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On the Generalized Hartmann Model for the Benzene Molecule

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Abstract

In this work we study a generalized integrable biparametric family of 4-D isotropic oscillators. This family allows to treat, in a unified way, oscillators defined by the potential given by Hartmann related with the benzene molecule and other ring shape ones. These potentials belong to two larger families of 3-D integrable Hamiltonian systems and have received a special attention since are related with the benzene molecule, as well as other models in quantum chemistry and nuclear physics. Using the Liouville-Arnold theorem and a particular analysis of the momentum map at its critical points, a complete topological classification of the different invariant sets of the phase flow of this problem is presented. By means of this topological study and the calculation of the action-angle variables the classification of periodic and quasiperiodic orbits are stated. The main aim of this paper is to study the stability properties of these orbits and to prove the integrability of the system correcting a mathematical mistake of [Ferrer et al., Cel. Mech. Dyn. Ast., in press].

1 Introduction

In this paper we study a 4-D integrable dynamical system defined by the parametric Hamiltonian function $\mathcal{H} : \mathbb{R}^8 \to \mathbb{R}$ determined by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1,$$

being

$$\mathcal{H}_0 = \frac{1}{2}\big(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 + \omega\big(q_1^2 + q_2^2 + q_3^2 + q_4^2\big)\big) \tag{1}$$

the isotropic oscillator, and

$$\mathcal{H}_1 = \frac{1}{2}\left(\frac{a}{q_1^2+q_2^2}+\frac{b}{q_3^2+q_4^2}\right) \tag{2}$$

another Hamiltonian where a, b and ω are three structural constants of the system that are considered to be positive.

This Hamiltonian is related with two families of 3-D integrable Hamiltonian systems, $\mathcal{H}=\frac{1}{2}\|X\|^2+V_i$ with axial symmetry, namely systems with potentials given by

$$V_1 = -\frac{\mu}{\sqrt{x_1^2+x_2^2+x_3^2}}+\frac{P}{x_1^2+x_2^2}+\frac{Qx_3}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2+x_3^2}}, \tag{3}$$

dubbed as Smorodinsky-Winternitzand potential, and

$$V_2 = \frac{\Omega^2}{2}\left(x_1^2+x_2^2+x_3^2\right)+\frac{P}{2x_3^2}+\frac{Q}{(x_1^2+x_2^2)} \tag{4}$$

(where μ, Ω, P are parameters, Q is the momentum and x_i are the variables of the problem).

The relation of those 3-D systems with a 4-D integrable dynamical system can be shown by means of two canonical extensions.

If we make a canonical transformation from cartesian to polar–symplectic coordinates in the classical way see [1], then (1) and (2) in the new variables are

$$\begin{aligned}
\mathcal{H}_0 &= \frac{1}{2}\left(P_1^2+P_2^2+\frac{A_1^2}{\rho_1^2}+\frac{A_2^2}{\rho_2^2}+\omega\left(\rho_1^2+\rho_2^2\right)\right),\\
\mathcal{H}_1 &= \frac{1}{2}\left(\frac{a}{\rho_1^2}+\frac{b}{\rho_2^2}\right).
\end{aligned}$$

Note that θ_1 and θ_2 are cyclic, thus A_1 and A_2 are first integrals.

The system is made separable in two subsystems of 1-DOF, defined by the Hamiltonian functions

$$\mathcal{H}_a = \frac{1}{2}\left(P_1^2+\frac{A_1^2}{\rho_1^2}+\omega\rho_1^2+\frac{a}{\rho_1^2}\right), \tag{5}$$

$$\mathcal{H}_b = \frac{1}{2}\left(P_2^2+\frac{A_2^2}{\rho_2^2}+\omega\rho_2^2+\frac{b}{\rho_2^2}\right), \tag{6}$$

with the same internal structure, such that

$$\mathcal{H} = \mathcal{H}_a+\mathcal{H}_b. \tag{7}$$

Due to the structure of the subsystem (5) and (6) it is enough to study $\mathcal{H}_a$ to deduce the corresponding results for $\mathcal{H}_b$. Finally, because (7) we obtain the complete study of $\mathcal{H}$.

Note that, $\mathcal{H}_a : \mathbf{E}_a \to \mathbb{R}$ where $\mathbf{E}_a = \mathbb{R}^+ \times S^1 \times \mathbb{R}^2$ is the phase space where the Hamiltonian is defined.

The particular case of system (2) when $a = b$ was consider by Kibler and Négadi [7] when they studied the Hartmann potential using the Kustaanheimo–Stiefel transformation. Therefore, in this sense the proposed Hamiltonian (1) represents a generalization of theirs.

Note that potentials V_i belong to a larger family of integrable systems which are known to be separable (see Makarov et al. [10]). These potentials have received special attention since the pioneer work of Hartmann et al. due to its relation to the benzene molecule, as well as other models in quantum chemistry and nuclear physics. When we take $Q = 0$ in the potential V_1 we have the Hartmann model [8].

In order to do a qualitative study of the dynamics associated to the Hamiltonian system, in a similar way to [9], we are going to consider the following sets

$$E_{h_a} = \mathcal{H}_a^{-1}(h_a) = \{z \in \mathbf{E}_a : \mathcal{H}_a(z) = h_a\},$$
$$J_{k_1} = \{z \in \mathbf{E}_a : A_1 = k_1\},$$
$$I_{h_a k_1} = E_{h_a} \cap J_{k_1},$$

where $z = (\rho_1, \theta_1, P_1, A_1) \in \mathbf{E}_a$ and $(h_a, k_1) \in \mathbb{R}^2$.

These sets are invariant by the flow associated to the Hamiltonian, being $\mathcal{H}_a$ and A_1 two first integrals of motion, independent and in involution.

The steps of this work for obtaining the topological classifications, which complete the studies presented in [5] and [3], are:

- The descriptions of the foliations of

  (i) the phase space $\mathbf{E}_a$ by the invariant sets E_{h_a},

  (ii) E_{h_a} by the invariant sets $I_{h_a k_1}$, and

  (iii) $I_{h_a k_1}$ by the flow of the Hamiltonian system.

- To calculate the action-angle variables for the systems $\mathcal{H}_a$ and $\mathcal{H}_b$.

- To obtain the region where the action-angle can be defined.

- To classify the periodic orbits of the systems $\mathcal{H}_a$ and $\mathcal{H}_b$.

- To obtain the classification of the orbits of $\mathcal{H}$.

- To use the canonical transformations to reduce the 4-D system into (3), in order to see how the orbits of $\mathcal{H}$ are transformed.

The aim of the paper is to state the stability of the periodic and quasiperiodic orbits previously described and to prove the integrability of the systems correcting a mathematical mistake stated in [5] where the integrability of the problem is not proved in a correct way.

The paper is organized as follows. In Sec. 2 we study the connection between the 4-D isotropic oscillator and ring-shaped potentials. In Sec. 3 we study the topology of the Hamiltonian $\mathcal{H}_a$ and $\mathcal{H}$. In Sec. 4 we classify the periodic orbits of the ring-shaped potentials. In Sec. 5 we study the biparametric oscillator and its integration. In Sec. 6 we transform some orbits of $\mathcal{H}$ to the 3-D system of Hartmann and finally in Sec. 7 we study the stability of the periodic and quasiperiodic orbits of the system.

2 Relations between Isotropic Oscillators and Ring–Shaped Potentials

The aim of this section is to show the relation of the 3–D systems previously stated with a 4–D integrable dynamical system defined by the Hamiltonian function (1). In the next two subsections we shall establish, respectively, the relation between the oscillator and the systems defined by the potentials (3) and (4). The key point for doing this is to use in each case adequate four dimensional transformations and their canonical extensions which allow to relate the the parameters P and Q of the potentials with integrals and parameters of the 4–D oscillator via a linear system. We shall use too some results from [5].

