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Erratum to 'The trees on $n \geq 9$ vertices with the first to seventeenth greatest Wiener indices are chemical trees'*

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Abstract: Deng once determined the trees on $n \geq 9$ vertices with the first to seventeenth greatest Wiener indices [H. Y. Deng, The trees on $n \geq 9$ vertices with the first to seventeenth greastest Wiener indices are chemical trees, MATCH Comm. Math. Comput. Chem. 57(2007) 393-402]. Unfortunately, some mistakes are found in Deng's paper. This paper will present a correct order of the first to fifteenth greatest Wiener indices for trees on $n \geq 28$.

1 Introduction

Throughout this paper, we only concern with connected, undirected simple graphs. Let $N(u)$ be the first neighbor vertex set of u, then $d(u) = |N(u)|$ is called the degree of u. As usual, P_n and S_n denote the path and star of order n, respectively.

The *distance* $d_G(u, v)$ between the vertices u and v of the graph G is equal to the length of (number of edges in) the shortest path that connects u and v. There are two important

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graph-based structure-descriptors, called Wiener index and hyper-Wiener index, based on distances in a graph. The *Wiener index* $W(G)$ is denoted by [1]

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v),$$

and the *hyper-Wiener index* $WW(G)$ is defined as [2]

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)^2.$$

The Wiener index and hyper-Wiener index are much studied in the chemical literatures because of their important applications in chemistry. Recently, the Wiener index and hyper-Wiener index gained much popularity. More and more mathematicians and chemists became interested in them and devoted themselves to the study. The mathematical properties of Wiener index and hyper-Wiener index with their applications in chemistry can be referred to [3-10] and the references cited therein.

We now introduce some more key notations. Recall that a vertex u of a tree T is called a *branching point* of T if $d(u) \geq 3$. Furthermore, u is said to be an *out-branching point* if at most one of the components of $T - u$ is not a path; otherwise, u is an *in-branching point* of T.

Let $T(n; n_1, n_2, ..., n_m)$ denote the starlike tree of order n obtained by inserting $n_1 - 1$, ..., $n_m - 1$ vertices into m edges of the star S_{m+1} of order $m + 1$ respectively, where $n_1 + \cdots + n_m = n - 1$. Note that any tree with only one branching point is a starlike tree.

If T is a tree of order n with exactly two branching points u_1 and u_2, with $d(u_1) = r$ and $d(u_2) = t$. The orders of $r-1$ components, which are paths, of $T - u_1$ are p_1, ..., p_{r-1}, the order of the component which is not a path of $T - u_1$ is $p_r = n - p_1 - \cdots - p_{r-1} - 1$. The orders of $t - 1$ components, which are paths, of $T - u_2$ are q_1, ..., q_{t-1}, the order of the component which is not a path of $T - u_2$ is $q_t = n - q_1 - \cdots - q_{t-1} - 1$. We denote this tree by $T(n; p_1, ..., p_{r-1}; q_1, ..., q_{t-1})$, where $r \leq t$, $p_1 \geq \cdots \geq p_{r-1}$ and $q_1 \geq \cdots \geq q_{t-1}$.

Gutman first obtained the trees on n vertices with the smallest and greatest hyper-Wiener index (i.e., the star and path) in [11]. Very recently, the trees with the seven smallest and the fifteen greatest hyper-Wiener indices were determined in [12], respectively. Also, among all connected graphs of order n ($n > 2k$), the first up to $(k + 1)$-th smallest Wiener indices and the first up to $(k + 1)$-th smallest hyper-Wiener indices are determined in [13], respectively. Gutman et al. also considered the similar order of Wiener index among the starlike trees in [14]. After then, the first up to fifteenth smallest Wiener indices among trees of order n were identified in [15]. Recently, Deng considered the order of greatest Wiener indices among trees and he once obtained

Theorem A [16] Suppose $n \geq 9$ and T is a tree of order n. Then, $W(T(n; n-3, 1, 1)) > W(T(n; n-4, 2, 1)) > W(T(n; 1, 1; 1, 1)) > W(T(n; n-5, 3, 1)) > W(T(n; n-4, 1, 1, 1)) \geq W(T(n; 1, 1; 2, 1)) > W(T(n; n-6, 4, 1)) > W(T(n; n-5, 2, 2)) > W(T(n; 1, 1; 3, 1)) > W(T(n; 2, 1; 2, 1)) > W(T(n; 1, 1; 1, 1, 1)) > W(T(n; n-7, 5, 1)) > W(T(n; 1, 1; 4, 1)) > W(T(n; n-5, 2, 1, 1)) \geq W(T(n; 1, 1; 2, 2)) \geq W(T(n; 2, 1; 3, 1)) > W(T_D) \geq W(T)$, where T_D is shown in Fig. 1.

