

Taking into account the rules for the complex-valued elliptic functions [35] one can calculate a real-valued potential function for the special case of the modulus $k = 1$ in form of

$$f_3(\xi) = -\frac{1}{\eta}\ln[1-\sec[\Psi]\times\tan[\Psi]], \quad \Psi = \frac{gd(\xi)}{2}, \qquad (9a)$$

where we have made use of the Gudermannian function once again. Finally, for the case $k = 0$ we derive the closed-form solution

$$f_4(\xi) = -\frac{1}{\eta}\ln\left[1-2\cosh\left[\frac{\xi}{2}\right]\times\sinh\left[\frac{\xi}{2}\right]\right] = -\frac{1}{\eta}\ln[1-\sinh\xi]\,. \qquad (9b)$$

Similiarly to Eq. (7b) the argument may not assume the values $\xi_1 = 0$ and $\xi_2 = \sinh^{-1}(1) \approx 0{,}88$. For practical calculations and estimations we find it useful to calculate an ascending power series

$$f_4(\xi) = \frac{\xi}{43} + \frac{\xi^2}{86} + \frac{\xi^3}{86} + \frac{5\xi^4}{516} + \frac{3\xi^5}{344} + \frac{16\xi^6}{1935} + \frac{83\xi^7}{10320} + \,........ \qquad (10)$$

However the convergence is relatively slow which is seen by considering higher terms but otherwise the series shows that the potential is linear in first order approximation.

5. Some selected properties

For any numerical intensions we have for e_0 / kT approximately $\eta \approx 43$ and the valency number is assumed to be $z = 1$.

For the solution Eq. (7a) we find that the first derivative exists and takes a finite value, further it is seen that $\lim_{\xi \to \pm 0} f_1(\xi) \in R^+$ holds. This finite value can be interpreted as finite potential value in an electrolyte. Taking into account the left figure of Fig.2 one can see a surprising similarity with potential curves known in molecular and/or nucleus physics. A sharp minimum is present and after passing this point the curve runs to infinity representing a domain of saturation. It is shown that $\lim_{\xi \to \pm\infty} f_1(\xi) \to +\infty$ holds. The second derivation can be used to prove the stability in the considered domain concluding the correctness. The right plot of Fig.2, that is the solution Eq. (7b) shows a typical periodic behaviour which arises from the overlay of the periodic elliptic functions. The first and the second derivatives exist and numerically we have $\lim_{\xi \to \pm 0} f_2(\xi) = -0{,}65$ but the function remains indeterminate as $\xi \to \pm\infty$.

We stress that by applying the above given procedure a periodic run could observed for the first time. It is also shown the function is stable.

A remarkable fact is the following: If we substitute in Eq. (9a) the first term of the series of the Gudermannian, that is

$$\frac{gd(x)}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \tanh^{2k+1} \frac{x}{2}, \tag{10}$$

we find the following expression

$$f_3(\xi) = -\frac{1}{43} \ln\left[1 - \sec\left[\tanh\left[\frac{\xi}{2}\right]\right] \times \tan\left[\tanh\left[\frac{\xi}{2}\right]\right]\right]. \tag{11}$$

The derivations exist and $\lim_{\xi \to \pm 0} f_3(\xi) = 0$ hold. Otherwise, by approaching to infinity we have $\lim_{\xi \to \pm\infty} f_3(\xi) \approx 0{,}022$. This finite value represents the domain of saturation. It is also possible to perform a series representation which gives

$$f_3(\xi) = \frac{\xi}{86} + \frac{\xi^2}{344} + \frac{5\xi^3}{2064} + \frac{3\xi^4}{2752} + \frac{61\xi^5}{165120} + \frac{3\xi^6}{13760} + \frac{3457\xi^7}{27740160} + \ldots\ldots \tag{11a}$$

Taking into account higher terms one can see that the convergence is relatively fast if we compare with the series, Eq. (10).

For the last function, $f_4(\xi)$, Eq. (9b) we find that the first and the second derivatives exist and it is further shown that the function assumes a constant value at the point $\xi = 0$. In detail the limit $\lim_{\xi\to\pm 0} f_4(\xi) = 0$ holds but otherwise, by approaching infinity we have $\lim_{\xi\to\pm\infty} f_4(\xi) = -\infty$. This behaviour covers the fact of a 'hard sphere' potential. A graphical overview is given in the Fig.4.

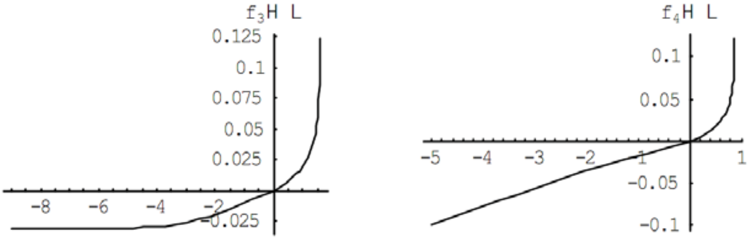

Fig.4 Planar plots of the solutions Eq. (10) left and Eq. (9b) right. Both of the solutions behave linearly in the negative domain and after passing the origin they tend towards infinity. This fact can be interpreted as a 'hard sphere'-like behaviour. The solution Eq. (10) comes from a domain of saturation which differs from the run of Eq. (9b).