2.1 The oscillator and the generalized Hartmann potentials

We show first the relation of the Hamiltonian system defined by (1) and the generalized Hartmann potentials defined by the potentials V_1. In order to do that we make use the transformation: $(r, \phi, \lambda, \psi) \to (q_1, q_2, q_3, q_4)$ given by

$$\begin{aligned} q_1 &= \sqrt{r}\sin\frac{\phi}{2}\cos\frac{\lambda-\psi}{2}, \quad q_3 = \sqrt{r}\cos\frac{\phi}{2}\sin\frac{\lambda+\psi}{2}, \\ q_2 &= \sqrt{r}\sin\frac{\phi}{2}\sin\frac{\lambda-\psi}{2}, \quad q_4 = \sqrt{r}\cos\frac{\phi}{2}\cos\frac{\lambda+\psi}{2}, \end{aligned} \tag{8}$$

with $(r, \phi, \lambda, \psi) \in R^+ \times (0, \pi) \times [0, 2\pi] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and whose jacobian is $-r \sin\phi/8$. Later on we will need the inverse transformation given by

$$
\begin{aligned}
r &= q_1^2 + q_2^2 + q_3^2 + q_4^2,\\
\sin\phi &= \frac{2\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}}{q_1^2+q_2^2+q_3^2+q_4^2}, \quad \cos\phi = \frac{q_3^2+q_4^2-q_1^2-q_2^2}{q_1^2+q_2^2+q_3^2+q_4^2},\\
\sin\lambda &= \frac{q_1q_3 + q_2q_4}{\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}}, \quad \cos\lambda = \frac{q_1q_4 - q_2q_3}{\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}},\\
\sin\psi &= \frac{q_1q_3 - q_2q_4}{\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}}, \quad \cos\psi = \frac{q_1q_4 + q_2q_3}{\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}}.
\end{aligned}
$$

The canonical extension associated to the transformation (8) is readily obtained as a Mathieu transformation, satisfying

$$\sum Q_i dq_i = R dr + \Phi d\phi + \Lambda d\lambda + \Psi d\psi.$$

R, Φ, Λ and Ψ are called the momenta of the problem. Note that the relations among these momenta are given by

$$
\begin{aligned}
R &= \frac{1}{2\sum q_i^2}(q_1Q_1 + q_2Q_2 + q_3Q_3 + q_4Q_4),\\
\Phi &= \frac{(q_1Q_1+q_2Q_2)(q_3^2+q_4^2)-(q_3Q_3+q_4Q_4)(q_1^2+q_2^2)}{2\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}},\\
\Lambda &= \frac{1}{2}(-q_2Q_1 + q_1Q_2 + q_4Q_3 - q_3Q_4),\\
\Psi &= \frac{1}{2}(q_2Q_1 - q_1Q_2 + q_4Q_3 - q_3Q_4).
\end{aligned}
$$

The Hamiltonian (1) in the new variables may be written as

$$
\begin{aligned}
\mathcal{H} = 4r \Bigg[&\frac{\omega}{8} + \frac{1}{2}\left(R^2 + \frac{\Phi^2}{r^2} + \frac{\Lambda^2}{r^2\sin^2\phi}\right)\\
&+ \frac{\Psi^2 - 2\Lambda\Psi\cos\phi}{2r^2\sin^2\phi} + \frac{c + d\cos\phi}{2r^2\sin^2\phi}\Bigg],
\end{aligned}
\tag{9}
$$

where

$$c = \frac{a+b}{2}, \qquad d = \frac{a-b}{2}.$$

Note that λ and ψ are cyclic variables, with Λ and Ψ as first integrals. In other words the differential systems is

$$\frac{dr}{d\tau} = \frac{\partial\mathcal{H}}{\partial R}, \quad \frac{d\phi}{d\tau} = \frac{\partial\mathcal{H}}{\partial\Phi}, \quad \frac{dR}{d\tau} = -\frac{\partial\mathcal{H}}{\partial r}, \quad \frac{d\Phi}{d\tau} = -\frac{\partial\mathcal{H}}{\partial\phi}$$

and two quadratures

$$\lambda = \int(\partial\mathcal{H}/\partial\Lambda)\, d\tau, \qquad \psi = \int(\partial\mathcal{H}/\partial\Psi)\, d\tau.$$

Using Poincaré notation and introducing a change of independent variable $\tau \to s$ given by $d\tau = 4r\,ds$, the Hamiltonian takes the form

$$\begin{aligned}\mathcal{K} &= \frac{1}{4r}(\mathcal{H} - h_O)\\ &= \frac{\omega}{8} + \frac{1}{2}\left(R^2 + \frac{\Phi^2}{r^2} + \frac{\Lambda^2}{r^2\,\sin^2\phi}\right)\\ &\quad + \frac{\Psi^2 - 2\,\Lambda\Psi\,\cos\phi}{2\,r^2\,\sin^2\phi} + \frac{c + d\cos\phi}{2r^2\sin^2\phi} - \frac{h_O}{4r},\end{aligned} \tag{10}$$

where h_O is a fixed value of the Hamiltonian $\mathcal{H}$ for chosen initial conditions, and the flow is defined now on the manifold $\mathcal{K} = 0$. We prefer to use a slightly different form; we consider the Hamiltonian

$$\begin{aligned}\tilde{\mathcal{K}} =& \frac{1}{2}\left(R^2 + \frac{\Phi^2}{r^2} + \frac{\Lambda^2}{r^2\sin^2\phi}\right) - \frac{h_O}{4r} + \frac{(\Psi^2 + c)/2}{r^2\sin^2\phi}\\ &+ \frac{(d/2 - \Lambda\Psi)\,\cos\phi}{r^2\sin^2\phi}\end{aligned} \tag{11}$$

in the manifold $\tilde{\mathcal{K}} = -\dfrac{\omega}{8}$. Denoting

$$\mathcal{H}_K = \frac{1}{2}\left(R^2 + \frac{\Phi^2}{r^2} + \frac{\Lambda^2}{r^2\,\sin^2\phi}\right) - \frac{h_O}{4r}$$

the differential system defined by (11) is given by

$$\begin{aligned}\frac{dr}{ds} &= \frac{\partial\tilde{\mathcal{K}}_O}{\partial R} = R,\\ \frac{d\phi}{ds} &= \frac{\partial\tilde{\mathcal{K}}_O}{\partial\Phi} = \frac{\Phi}{r^2},\\ \frac{dR}{ds} &= -\frac{\partial\tilde{\mathcal{K}}_O}{\partial r} = -\frac{\partial\mathcal{H}_K}{\partial r}\\ &+2\frac{(\Psi^2 + c)/2}{r^3\,\sin^2\phi} + 2\frac{(d/2 - \Lambda\Psi)\,\cos\phi}{r^3\,\sin^2\phi}\\ \frac{d\Phi}{ds} &= -\frac{\partial\tilde{\mathcal{K}}_O}{\partial\phi} = -\frac{\partial\mathcal{H}_K}{\partial\phi} + \frac{(\Psi^2 + c)/2}{r^2}\cdot\frac{\partial}{\partial\phi}\left(\frac{1}{\sin^2\phi}\right)\\ &+\frac{(d/2 - \Lambda\Psi)}{r^2}\cdot\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{\sin^2\phi}\right)\end{aligned} \tag{12}$$

and two quadratures

$$\lambda = \int\frac{\partial\tilde{\mathcal{K}}_O}{\partial\Lambda}ds = \int\left(\frac{\Lambda}{r^2\,\sin^2\phi} - \frac{\Psi\,\cos\phi}{r^2\,\sin^2\phi}\right)ds, \tag{13}$$

$$\psi = \int\frac{\partial\tilde{\mathcal{K}}_O}{\partial\Psi}ds = \int\frac{\Psi - \Lambda\,\cos\phi}{r^2\,\sin^2\phi}ds. \tag{14}$$

If we consider now the differential system defined by the Hamiltonian with potential V_1, Eq. (3) in spherical variables (r, ϕ, λ),

$$\begin{aligned} x_1 &= r \sin\phi \, \cos\lambda, \\ x_2 &= r \sin\phi \, \sin\lambda, \\ x_3 &= r \cos\phi \end{aligned} \tag{15}$$

and their momenta (R, Φ, Λ), we check that those equations coincide with equations (12) and (13), when we restrict to the manifold $\Psi = 0$ and we take the following values for the coefficients

$$h_O = 4\mu, \qquad c = 2P, \qquad d = 2Q,$$

and we identify the variable s with the physical time t.

Thus, we have shown that the dynamics of the oscillator defined by Hamiltonian (1) corresponds to the family of the generalized Hartmann potentials. If we assume $\Psi = 0$, the particular case of the Hartmann model is obtained when $d = 0$, *i. e.* when we take for the oscillator the following values

$$\mathcal{H}_O = 4\mu \qquad \omega = -8\tilde{\mathcal{K}}_O \qquad a = b = P.$$

3 Qualitative Study of the Hamiltonian Flow

3.1 Study of the Hamiltonian $\mathcal{H}_a$

3.1.1 Introduction

In this part of the paper we study the Hamiltonian $\mathcal{H}_a$. Remember that the Hamiltonian $\mathcal{H}_a : \mathbf{E}_a \to \mathbb{R}$ is given by

$$\mathcal{H}_a = \frac{1}{2}\left(P_1^2 + \frac{A_1^2}{\rho_1^2} + \omega\rho_1^2 + \frac{a}{\rho_1^2}\right), \tag{16}$$

where $\mathbf{E}_a = \mathbb{R}^+ \times S^1 \times \mathbb{R}^2$ is the phase space where the Hamiltonian (16) is defined and $(\rho_1, \theta, P_1, A_1) \in \mathbf{E}_a$. These kind of coordinates are the most appropriated because they let us note that the Hamiltonian is integrable. The parameters $a > 0$ and $\omega > 0$ are two structural constants of the system.