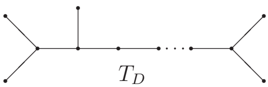

Fig. 1. The tree T_D.

Unfortunately, as the following example shown, Theorem A is not correct. In the sequel, $\binom{n+1}{3}$ is written as C^3_{n+1}.

Example 1.1. By an elementary computation (or the application of Lemma 2.4), it follows that $W(T(n; 1, 1; n-6, 1)) = C^3_{n+1} - 5n + 27$ and $W(T_D) = C^3_{n+1} - 5n + 21$. Thus, $W(T(n; 1, 1; n-6, 1)) > W(T_D)$, a contradiction to Theorem A.

Thus, we give a correct order of the greatest Wiener indices of trees in this paper, namely

Theorem 1.1 *Suppose $n \geq 28$ and T is a tree of order n. Then,* $W(P_n) > W(T(n; n-3, 1, 1)) > W(T(n; n-4, 2, 1)) > W(T(n; 1, 1; 1, 1)) > W(T(n; n-5, 3, 1)) > W(T(n; n-4, 1, 1, 1)) = W(T(n; 1, 1; 2, 1)) > W(T(n; n-6, 4, 1)) > W(T(n; n-5, 2, 2)) > W(T(n; 1, 1; n-5, 1)) = W(T(n; 1, 1; 3, 1)) > W(T(n; 2, 1; 2, 1)) > W(T(n; 1, 1; 1, 1, 1)) > W(T(n; n-7, 5, 1)) > W(T(n; 1, 1; n-6, 1)) = W(T(n; 1, 1; 4, 1)) > W(T(n; n-5, 2, 1, 1)) = W(T(n; 1, 1; 2, 2)) = W(T(n; 2, 1; 3, 1)) > W(T_D) > W(T)$.

Remark. By comparing with Theorem A and Theorem 1.1, two trees, say $T(n; 1, 1; n-5, 1)$ and $T(n; 1, 1; n-6, 1)$, are left out in [16].

2 The proof of Theorem 1.1

Given a simple and undirected graph $G = (V, E)$. Let $G - u$ (resp. $G - uv$) denote the graph obtained from G by deleting the vertex $u \in V(G)$ (resp. the edge $uv \in E(G)$). Similarly, $G + uv$ is a graph yielded from G by adding an edge $uv \notin E(G)$, where $u, v \in V(G)$.

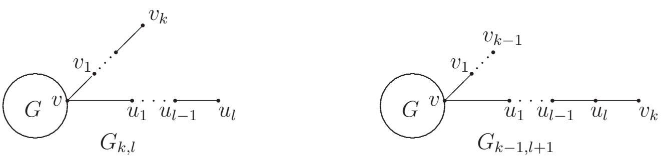

Fig. 2. The graphs $G_{k,l}$ and $G_{k-1,l+1}$.

Suppose v is a vertex of graph G. As shown in Fig. 2, let $G_{k,l}$ ($l \geq k \geq 1$) be the graph obtained from G by attaching two new paths P: $v(= v_0)v_1v_2 \cdots v_k$ and Q: $v(= u_0)u_1u_2 \cdots u_l$ of length k and l, respectively, at v, where v_1, v_2, ..., v_k and u_1, u_2, ..., u_l are distinct new vertices. Let $G_{k-1,l+1} = G_{k,l} - v_{k-1}v_k + u_lv_k$.

Lemma 2.1 *Suppose G is a connected graph on $n \geq 2$ vertices, or an isolated vertex. If $l \geq k \geq 1$, then $W(G_{k,l}) \leq W(G_{k-1,l+1})$, the equality holds if and only if G is an isolated vertex.*

Proof. It is easy to see that

$$W(G_{k-1,l+1}) - W(G_{k,l}) = \sum_{w \in V(G_{k-1,l+1})} d_{G_{k-1,l+1}}(w, v_k) - \sum_{w \in V(G_{k,l})} d_{G_{k,l}}(w, v_k). \tag{1}$$

Let $V_1 = V(G) \setminus \{v\}$, then $V(G_{k,l}) \setminus V_1 = V(G_{k-1,l+1}) \setminus V_1$. Let $V_2 = V(G_{k,l}) \setminus V_1$. Clearly,

$$\sum_{w \in V(G_{k,l})} d_{G_{k,l}}(w, v_k) = \sum_{w \in V_1} d_{G_{k,l}}(w, v_k) + \sum_{w \in V_2} d_{G_{k,l}}(w, v_k). \tag{2}$$

$$\sum_{w \in V(G_{k-1,l+1})} d_{G_{k-1,l+1}}(w, v_k) = \sum_{w \in V_1} d_{G_{k-1,l+1}}(w, v_k) + \sum_{w \in V_2} d_{G_{k-1,l+1}}(w, v_k). \tag{3}$$