6. Conclusion and outlook

Transient electrodynamical (and therefore electrochemical) problems are notoriously difficult to solve (in the general case one needs Green's tensors and/or vector potentials) and uniqueness is not always given (solutions depending upon roots are involved so that branch cuts must be taken into account).

In this study we showed that the nPDE, Eq. (1) in the meaning of a modified Poisson-Boltzmann Equation can be solved analytically by algebraic methods. The highly nonlinear equation was introduced by the author [28] recently to describe electrochemical processes and/or charge transfer on electrodes.

We applied an algebraic approach containing elliptic functions explicitly. It is remarkable that classes of solutions derived by this special method differ completely from solutions of the DHT in their behaviour. It is a special hallmark of algebraic methods that one cannot predict appropriate solutions in the sense of the solubility of the nonlinear algebraic system of polynomial equations. Several other approaches are used but none of them leads to useful results.

From this standpoint one can argue that the results represent a peculiar contribution. In principle we found two kinds of potentials which refer to different solutions of the nonlinear algebraic polynomial system: They are real as well as complex-valued. However, suitable transformations are appropriate to generate potentials of practical interest. Moreover we stress that by applying the algebraic approach under consideration unexpected classes of functions result. One can say that the periodicity (the Jacobian are double-periodic generally) of the elliptic functions will be transferred into the solutions of the nPDE, Eq. (1). In the following studies we shall connect the calculated potentials with real advanced situations in order to develop further our model under consideration.

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References

[1] E. Infeld, G. Rowlands, *Nonlinear Waves, Solitons and Chaos*, Cambridge Univ. Press, Cambridge, 1990.
[2] A. Huber, *Int. J. Eng. Sci. Technol.* **2** (2010) 1.
[3] C. Bai, *Phys. Lett.* **A 288** (2001) 191.
[4] S. A. Elwakil, S. K. El-Labany, *Chaos Solitons Fractals* **17** (2003) 121.
[5] S.A. Elwakil, S. K. El-Labany, *Chaos Solitons Fractals* **19** (2004) 1083.
[6] S. A. Elwakil, S. K. El-Labany, *Chaos Solitons Fractals* **22** (2004) 13.
[7] A. Huber, *Appl. Math. Comp.* **166** (2005) 464.
[8] Y. Chen, *Int. J. Mod. Phys.* **C 14** (2003) 471.
[9] D. Huang, *Chaos Solitons Fractals* **22** (2004) 243.
[10] A. Huber, *Chaos Solitons Fractals* **28** (2006) 972.
[11] A. A. Darwish, *Chaos Solitons Fractals* **33** (2007) 1263.
[12] S. A. Khuri, *Chaos Solitons Fractals* **32** (2007) 252.
[13] J. Nickel, *Chaos Solitons Fractals* **33** (2007) 1376.
[14] A. Huber, *Appl. Math. Comp.* **215** (2009) 2337.
[15] A. Huber, *J. Comput. Meth. Sci. Eng.* **9** (2009) 141.
[16] A. Huber, *Int. J. Res. Rev. Appl. Sci.* **3** (2010) 47.
[17] A. Huber, *Chaos Solitons Fractals* **34** (2007) 765.
[18] G. Gouy, *Ann. Phys.* **7** (1917) 129.
[19] D. L. Chapman, *Philos. Mag.* **25** (1913) 475.
[20] P. Debye, E. Hückel, *Z. Phys.* **24** (1923) 185.
[21] B. Felderhof, *J. Chem. Phys.* **48** (1968) 1178.
[22] D. Saville, *Ann. Rev. Fluid Mech.* **9** (1977) 321.
[23] P. B. van der Weg, *J. Coll. Interface Sci.* **339** (2009) 542.
[24] J. C. Justice, *J. Phys. Chem.* **100** (1996) 1246.
[25] V. Lakbno, O.Vasil'ev, *Chem. Phys.* **153** (1991) 147.
[26] J. Bockris, A. Reddy, *Modern Electrochemistry*, Kluwer, New York, 1998.

[27] G. Kortüm, *Lehrbuch der Elektrochemie*, Verlag Chemie, Weinheim, 1972.

[28] A. Huber, *J. Math. Chem.* 48 (2010) 856.

[29] H. A. Haus, J. R. Melcher, *Electromagnetic Fields and Energy*, Prentice Hall, New York, 1989.

[30] P. Drazin, R. Johnson, *Solitons: An Introduction*, Cambridge Univ. Press, Cambridge, 1989.

[31] A. Huber, *Int. J. Diff. Eq. Dyn. Syst.* **15** (2007) 27.

[32] J. Weiss, M. Tabor, G. Carnevale, *J. Math. Phys.* **24** (1983) 975.

[33] M. Ablowitz, A. Ramani, H. Segur, *J. Math. Phys.* **21** (1980) 1006.

[34] I. Gradshteyn, R. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York, 1965.

[35] P. F. Byrd, M. D. Friedman, *Handbook of Elliptic Integrals*, Springer–Verlag, Berlin, 1954.