The Hamiltonian system is

$$\begin{aligned} \frac{d\rho_1}{dt} &= \frac{d\mathcal{H}_a}{dP_1} = P_1, & \frac{d\theta}{dt} &= \frac{d\mathcal{H}_a}{dA_1} = \frac{A_1}{\rho_1^2}, \\ \frac{dP_1}{dt} &= -\frac{d\mathcal{H}_a}{d\rho_1} = \frac{\omega\rho_1^4 - A_1^2 - a}{\rho_1^3}, & \frac{dA_1}{dt} &= -\frac{d\mathcal{H}_a}{d\theta} = 0. \end{aligned} \tag{17}$$

We denoted by $g : \mathbb{R}^+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ the map define by $g(\rho_1, P_1, A_1) = \mathcal{H}(\rho_1, \theta, P_1, A_1)$. If $h_a \in \mathbb{R}$ is a regular value of the map g and $g^{-1}(h_a) \neq \emptyset$, then $g^{-1}(h_a)$ is a surface of $\mathbb{R}^+ \times \mathbb{R}^2$ called energy surface.

In order to do a qualitative study of the dynamics associated to the Hamiltonian system, in a similar way to [9], we are going to consider the following sets:

$$E_{h_a} = \mathcal{H}^{-1}(h) = \{z \in \mathbf{E}_a : \mathcal{H}(z) = h_a\},$$

$$J_{k_1} = \{z \in \mathbf{E}_a : A_1 = k_1\}, \ k_1 \in \mathbb{R},$$

$$I_{h_a k_1} = E_{h_a} \cap I_{k_1}, \ (h_a, k_1) \in \mathbb{R}^2.$$

with $z = (\rho_1, \theta_1, P_1, A_1) \in \mathbf{E}_a$ and $(h_a, k_1) \in \mathbb{R}^2$.

Note that $E_{h_a} \approx g^{-1}(h_a) \times S^1$ and is called constant energy manifold, J_{k_1}constant momentum manifold and $I_{h_a k_1}$ constant energy-momentum manifold. These sets are invariant by the flow associated to the Hamiltonian, being $\mathcal{H}_a$ and A_1 two first integrals of motion, independent and in involution. Hence, the Hamiltonian system (17) is integrable.

The main results of this section are the description of the foliation of the phase space $\mathbf{E}_a$ by the invariant sets E_{h_a}, the energy sets E_{h_a} by the invariant sets $I_{h_a k_1}$ and $I_{h_a k_1}$ by the flow of the Hamiltonian system. This foliation provides a good description of the phase space when $(h_a, k_1) \in \mathbb{R}^2$ and depends on the different values of a and ω.

The main tool for this study is the Liouville-Arnold theorem:

Theorem 1 (Liouville-Arnold). *The Hamiltonian system with two degrees of freedom defined on the phase space* $\mathbf{E}$ *has the Hamiltonian* $\mathcal{H}$ *and angular momentum* p_θ *as two independent first integrals in involution. If* $I_{hk} = \emptyset$ *and* (h, k) *is a regular value of the map* $(\mathcal{H}, p_\theta)$*, then the following statements hold:*

(a) I_{hk} *is a two-dimensional submanifold of* $\mathbf{E}$ *invariant under the flow of the Hamiltonian system.*

(b) *If the flow on a connected component* I^*_{hk} *of* I_{hk} *is complete, then*I^*_{hk} *is diffeomorphic either to the torus* $S^1 \times S^1$*, or to the cylinder* $S^1 \times \mathbb{R}$*. We note that if* I^*_{hk} *is compact (i.e.* $I^*_{hk} \approx S^1 \times S^1$*), then the flow on it is always complete.*

(c) *Under the hypothesis (b) the flow on* I^*_{hk} *is conjugated to a linear flow either on* $S^1 \times S^1$*, or on* $S^1 \times \mathbb{R}$.

For more details about Hamiltonian systems and the proof of the previous theorem see Abraham and Marsden [1] and Arnold [2]. In general, under the assumptions of statement (b), I^*_{hk} can also be diffeomorphic to the plane $\mathbb{R}^2$.

The Liouville-Arnold theorem shows that, for integrable Hamiltonian systems, the invariant sets associated with the intersections of all independent first integrals in involution are generically submanifolds of the phase space. Moreover, if the flow on such submanifolds is complete, then these submanifolds are diffeomorphic to the union of generalized cylinders and the flow on them is conjugated to a linear flow.

Let us define what are for us a critical and a regular value and an equilibrium point. The value (h_a, k_1) is a critical value for the momentum mapping $(\mathcal{H}_a, A_1)$, if this map is not differentiable in this value. If (h_a, k_1) is not critical, it is named regular value. We obtain an equilibrium point for the map $\mathcal{H}_a : \mathbf{E}_a \longrightarrow \mathbb{R}$ when we equal the system (17) to zero.

We applied this theorem to the momentum map $(\mathcal{H}_a, A_1) : \mathbf{E}_a \times \mathbb{R} \longrightarrow \mathbb{R}^2$ at regular values. When it is not stated by the Liouville-Arnold theorem we have to do a particular study for the sets $I_{h_a k_1}$ for critical values $(h_a, k_1) \in \mathbb{R}^2$ of the momentum mapping. These values correspond either to equilibrium points of $\mathcal{H}_a$ or to values where $A_1 = k_1$ is a maximum or a minimum of the energy surface.

3.1.2 Equilibrium points

In order to calculate the equilibrium points we need to equal the Hamiltonian system (17) to zero.

This system has only a real solution for $\rho_1 > 0$ and $\forall a, \omega$. This solution is

$$\begin{aligned} P_1 &= 0, & A_1 &= 0, \\ \rho_1 &= \sqrt[4]{\frac{a}{\omega}}, & \theta_1 &= \theta_1. \end{aligned}$$

If we substitute these values in our Hamiltonian $\mathcal{H}_a$ we obtain $h_{a_e} = \sqrt{a\omega}$, that is the value of the equilibrium point.

3.1.3 Function potential. Hill regions

The Hamiltonian (16), in polar symplectic coordinates, can be written as follows

$$\mathcal{H}_a = \frac{1}{2}\left(P_1^2 + \frac{A_1^2}{\rho_1^2}\right) + \tilde{\mathcal{V}}(\rho_1),$$

where

$$\tilde{\mathcal{V}}\left(\rho_1\right) = \omega\rho_1^2 + \frac{a}{\rho_1^2} \tag{18}$$

is the potential. Moreover, the Hill regions are completely characterized by means of this potential function.

Now, the regions of the phase space where exits real motion are determined for the critical points of $\widetilde{\mathcal{V}}$ and these come given by the real positive roots of the polynomial equation

$$\omega\rho_1 + \frac{a}{\rho_1^3} = 0. \tag{19}$$

that we obtain doing the derivate of (18) with respect to ρ_1 equal to zero.

Using the Sturm algorithm we discuss, according to the values of the parameters a and ω, the number of positive real roots respect to ρ_1 of the equation (19). Is it easy to see that $\forall a, \omega$ we have only a real positive root because

$$\rho_{1e} = \sqrt[4]{\frac{a}{\omega}}. \tag{20}$$

ρ_{1e} is called critical point of $\widetilde{\mathcal{V}}$.

Let $\pi : \mathbf{E}_a \longrightarrow \mathbb{R}^+ \times S^1$ be the natural projection. For each $h_a \in \mathbb{R}$ the Hill region R_{h_a} of E_{ha} is defined by $R_{h_a} = \pi\left(E_{ha}\right)$.

$$R_{h_a} = \{(\rho_1, \theta_1) \in \mathbb{R}^+ \times S^1 : \widetilde{\mathcal{V}} \leq h_a\} =$$

$$\{(\rho_1, \theta_1) \in \mathbb{R}^+ \times S^1 : \omega\rho_1^2 + \frac{a}{\rho_1^2} \leq h_a\}.$$

Note that $R_{h_a} \approx S^1 \times g^{-1}(h_a)$ where $\approx$ means diffeomorphic to.

In which it follows the value of the potential, in the critical point ρ_{1e}, will be denoted by $h_{a_e} = \sqrt{a\omega} = \widetilde{\mathcal{V}}(\rho_{1e})$. The values ρ_1^j, $j = 1, 2$ will be the intersection points between the graph of the potential and $\widetilde{\mathcal{V}} = h_a$.

To a clearer understanding of the topology of the Hill regions the following figure is presented. The table give us the topological classification of the Hill regions.

Table 1: Topological classification of the Hill regions for the different values of h where h_a is the critical point.

$\forall a, \omega$	$h < h_{a_e}$	$R_{h_a} \approx \emptyset$
	$h = h_{a_e}$	$R_{h_a} \approx \{h_{a_e}\} \times S^1$
	$h_{a_e} < h$	$R_{h_a} \approx [\rho_1^1, \rho_1^2] \times S^1$

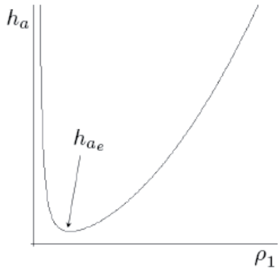

Figure 1: Graph of the potential, where h_a is the critical point.