Note that the subgraph of $G_{k,l}$ induced by V_2 is a path of length $k+l$, which is isomorphic to the subgraph of $G_{k-1,l+1}$ induced by V_2, thus

$$\sum_{w \in V_2} d_{G_{k,l}}(w, v_k) = \sum_{w \in V_2} d_{G_{k-1,l+1}}(w, v_k). \tag{4}$$

Therefore, by combining equalities (1)-(4), we have

$$W(G_{k-1,l+1}) - W(G_{k,l}) = \sum_{w\in V_1} d_{G_{k-1,l+1}}(w,v_k) - \sum_{w\in V_1} d_{G_{k,l}}(w,v_k)$$

$$= \sum_{w\in V_1}(d_{G_{k-1,l+1}}(w,v_k) - d_{G_{k,l}}(w,v_k)). \quad (5)$$

If G is an isolated vertex, then $V_1 = \emptyset$. By equality (5), it follows that $W(G_{k-1,l+1}) = W(G_{k,l})$. If G is not an isolated vertex, since $l \geq k$, then $d_{G_{k-1,l+1}}(w,v_k) > d_{G_{k,l}}(w,v_k)$ holds for every $w \in V_1$. Thus, the result follows from equality (5). ∎

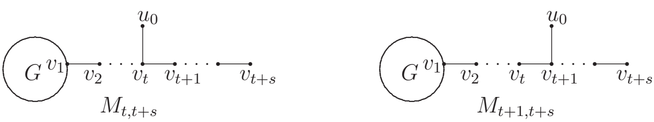

Fig. 3. The graphs $M_{t,t+s}$ and $M_{t+1,t+s}$.

Suppose v_1 is a vertex of graph G, and v_2, ..., v_{t+s}, u_0 are distinct new vertices (not in G). Let G' be the graph obtained from G by attaching a new path P: $v_1v_2\cdots v_{t+s}$. Let $M_{t,t+s} = G' + v_tu_0$ and $M_{t+i,t+s} = G' + v_{t+i}u_0$, where $1 \leq i \leq s$. For instance, $M_{t,t+s}$ and $M_{t+1,t+s}$ are depicted in Fig. 3.

Lemma 2.2 *Suppose G is a connected graph on $n \geq 2$ vertices, or an isolated vertex. If $t \geq s \geq 1$, then $W(M_{t,t+s}) \leq W(M_{t+1,t+s})$. Moreover, the equality holds if and only if $t = s$ and G is an isolated vertex.*

Proof. For convenience, sometimes we write $M_{t,t+s}$ as M, and $M_{t+1,t+s}$ as M' in the proof of this lemma. Let $V_1 = V(G) \setminus \{v_1\}$, then $V(M_{t,t+s}) \setminus V_1 = V(M_{t+1,t+s}) \setminus V_1$. Let $V_2 = V(M_{t,t+s}) \setminus V_1$. Thus,

$$W(M_{t+1,t+s}) - W(M_{t,t+s}) = \sum_{w\in V(M')} d_{M'}(w,u_0) - \sum_{w\in V(M)} d_M(w,u_0)$$

$$= \sum_{w\in V_1}(d_{M'}(w,u_0) - d_M(w,u_0)) + \sum_{w\in V_2} d_{M'}(w,u_0) - \sum_{w\in V_2} d_M(w,u_0). \quad (6)$$

Note that $d_{M'}(w,u_0) > d_M(w,u_0)$ holds for every $w \in V_1$, and

$$\sum_{w\in V_2} d_{M'}(w,u_0) - \sum_{w\in V_2} d_M(w,u_0) = t+1-(s+1) = t-s \geq 0.$$

Thus, the result follows by equality (6). ∎

From Lemma 2.2, it immediately follows that

Proposition 2.1 *Suppose G is a connected graph on $n \geq 2$ vertices, or an isolated vertex. If $t \geq s \geq 1$, then $W(M_{t,t+s}) \leq W(M_{t+i,t+s})$, where $1 \leq i \leq s$.*

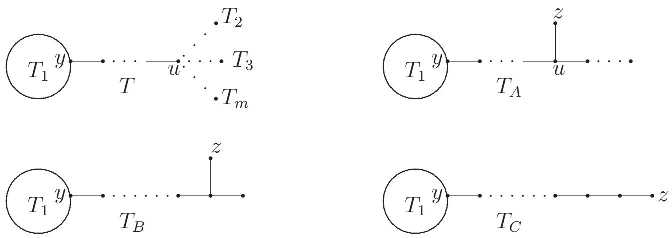

Fig. 4. The trees T, T_A, T_B and T_C.