3.1.4 Relative extremes of the energy surface

Using the implicit function theorem we can see that the extremes of the energy surface $g^{-1}(h_a)$ come given by the real roots of the polynomial equation

$$A_1^4 + \left(2a - \frac{h_a}{\omega}\right) A_1^2 + a^4 - \frac{ah_a}{\omega} = 0. \tag{21}$$

The Sturm Algorithm is used to calculate the number of real roots of (21) respect to A_1. It is easy to see that (21) has two or zero real positive roots denoted by $A_{1_i}, i = 1, 2$ and corresponding to the extremes of the energy surface $m_{a_i}, i = 1, 2$. The extremes of $g^{-1}(h_a)$ come given by $m_{a_i} = \mathcal{H}_a\left(\rho_{1_i}, \_, P_{1_i}, A_{1_i}\right), i = 1, 2$, where $P_{1_i} = 0, \rho_{1_i} = \sqrt[4]{\dfrac{a + A_{1_i}^2}{\omega}}$.

3.1.5 Topological classification of E_{h_a} and $I_{h_a k_1}$

In this section we study the topology of the invariant manifolds $\mathcal{H}_a^{-1}(h) = E_{h_a}$ and $I_{h_a k_1}$. To give the topological classification of these invariant sets we need some notation and some new results.

Note that $z_e = (\rho_{1_e}, \theta_{1_e}, P_{1_e}, A_{1_e}) \in \mathbf{E}_a$ is an equilibrium point of the Hamiltonian flow if and only if $\widetilde{z}_e = (\rho_{1_e}, \theta_{1_e})$ is a critical point of the potential. Moreover $\pi\left(z_e\right) = \widetilde{z}_e$, where $\pi : \mathbf{E}_a \longrightarrow \mathbb{R}^+ \times S^1$ is the natural projection. For this reason we know that we have only a family of equilibrium points.

Let S^{n-1} be the sphere in $\mathbb{R}^n$, with $n > 1$ and $A_{1_j}, j = 1, 2$ the values of A_1 that correspond to the extremes of the energy surface $g^{-1}\left(h_a\right)$. Finally h_{a_e} is the value of the equilibrium point for the Hamiltonian $\mathcal{H}_a$.

The different cases can be shown by means of Figure 2.

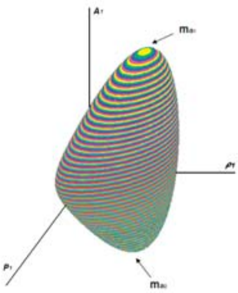

Figure 2: Energy surface for $H_a^{-1}(h_a)/S^1$, with $h_{a_e} < h_a$ where m_{a_1} and m_{a_2} are the extremes of the surface.

We obtain the topological classification for E_{h_a} and $I_{h_a k_1}$.

Table 2: Topological classification of the sets E_{h_a} and $I_{h_a k_1}$, for all the values of a and ω, that we considered to be positive, and (h_a, k_1) varying.

h_a	E_{h_a}	$I_{h_a k_1}$	
$h_a < h_{a_e}$	$\emptyset$	$\forall k_1$	$\emptyset$
$h_a = h_{a_e}$	S^1	$k_1 = 0$	S^1
$h_{a_e} < h_a$	S^3	$k_1 > A_{1_1}$	$\emptyset$
		$k_1 = A_{1_1}$	S^1
		$A_{1_2} < k_1 < A_{1_1}$	$S^1 \times S^1$
		$k_1 = A_{1_2}$	S^1
		$k_1 < A_{1_2}$	$\emptyset$

3.2 Study of the Hamiltonian $\mathcal{H}$

Remember that

$$\mathcal{H} = \mathcal{H}_a + \mathcal{H}_b,$$

Due to the decomposition of $\mathcal{H}$, we use the results obtained in the study of $\mathcal{H}_a$ to deduce the corresponding results for $\mathcal{H}$.

Let S^{n-1} be the sphere in $\mathbb{R}^n$, with $n > 1$, $A_{1_j}, j = 1, 2$ the value of A_1 that correspond to the extremes of the energy surface $g^{-1}(h_a)$ and h_{a_e} the value of the equilibrium point of the Hamiltonian $\mathcal{H}_a$. Finally, $A_{2_j}, j = 1, 2$ correspond to the values of P_2 that correspond to the extremes of the energy surface $g^{-1}(h_b)$ and h_{b_e} the value of the equilibrium point of the Hamiltonian $\mathcal{H}_b$.

The topological classification of $\mathcal{H}$ is presented in the tables from the 3 to the 11.

Table 3: Topological classification of the sets E_h and I_{hk}, for all the values of a, b and ω, that we considered to be positive, $\forall k_1$, $h_a < h_{a_e}$ and h_b, k_2 varying.

h_a	h_b	k_1	k_2	E_h	I_{hk}
	$h_b < h_{b_e}$	$\forall k_1$	$\forall k_2$	$\emptyset$	$\emptyset$
	$h_b = h_{b_e}$	$\forall k_1$	$k_2 = 0$	S^1	S^1
			$k_2 > A_{2_1}$		$\emptyset$
			$k_2 = A_{2_1}$		S^1
$h_a < h_{a_e}$	$h_{b_e} < h_b$	$\forall k_1$	$A_{2_2} < k_2 < A_{2_1}$	S^3	$S^1 \times S^1$
			$k_2 = A_{2_2}$		S^1
			$k_2 < A_{2_2}$		$\emptyset$

Table 4: Topological classification of the sets E_h and I_{hk}, for all the values of a, b and ω, that we considered to be positive, $k_1 = 0$, $h_a = h_{a_e}$ and h_b, k_2 varying.

h_a	h_b	k_1	k_2	E_h	I_{hk}
	$h_b < h_{b_e}$	$k_1 = 0$	$\forall k_2$	S^1	S^1
	$h_b = h_{b_e}$	$k_1 = 0$	$k_2 = 0$	$S^1 \cup S^1$	$S^1 \cup S^1$
			$k_2 > A_{2_1}$		S^1
			$k_2 = A_{2_1}$		$S^1 \cup S^1$
$h_a = h_{a_e}$	$h_{b_e} < h_b$	$k_1 = 0$	$A_{2_2} < k_2 < A_{2_1}$	$S^1 \cup S^3$	$S^1 \cup \{S^1 \times S^1\}$
			$k_2 = A_{2_2}$		$S^1 \cup S^1$
			$k_2 < A_{2_2}$		S^1

4 Periodic Orbits of Ring-Shaped Potentials

Now, we want to characterize the periodic orbits of the Hamiltonian system (1). In order to carry out this characterization it is necessary to calculate the action-angle variables for this system. First, we calculate the action-angle variables for the susbsystem $\mathcal{H}_a$ and, for the structure of both subsystems, we obtain the expression of $\mathcal{H}$ in these variables.

4.1 Action-angle variables for the subsystem $\mathcal{H}_a$

By means of Hamilton-Jacobi Theory we are able to define the action-angle variables

$$J_{\theta_1} = A_1 = k_1, \tag{22}$$

Table 5: Topological classification of the sets E_h and I_{hk}, for all the values of a, b and ω, that we considered to be positive, $\forall k_2$, $h_{a_e} < h_a$, $h_b < h_{b_e}$ and k_1 varying.

h_a	h_b	k_1	k_2	E_h	I_{hk}
$h_{a_e} < h_a$	$h_b < h_{b_e}$	$k_1 > A_{1_1}$ $k_1 = A_{1_1}$ $A_{1_2} < k_1 < A_{1_1}$ $k_1 = A_{1_2}$ $k_1 < A_{1_2}$	$\forall k_2$	S^3	$\emptyset$ S^1 $S^1 \times S^1$ S^1 $\emptyset$

Table 6: Topological classification of the sets E_h and I_{hk}, for all the values of a, b and ω, that we considered to be positive, $k_2 = 0$, $h_{a_e} < h_a$, $h_b = h_{b_e}$, k_1 varying.

h_a	h_b	k_1	k_2	E_h	I_{hk}
$h_{a_e} < h_a$	$h_b = h_{b_e}$	$k_1 > A_{1_1}$ $k_1 = A_{1_1}$ $A_{1_2} < k_1 < A_{1_1}$ $k_1 = A_{1_2}$ $k_1 < A_{1_2}$	$k_2 = 0$	$S^3 \cup S^1$	S^1 $S^1 \cup S^1$ $\{S^1 \times S^1\} \cup S^1$ $S^1 \cup S^1$ S^1

$$J_{\rho_1} = \frac{1}{2\pi} \int_{\rho_{1_2}}^{\rho_{1_1}} \frac{\sqrt{-k_1^2 - a + 2h_a \rho_1^2 - \omega \rho_1^4}}{\rho_1} d\rho_1, \tag{23}$$

where ρ_{1_1} and ρ_{1_2} are the two positive real roots of the polynomial

$$P(\rho_1) = -k_1^2 - a + 2h_a \rho_1^2 - \omega \rho_1^4, \tag{24}$$

for this reason, the transformation from polar-symplectic variables to action-angle variables can be only defined in the region where the polynomial (24) has two different positive real roots according to the parameters a, ω, k_1 and h_a.