For convenience, we introduce a transfer operation: $T \rightarrow T_A \rightarrow T_B \rightarrow T_C$, as shown in Fig. 4, where T is a tree of order n, u is an out-branching point of T, $d(u) = m$, and all the components T_1, T_2, ... T_m of $T - u$ except T_1 are paths.

Lemma 2.3 [12,16] *Let u be an out-branching point of a tree T of order n, $d(u) = m$ ($m \geq 3$), and let all components T_1, T_2, ..., T_m of $T - u$ except T_1 be paths. Then,*

$$W(T) \leq W(T_A) \leq W(T_B) < W(T_C),$$

and $W(T) = W(T_A)$ (or $W(T_B)$) if and only if $T = T_A$ (or T_B).

Proof. By Lemma 2.1, it is easy to see that $W(T) \leq W(T_A)$ with the equality holding if and only if $T = T_A$. Moreover, Lemma 2.1 implies that $W(T_B) < W(T_C)$. Next we shall prove that $W(T_A) \leq W(T_B)$ with the equality holding if and only if $T_A = T_B$.

Let T_u denote the component of $T_A - y$, which contains u. Set $V_1 = V(T_u) \cup \{y\}$. Then, $V(T_A) \setminus V_1 = V(T_B) \setminus V_1$. Let $V_2 = V(T_A) \setminus V_1$. It is easy to see that

$$\begin{aligned}
&\sum_{\{w,v\} \subseteq V(T_B)} d_{T_B}(w,v) - \sum_{\{w,v\} \subseteq V(T_A)} d_{T_A}(w,v) \\
&= \sum_{w \in V(T_B)} d_{T_B}(w,z) - \sum_{w \in V(T_A)} d_{T_A}(w,z) \\
&= \sum_{w \in V_1} d_{T_B}(w,z) - \sum_{w \in V_1} d_{T_A}(w,z) + \sum_{w \in V_2} (d_{T_B}(w,z) - d_{T_A}(w,z)). \qquad (7)
\end{aligned}$$

Note that $\sum_{w \in V_1} d_{T_B}(w,z) \geq \sum_{w \in V_1} d_{T_A}(w,z)$, and $d_{T_B}(w,z) \geq d_{T_A}(w,z)$ holds for every $w \in V_2$, then the conclusion follows from equality (7). ∎

Lemma 2.4 [16,17] *Let T be a tree of order n, u_1, u_2, ..., u_k be the all branching points of T, $d(u_i) = m_i$ $(i = 1, 2, ..., k)$, T_{i1}, T_{i2}, ..., T_{im}, be the components of $T - u_i$, and the order of T_{ij} is equal to n_{ij} $(j = 1, 2, ..., m_i;\ i = 1, 2, ..., k)$. Then,*

$$W(T) = C_{n+1}^3 - \sum_{i=1}^{k} \sum_{1 \le p < q < r \le m_i} n_{ip} n_{iq} n_{ir},$$

where $n_{i1} + n_{i2} + \cdots + n_{im_i} = n - 1$, $i = 1, 2, ..., k$.

By Lemma 2.4, we have

$W(P_n) = C_{n+1}^3$,

$W(T(n; n-3, 1, 1)) = C_{n+1}^3 - n + 3$,

$W(T(n; n-4, 2, 1)) = C_{n+1}^3 - 2n + 8$,

$W(T(n; 1, 1; 1, 1)) = C_{n+1}^3 - 2n + 6$,

$W(T(n; n-5, 3, 1)) = C_{n+1}^3 - 3n + 15$,

$W(T(n; n-4, 1, 1, 1)) = W(T(n; 1, 1; 2, 1)) = C_{n+1}^3 - 3n + 11$,

$W(T(n; n-6, 4, 1)) = C_{n+1}^3 - 4n + 24$,

$W(T(n; n-5, 2, 2)) = C_{n+1}^3 - 4n + 20$,

$W(T(n; 1, 1; n-5, 1)) = W(T(n; 1, 1; 3, 1)) = C_{n+1}^3 - 4n + 18$,

$W(T(n; 2, 1; 2, 1)) = C_{n+1}^3 - 4n + 16$,

$W(T(n; 1, 1; 1, 1, 1)) = C_{n+1}^3 - 4n + 14$,

$W(T(n; n-7, 5, 1)) = C_{n+1}^3 - 5n + 35$,

$W(T(n; 1, 1; n-6, 1)) = W(T(n; 1, 1; 4, 1)) = C_{n+1}^3 - 5n + 27$,

$W(T(n; n-5, 2, 1, 1)) = W(T(n; 2, 1; 3, 1)) = W(T(n; 1, 1; 2, 2)) = C_{n+1}^3 - 5n + 23$,

$W(T_D) = C_{n+1}^3 - 5n + 21$.