In order to calculate J_{ρ_1} we use the Cauchy Residue theorem and we obtain the expression

$$J_{\rho_1} = \frac{1}{2} \left(\frac{h_a}{\sqrt{\omega}} - \sqrt{J_{\theta_1}^2 + a} \right). \tag{25}$$

Obtaining the value of h_a in (25) we have the Hamiltonian $\mathcal{H}_a$ expressed in action-angle variables

$$\mathcal{H}_a = \sqrt{\omega} \left(2J_{\rho_1} + \sqrt{J_{\theta_1}^2 + a} \right). \tag{26}$$

The condition that must satisfy the parameters a, ω, k_1 and h_a in order to have defined the action-angle variables is

$$h_a > \sqrt{(a + k_1^2)\,\omega}.$$

Table 7: Topological classification of the sets E_h and I_{hk}, for all the values of a, b and ω, that we considered to be positive, $k_1 > A_{1_1}$, $h_{a_e} < h_a$, $h_{b_e} < h_b$ and k_2 varying.

h_a	h_b	k_1	k_2	E_h	I_{hk}
$h_{a_e} < h_a$	$h_{b_e} < h_b$	$k_1 > A_1$	$k_2 > A_{2_1}$ $k_2 = A_{2_1}$ $A_{2_2} < k_2 < A_{2_1}$ $k_2 = A_{2_2}$ $k_2 < A_{2_2}$	$S^3 \cup S^3$	$\emptyset$ S^1 $S^1 \times S^1$ S^1 $\emptyset$

Table 8: Topological classification of the sets E_h and I_{hk}, for all the values of a, b and ω, that we considered to be positive, $k_1 = A_{1_1}$, $h_{a_e} < h_a$, $h_{b_e} < h_b$ and k_2 varying.

h_a	h_b	k_1	k_2	E_h	I_{hk}
$h_{a_e} < h_a$	$h_{b_e} < h_b$	$k_1 = A_{1_1}$	$k_2 > A_{2_1}$ $k_2 = A_{2_1}$ $A_{2_2} < k_2 < A_{2_1}$ $k_2 = A_{2_2}$ $k_2 < A_{2_2}$	$S^3 \cup S^3$	S^1 $S^1 \cup S^1$ $S^1 \cup \{S^1 \times S^1\}$ $S^1 \cup S^1$ $\emptyset$

The region where the action-angle variables can be defined is

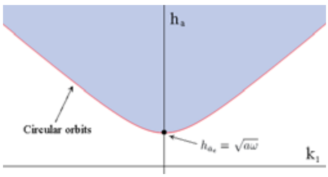

Figure 3: Region where $(J_{\rho_1}, J_{\theta_1})$ can be defined.

We can observe in this figure that the line $h_a = \sqrt{(a + k_1^2)\,\omega}$ corresponds with the extremes of the energy surface E_{h_a}, *i. e.* in these points we have circular orbits. For the case $k_1 = 0$, the energy takes the value $h_a = \sqrt{a\omega}$, *i. e.* we are in the equilibrium point of the system. Due to the topology of the energy surface, we can assure that in the gray region the orbits are going to be bounded, therefore, we will have periodic or quasiperiodic orbits.

Table 9: Topological classification of the sets E_h and I_{hk}, for all the values of a, b and ω, that we considered to be positive, $A_2 < k_1 < A_{1_1}$, $h_{a_e} < h_a$, $h_{b_e} < h_b$ and k_2 varying.

h_a	h_b	k_1	k_2	E_h	I_{hk}
$h_{a_e} < h_a$	$h_{b_e} < h_b$	$A_{1_2} < k_1 < A_{1_1}$	$k_2 > A_{2_1}$ $k_2 = A_{2_1}$ $A_{2_2} < k_2 < A_{2_1}$ $k_2 = A_{2_2}$ $k_2 < A_{2_2}$	$S^3 \cup S^3$	$\{S^1 \times S^1\}$ $\{S^1 \times S^1\} \cup S^1$ $\{S^1 \times S^1\} \cup \{S^1 \times S^1\}$ $\{S^1 \times S^1\} \cup S^1$ $\{S^1 \times S^1\}$

Table 10: Topological classification of the sets E_h and I_{hk}, for all the values of a, b and ω, that we considered to be positive, $k_1 = A_{1_2}$, $h_{a_e} < h_a$, $h_{b_e} < h_b$ and k_2 varying.

h_a	h_b	k_1	k_2	E_h	I_{hk}
$h_{a_e} < h_a$	$h_{b_e} < h_b$	$k_1 = A_{1_2}$	$k_2 > A_{2_1}$ $k_2 = A_{2_1}$ $A_{2_2} < k_2 < A_{2_1}$ $k_2 = A_{2_2}$ $k_2 < A_{2_2}$	$S^3 \cup S^3$	S^1 $S^1 \cup S^1$ $S^1 \cup \{S^1 \times S^1\}$ $S^1 \cup S^1$ S^1

4.2 Classification of periodic orbits of $\mathcal{H}_a$

4.2.1 Circular orbits

The circular orbits have constant radius, $\rho_1 = C$. These orbits are the solution of

$$\begin{aligned} \frac{d\rho_1}{dt} &= \frac{d\mathcal{H}_a}{dP_1} = P_1 = 0, \\ \frac{dP_1}{dt} &= -\frac{d\mathcal{H}_a}{d\rho_1} = \frac{\omega\rho_1^4 - A_1^2 - a}{\rho_1^3} = 0, \end{aligned}$$

where

$$\rho_1(t) = \sqrt[4]{\frac{a + A_1^2}{\omega}}, \quad \theta_1(\tau) = \frac{\sqrt{\omega}A_1}{\sqrt{a + A_1^2}}\, t.$$

They correspond to the maximum and minimum of the energy surface.

4.2.2 Rectilinear orbits

The rectilinear orbits have constant angle, $\theta_1 = C$, therefore

Table 11: Topological classification of the sets E_h and I_{hk}, for all the values of a, b and ω, that we considered to be positive, $k_1 < A_{1_2}$, $h_{a_e} < h_a$, $h_{b_e} < h_b$ and k_2 varying.

h_a	h_b	k_1	k_2	E_h	I_{hk}
			$k_2 > A_{2_1}$		$\emptyset$
			$k_2 = A_{2_1}$		S^1
$h_{a_e} < h_a$	$h_{b_e} < h_b$	$k_1 < A_{1_2}$	$A_{2_2} < k_2 < A_{2_1}$	$S^3 \cup S^3$	$S^1 \times S^1$
			$k_2 = A_{2_2}$		S^1
			$k_2 < A_{2_2}$		$\emptyset$

$$\frac{d\theta_1}{dt} = \frac{d\mathcal{H}_a}{dA_1} = \frac{A_1}{\rho_1^2} = 0,$$

where $A_1 = 0$.

The rectilinear trajectories are the solutions of the system

$$\frac{d\rho_1}{dt} = P_1 \quad \text{and} \quad \frac{dP_1}{dt} = \frac{a - \omega\rho_1^4}{\rho_1^3}.$$

4.2.3 Periodic and quasiperiodic orbits

An orbit of $\mathcal{H}_a$ is periodic if

$$n = \frac{n_1}{n_2} = \frac{\partial\mathcal{H}_a/\partial J_{\rho_1}}{\partial\mathcal{H}_a/\partial J_{\theta_1}} = \frac{2\sqrt{a + J_{\theta_1}^2}}{J_{\theta_1}} = \frac{p_1}{q_1}$$

is a rational number. In the other case we have quasiperiodic orbits.

Obtaining the value of J_{θ_1} respect to p_1 and q_1, we obtain the condition

$$J_{\theta_1} = \pm\frac{2\sqrt{a}q_1}{\sqrt{p_1^2 - 4q_1^2}},$$

where p_1 and q_1 are integer numbers.

Now, we return again to the region where the action-angle variables are defined to classify the periodic orbits.

In this figure, we can observe that, for $k_1 = 0$ (central vertical line), we obtain rectilinear orbits. These orbits are bounded too. And now, for each pair of integer numbers (p_1, q_1), we have two vertical lines where the system has periodic orbits.

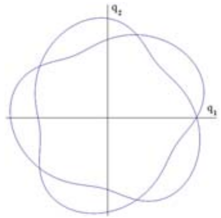

Figure 4: Example of periodic orbit for the parameters $a = 2$, $\omega = 1$, $p = 5$, $q = 2$, $J_{\rho_1} = \frac{1}{10}$ and $J_{\theta_1} = \frac{4}{3}$.

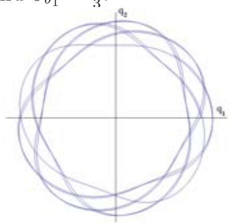

Figure 5: Example of quasiperiodic orbit for the parameters $a = 2$, $\omega = 1$, $p = 5$, $q = \sqrt{3.9}$, $J_{\rho_1} = \frac{1}{10}$ and $J_{\theta_1} = 1.88$.

4.3 Characterization of the orbits of $\mathcal{H}$

Remember that

$$\mathcal{H} = \mathcal{H}_a + \mathcal{H}_b,$$

where the result of $\mathcal{H}_a$ are analogous for $\mathcal{H}_b$.

4.3.1 Action-angle variables for $\mathcal{H}$

Taking into account that

$$\mathcal{H}_a = \sqrt{\omega}\left(2J_{\rho_1} + \sqrt{J_{\theta_1}^2 + a}\right),$$
$$\mathcal{H}_b = \sqrt{\omega}\left(2J_{\rho_2} + \sqrt{J_{\theta_2}^2 + b}\right),$$

we obtain the Hamiltonian $\mathcal{H}$ in action-angle variables

$$\mathcal{H} = \sqrt{\omega}\left(2J_{\rho_1} + \sqrt{J_{\theta_1}^2 + a} + 2J_{\rho_2} + \sqrt{J_{\theta_2}^2 + b}\right).$$

Now, the periodic orbits must verify

$$\frac{n_1}{n_2} = \frac{p_1}{q_1}, \quad \frac{n_2}{n_3} = \frac{q_1}{p_2}, \quad \frac{n_3}{n_4} = \frac{p_2}{q_2},$$

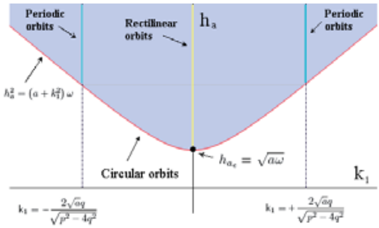

Figure 6: Characterization of the periodic orbits of the Hamiltonian $\mathcal{H}_a$.

where

$$n_1 = \frac{\partial \mathcal{H}_a}{\partial J_{\rho_1}}, \quad n_2 = \frac{\partial \mathcal{H}_a}{\partial J_{\theta_1}}, \quad n_3 = \frac{\partial \mathcal{H}_b}{\partial J_{\rho_2}}, \quad n_4 = \frac{\partial \mathcal{H}_b}{\partial J_{\theta_2}}$$

and p_1, q_1, p_2, q_2 are integer numbers.

We obtain the conditions

$$\begin{gathered} J_{\theta_1} = \pm \frac{2\sqrt{a}q_1}{\sqrt{p_1^2 - 4q_1^2}}, \quad J_{\theta_1} = \pm \frac{2\sqrt{a}q_1}{\sqrt{p_2^2 - 4q_1^2}}, \\ J_{\theta_2} = \pm \frac{2\sqrt{b}q_2}{\sqrt{p_2^2 - 4q_2^2}}, \end{gathered} \tag{27}$$

with p_1, q_1, p_2 and q_2 integer numbers.

We can characterize the periodic orbits of $\mathcal{H}$ considering the periodic orbits of $\mathcal{H}_a$ and $\mathcal{H}_b$.

Table 12: Characterization of the orbits of $\mathcal{H}_a$

Periodic orbits of $\mathcal{H}_a$	
Fixed point	$A_1 = 0$ and $h_a = +\sqrt{a\omega}$
Rectilinear orbits	$A_1 = 0$ and $h_a > +\sqrt{a\omega}$
Circular orbits	$\rho_1(t) = \sqrt[4]{\frac{a + A_1^2}{\omega}}$ and $h_a > +\sqrt{a\omega}$
Periodic orbits	$J_{\theta_1} = \pm \frac{2\sqrt{a}q_1}{\sqrt{p_1^2 - 4q_1^2}}$ and $h_a > +\sqrt{a\omega}$ with p_1, q_1 integer number

Table 13: Characterization of the orbits of $\mathcal{H}_b$

Periodic orbits of $\mathcal{H}_b$	
Fixed point	$A_2 = 0$ and $h_b = +\sqrt{b\omega}$
Rectilinear orbits	$A_2 = 0$ and $h_b > +\sqrt{b\omega}$
Circular orbits	$\rho_2(t) = \sqrt[4]{\dfrac{b + A_2^2}{\omega}}$ and $h_b > +\sqrt{b\omega}$
Periodic orbits	$J_{\theta_2} = \pm \dfrac{2\sqrt{b}q_1}{\sqrt{p_2^2 - 4q_2^2}}$ and $h_b > +\sqrt{b\omega}$ with p_2, q_2 integer number

Table 14: Characterization of the orbits of $\mathcal{H}$ res- pect to the orbits of $\mathcal{H}_a$ and $\mathcal{H}_b$. Note that, the periodic orbits must verify the conditions (27).

Periodic orbits of $\mathcal{H}$						
$\mathcal{H}_a$		$\mathcal{H}_b$		$\mathcal{H}_a$		$\mathcal{H}_b$
$\{h_{a_e}\}$	$\times$	$\{h_{b_e}\}$		S^1	$\times$	$\{h_{b_e}\}$
$\{h_{a_e}\}$	$\times$	$[\rho_2^1, \rho_2^2]$		S^1	$\times$	$[\rho_2^1, \rho_2^2]$
$\{h_{a_e}\}$	$\times$	S^1		S^1	$\times$	S^1
$\{h_{a_e}\}$	$\times$	T^2		S^1	$\times$	T^2
$[\rho_1^1, \rho_1^2]$	$\times$	$\{h_{b_e}\}$		T^2	$\times$	$\{h_{b_e}\}$
$[\rho_1^1, \rho_1^2]$	$\times$	$[\rho_2^1, \rho_2^2]$		T^2	$\times$	$[\rho_2^1, \rho_2^2]$
$[\rho_1^1, \rho_1^2]$	$\times$	S^1		T^2	$\times$	S^1
$[\rho_1^1, \rho_1^2]$	$\times$	T^2		T^2	$\times$	T^2

5 The Biparametric Oscillator and its Integration

Having already shown the relation of both families of ring-shaped systems with the oscillator, we focus now the integration of our oscillator. The Hamiltonian function (1) defines an integrable system, due to θ_1 and θ_2 are cyclic, thus A_1 and A_2 are first integrals. In other words, the system is made separable in two subsystems of 1-DOF, defined by the Hamiltonian functions (5) and (6). We integrate the differential system defined by (1) immediately. Let

$$\tilde{Q} = 2\mathcal{H}_a - \omega\rho_1^2 - \frac{A_1^2 + a}{\rho_1^2},$$

and consider the quantities a_1 and b_1 defined by

$$a_1 + b_1 = \sqrt{2\left(\frac{\mathcal{H}_a}{\omega} + \sqrt{\frac{A_1^2 + a}{\omega}}\right)},$$

$$a_1 - b_1 = \sqrt{2\left(\frac{\mathcal{H}_a}{\omega} - \sqrt{\frac{A_1^2 + a}{\omega}}\right)},$$

then we may write

$$\tilde{Q} = \frac{\omega}{\rho_1^2}(a_1^2 - \rho_1^2)(\rho_1^2 - b_1^2).$$

We see that the equation $\tilde{Q} = 0$ has real roots when $\mathcal{H}_a \geq \sqrt{\omega\,(A_1^2 + a)}$. The system defined by $\mathcal{H}_a$ reduces to

$$\dot{\rho}_1 = P_1 = \sqrt{\tilde{Q}}, \qquad \dot{\alpha}_1 = \frac{A_1}{\rho_1^2},$$

i.e., to two quadratures. From the first quadrature we obtain immediately

$$\rho_1(\tau) = \sqrt{a_1^2 \cos^2(\sqrt{\omega}\tau) + b_1^2 \sin(\sqrt{\omega}\tau)}$$

and the angle $\theta_1(\tau)$, after some computations, is given by

$$\theta_1(\tau)=\frac{k_1}{\sqrt{a+k_1^2}}\left[\arctan\left(\sqrt{\frac{h_a^2-\left(a+k_1^2\right)\omega}{\left(a+k_1^2\right)\omega}}\right) + \arctan\left(\frac{h_a\tan(\sqrt{\omega}\,\tau)-\sqrt{h_a^2-\left(a+k_1^2\right)\omega}}{\sqrt{\left(a+k_1^2\right)\omega}}\right)\right],$$

for $0 \leq \tau < \frac{\pi}{2}$. And

$$\theta_1(\tau)=\frac{k_1}{\sqrt{a+k_1^2}}\left[\arctan\left(\sqrt{\frac{h_a^2-\left(a+k_1^2\right)\omega}{\left(a+k_1^2\right)\omega}}\right) + \arctan\left(\frac{h_a\tan(\sqrt{\omega}\,\tau)-\sqrt{h_a^2-\left(a+k_1^2\right)\omega}}{\sqrt{\left(a+k_1^2\right)\omega}}\right) + n\pi\right],$$

for $\frac{\pi}{2}(2n-1) \leq \tau < \frac{\pi}{2}(2n+1)$, with $n \geq 1$

Similar expressions are obtained for ρ_2 and θ_2. With the quantities a_2 and b_2 given by

$$a_2 + b_2 = \sqrt{2\left(\frac{\mathcal{H}_b}{\omega} + \sqrt{\frac{A_2^2 + b}{\omega}}\right)},$$

$$a_2 - b_2 = \sqrt{2\left(\frac{\mathcal{H}_b}{\omega} - \sqrt{\frac{A_2^2 + b}{\omega}}\right)},$$

we have

$$\rho_2(\tau) = \sqrt{a_2^2 \cos^2(\sqrt{\omega}\tau) + b_2^2 \sin(\sqrt{\omega}\tau)},$$

where $\rho_2(0) = b_2$ and the angle $\theta_2 = \theta_2(\tau)$, after some computations, is given by

$$\theta_2(\tau)=\frac{k_2}{\sqrt{b+k_2^2}}\left[\arctan\left(\sqrt{\frac{h_b^2-\left(b+k_2^2\right)\omega}{\left(b+k_2^2\right)\omega}}\right)\right.$$
$$\left.+\arctan\left(\frac{h_b\tan(\sqrt{\omega}\;\tau)-\sqrt{h_b^2-\left(b+k_2^2\right)\omega}}{\sqrt{\left(b+k_2^2\right)\omega}}\right)\right],$$

for $0 \leq \tau < \frac{\pi}{2}$. And

$$\theta_2(\tau)=\frac{k_2}{\sqrt{b+k_2^2}}\left[\arctan\left(\sqrt{\frac{h_b^2-\left(b+k_2^2\right)\omega}{\left(b+k_2^2\right)\omega}}\right)\right.$$
$$\left.+\arctan\left(\frac{h_b\tan(\sqrt{\omega}\;\tau)-\sqrt{h_b^2-\left(b+k_2^2\right)\omega}}{\sqrt{\left(b+k_2^2\right)\omega}}\right)+n\pi\right],$$

for $\frac{\pi}{2}(2n-1) \leq \tau < \frac{\pi}{2}(2n+1)$, with $n \geq 1$.

Finally replacing such values in the previous equations we obtain the q_i variables.

6 Transformation of Some Orbits of $\mathcal{H}$ to the 3-D System of Hartmann

We make use of the canonical transformation

After some computations, the solutions of the Hamiltonian with potential V_1 are

$$x_1 = r\sin\phi\cos\lambda \quad x_2 = r\sin\phi\sin\lambda \quad x_3 = r\cos\phi$$

Now, we can show how some of the periodic orbits of $\mathcal{H}$ are transformed into the Hartmann system.

7 Stability of Periodic Orbits

In this section we study the stability of the different periodic orbits of the system (17). The variational equations of the previous system are

$$\begin{pmatrix} \dot{\delta\rho} \\ \dot{\delta\theta} \\ \dot{\delta P} \\ \dot{\delta A} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -\dfrac{2A^2}{\rho^3} & 0 & 0 & \dfrac{1}{\rho^2} \\ -\left(\dfrac{3(a+A^2)}{\rho^4}+\omega\right) & 0 & 0 & -\dfrac{2A^2}{\rho^3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta\theta \\ \delta P \\ \delta A \end{pmatrix} \tag{28}$$

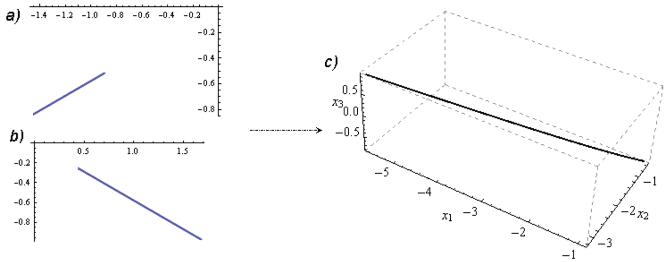

Figure 7: a) Rectilinear orbit of the subsystem $\mathcal{H}_a$ for the parameters $a = 2$, $\omega = 1$, $p_1 = 5$, $q_1 = 2$, $J_{\rho_1} = \frac{1}{10}$ and $J_{\theta_1} = 0$. b) Rectilinear orbit of the subsystem $\mathcal{H}_b$ for the parameters $b = 1$, $\omega = 1$, $p_2 = 5$, $q_2 = 1$, $J_{\rho_2} = \frac{1}{2}$ and $J_{\theta_2} = 0$. c) Rectilinear orbit $\times$ Rectilinear orbit reduced to the system of Hartmann.

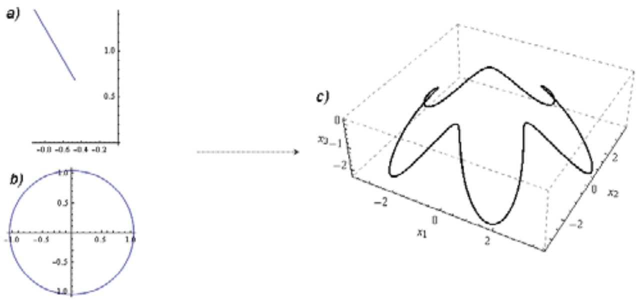

Figure 8: a) Rectilinear orbit of the subsystem $\mathcal{H}_a$ for the parameters $a = 3$, $\omega = 1$, $p_1 = 5$, $q_1 = 2$, $J_{\rho_1} = \frac{1}{5}$ and $J_{\theta_1} = 0$ b) Circular orbit of the subsystem $\mathcal{H}_b$ for the parameters $b = 1$, $\omega = 1$, $p_2 = 5$, $q_2 = 1$, $J_{\rho_2} = 0$ and $J_{\theta_2} = \frac{2}{\sqrt{21}}$. c) Rectilinear orbit $\times$ Circular orbit reduced to the system of Hartmann.

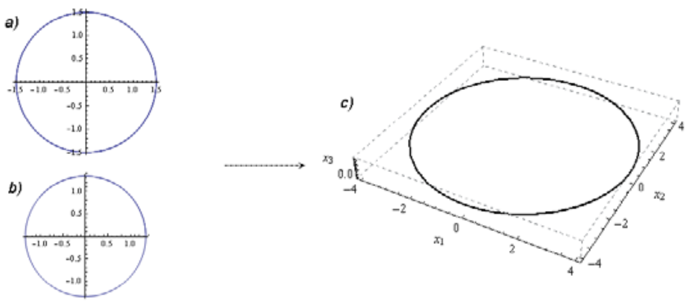

Figure 9: a) Circular orbit of the subsystem $\mathcal{H}_a$ for the parameters $a = 3$, $\omega = 1$, $p_1 = 5$, $q_1 = 2$, $J_{\rho_1} = 0$ and $J_{\theta_1} = \frac{1}{2}$ b) Circular orbit of the subsystem $\mathcal{H}_b$ for the parameters $b = 1$, $\omega = 1$, $p_2 = 5$, $q_2 = 1$, $J_{\rho_2} = 0$ and $J_{\theta_2} = 2$. c) Circular orbit $\times$ Circular orbit reduced to the system of Hartmann.

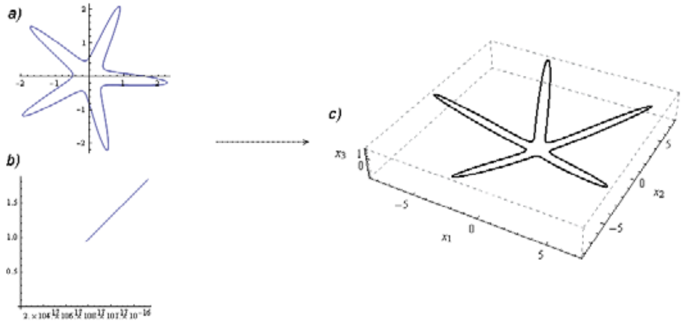

Figure 10: a) Rectilinear orbit of the subsystem $\mathcal{H}_a$ for the parameters $a=3$, $\omega=1$, $p_1=5$, $q_1=2$, $J_{\rho_1}=\frac{1}{5}$ and $J_{\theta_1}=0$. b) Periodic orbit of the subsystem $\mathcal{H}_b$ for the parameters $b=1$, $\omega=1$, $p_2=5$, $q_2=1$, $J_{\rho_2}=\frac{4}{5}$ and $J_{\theta_2}=\frac{2}{\sqrt{21}}$. c) Rectilinear orbit $\times$ Periodic orbit reduced to the system of Hartmann.

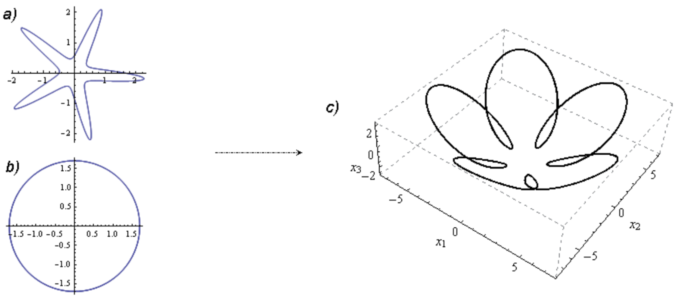

Figure 11: a) Circular orbit of the subsystem $\mathcal{H}_a$ for the parameters $a=3$, $\omega=1$, $p_1=5$, $q_1=2$, $J_{\rho_1}=0$ and $J_{\theta_1}=\frac{4}{\sqrt{3}}$ b) Periodic orbit of the subsystem $\mathcal{H}_b$ for the parameters $b=1$, $\omega=1$, $p_2=5$, $q_2=1$, $J_{\rho_2}=\frac{4}{5}$ and $J_{\theta_2}=\frac{2}{\sqrt{21}}$. c) Rectilinear orbit $\times$ Circular orbit reduced to the system of Hartmann.

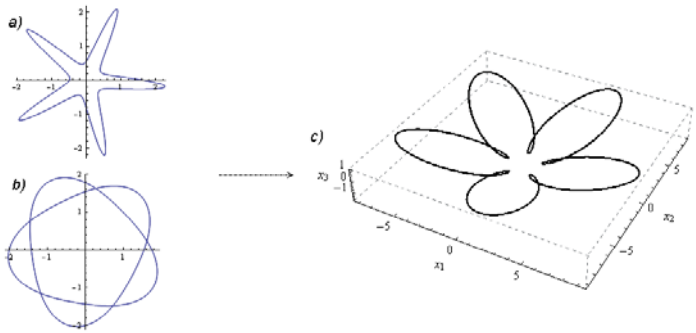

Figure 12: a) Periodic orbit of the subsystem $\mathcal{H}_a$ for the parameters $a = 3$, $\omega = 1$, $p_1 = 5$, $q_1 = 2$, $J_{\rho_1} = 0$ and $J_{\theta_1} = \frac{4}{\sqrt{3}}$ b) Periodic orbit of the subsystem $\mathcal{H}_b$ for the parameters $b = 1$, $\omega = 1$, $p_2 = 5$, $q_2 = 1$, $J_{\rho_2} = \frac{4}{5}$ and $J_{\theta_2} = \frac{2}{\sqrt{21}}$. c) Periodic orbit $\times$ Circular orbit reduced to the system of Hartmann.

The previous system in the case of a periodic rectilinear orbit $\gamma(t) = (\rho, \theta_0, \dot{\rho}, 0)$ with $\rho(t) = \sqrt{h - \sqrt{(h^2 - a\omega)}\sin(2t)}$ simplifies to

$$\begin{aligned} \dot{\delta\rho} &= \delta P \\ \dot{\delta\theta} &= \frac{1}{\rho^2}\delta A \\ \dot{\delta P} &= -\left(\frac{3a}{\rho^4} + \omega\right)\delta\rho \\ \delta A &= cte \end{aligned}$$

The previous system is integrable by quadratures. We won't make the complete integration of this system. We will only keep in mind that

$$\delta\theta(t) = \delta\theta_0 + cte \int_0^t \frac{ds}{h - \sqrt{(h^2 - a\omega)}\sin(2s)}$$

is not bounded, and we conclude that the *rectilinear periodic solutions are unstable.*

In the case of the circular periodic solutions $\gamma(t) = (\sqrt[4]{\frac{a+k^2}{\omega}}, \frac{\sqrt{\omega}kt}{\sqrt{a+k^2}}, 0, k)$ we have that (28) is

$$\begin{pmatrix} \dot{\delta\rho} \\ \dot{\delta\theta} \\ \dot{\delta P} \\ \dot{\delta A} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -\frac{2k^2}{\rho^3} & 0 & 0 & \frac{1}{\rho^2} \\ -4\omega & 0 & 0 & -\frac{2A^2}{\rho^3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta\theta \\ \delta P \\ \delta A \end{pmatrix}$$

a linear system with constants coefficients. The characteristic polynomial is $p(s) = s^2(s^2+4\omega)$. The stability index of the system K (see [4] for details) are $K = 2\cos(4\pi\sqrt{\omega})$ then these orbits are *linearly stable*. Observe that when $\omega = \frac{(1+2k)^2}{64}$ bifurcations of these circular orbits take place.

In the case of the remaining periodic orbits, the stability of the system (28) are directly related to the study of the stability of the following equation

$$\ddot{\delta\rho} + \left(\frac{3(a+k^2)}{\rho^4} + \omega\right)\delta\rho = 0$$

with $\rho(t) = \sqrt{\frac{1}{\omega}\left(h - \sqrt{h^2-\omega(a+k^2)}\sin(2t\sqrt{\omega})\right)}$, $A = k$.

This linear differential equation of second order with periodic coefficients will be studied numerically by means of the Floquet theory (see [11] for details). For example for the case in that $k = \frac{2\sqrt{21}}{21}$, $\omega = 1$, $a = 1$, we are interested in the stability of this family of orbits.

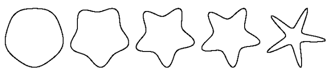

Figure 13: Orbits for $k = \frac{2\sqrt{21}}{21}$, $\omega = 1$, $a = 1$ with h varying. Observe you that when h increases the orbits are very 'starry'.

Computing numerically the trace of the monodromy matrix we obtain the following diagram of stability.

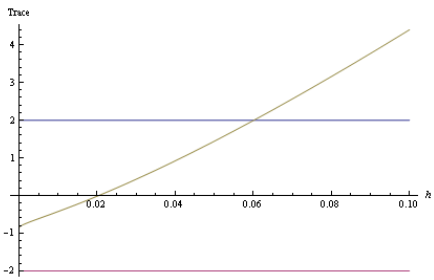

Figure 14: Trace of the monodromy matrix varying h. For $0 \le h < 0.0601811$ this orbits are linearly stable. For $h \ge 0.0601811$ the orbit are unstable.

Conclusions

In the present paper we have established the relation of two families of ring-shaped type systems with a 4-D isotropic oscillator. This allows a unified treatment which is of interest both in quantum and classical studies. By means of this relation we have deduced some results on the periodic and quasiperiodic orbits of the family of 3-D systems related with generalization of the Hartmann model for the Benzene molecule.

First, we have shown the relation of the Hamiltonian system defined by (1) and the generalized Hartmann potentials defined by the potential V_1.

We have made the qualitative study of the Hamiltonian flow (1). We have described the Hamiltonian dynamics of $\mathcal{H}$. A complete topological classification of the invariant sets I_h, I_k and I_{hk} is given by means of Liouville–Arnold theorem and some specifics techniques. The action–angle variables have been obtained and the region of the phase space where they can be defined. The system is made separable in two subsystems of 1-DOF, defined by the Hamiltonian functions. Due to the structure of the subsystem (5) and (6) it has been enough to study $\mathcal{H}_a$ to deduce the corresponding results for $\mathcal{H}_b$. Finally, due to (7) we have obtained the complete study of $\mathcal{H}$.

We have classified the periodic orbits of ring-shaped potentials. In order to carry out this characterization it has been necessary to calculate the action-angle variables for this system. First, we have calculated the action-angle variables for the subsystem $\mathcal{H}_a$ and, for the structure of both subsystems, the expression of $\mathcal{H}$ in these variables is obtained.

The periodic orbits of $\mathcal{H}_a$ are circular orbits with constant radius and they correspond to the maximum and minimum of the energy surface; the rectilinear orbits, have constant angle ($\theta_1 = C$); the periodic and quasiperiodic orbits must verify or not a particular condition that depends on $n_1 = \partial\mathcal{H}_a/\partial J_{\rho_1}$ and $n_2 = \partial\mathcal{H}_a/\partial J_{\theta_1}$. If the quotient of n_1/n_2 is a rational number we obtain periodic orbits, in the other case we have quasiperiodic orbits.

Having already shown the relation of both families of ring-shaped systems with the oscillator, we have faced on the integration of our oscillator correcting a mathematical mistake stated in [5]. The Hamiltonian function (1) defines an integrable system, due to θ_1 and θ_2 are cyclic, thus A_1 and A_2 are first integrals. In other words, the system is made separable in two subsystems of 1-DOF, defined by the Hamiltonian functions (5) and (6). We have integrated the differential system defined by (1) immediately, following closely the steps of the classical Deprit's method. Later, we have studied the transformation of some orbits of $\mathcal{H}$ to the 3-D system of Hartmann anf finally we conclude with the study of the stability of the periodic and quasiperiodic orbits of the system. After this study the problem of the characterization periodic dynamics of this system is closed in the line of results presented in the papers [6] or [7].

Acknowledgements

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