Thus, we have

Lemma 2.5 *If $n \ge 28$, then $W(P_n) > W(T(n; n-3, 1, 1)) > W(T(n; n-4, 2, 1)) > W(T(n; 1, 1; 1, 1)) > W(T(n; n-5, 3, 1)) > W(T(n; n-4, 1, 1, 1)) = W(T(n; 1, 1; 2, 1)) > W(T(n; n-6, 4, 1)) > W(T(n; n-5, 2, 2)) > W(T(n; 1, 1; n-5, 1)) = W(T(n; 1, 1; 3, 1)) > W(T(n; 2, 1; 2, 1)) > W(T(n; 1, 1; 1, 1, 1)) > W(T(n; n-7, 5, 1)) > W(T(n; 1, 1; n-6, 1)) = W(T(n; 1, 1; 4, 1)) > W(T(n; n-5, 2, 1, 1)) = W(T(n; 2, 1; 3, 1)) = W(T(n; 1, 1; 2, 2)) > W(T_D)$.*

By Lemma 2.4, it also follows that

$W(T(n;2,1;1,1,1)) = C_{n+1}^3 - 5n + 19,$

$W(T(n;1,1;n-6,2)) = W(T(n;n-6,3,1,1)) = W(T(n;1,1;3,2)) = C_{n+1}^3 - 7n + 39,$

$W(T(n;n-6,1;1,1,1)) = C_{n+1}^3 - 7n + 35,$

$W(T(n;n-7,1;1,1,1)) = C_{n+1}^3 - 8n + 46,$

$W(T(n;1,1;n-7,2)) = C_{n+1}^3 - 9n + 59,$

$W(T(n;n-8,6,1)) = C_{n+1}^3 - 6n + 48,$

$W(T(n;1,1;n-7,1)) = W(T(n;1,1;5,1)) = C_{n+1}^3 - 6n + 38,$

$W(T(n;n-6,3,2)) = C_{n+1}^3 - 6n + 36,$

$W(T(n;2,1;n-6,1)) = W(T(n;2,1;4,1)) = C_{n+1}^3 - 6n + 32,$

$W(T(n;3,1;3,1)) = C_{n+1}^3 - 6n + 30,$

$W(T(n;2,1;2,2)) = C_{n+1}^3 - 6n + 28,$

$W(T(n;n-5,1,1,1,1)) = W(T(n;1,1;2,1,1)) = C_{n+1}^3 - 6n + 26.$

The next lemma can be obtained directly from the above equalities.

Lemma 2.6 *If $n \geq 28$, then* (1) $W(T_D) > W(T(n;2,1;1,1,1)) > W(T(n;1,1;n-6,2)) = W(T(n;n-6,3,1,1)) = W(T(n;1,1;3,2)) > W(T(n;n-6,1;1,1,1)) > W(T(n;n-7,1;1,1,1)) > W(T(n;1,1;n-7,2))$; (2) $W(T_D) > W(T(n;n-8,6,1)) > W(T(n;1,1;n-7,1)) = W(T(n;1,1;5,1)) > W(T(n;n-6,3,2)) > W(T(n;2,1;n-6,1)) = W(T(n;2,1;4,1)) > W(T(n;3,1;3,1)) > W(T(n;2,1;2,2)) > W(T(n;n-5,1,1,1,1)) = W(T(n;1,1;2,1,1))$.

Lemma 2.7 *If $n \geq 28$ and T is a tree with exactly one branching point of degree $m \geq 5$, then $W(T) \leq W(T(n;n-5,1,1,1,1)) < W(T_D)$.*

Proof. By hypothesis, $T = T(n;n_1,n_2,...,n_m)$. Without loss of generality, assume $n_1 \geq n_2 \geq \cdots \geq n_m$. We prove the lemma by induction on m.

If $m = 5$, by Lemma 2.1 and Lemma 2.6 it follows that $W(T) = W(T(n;n_1,n_2,n_3,n_4,n_5)) \leq W(T(n;n_1+n_5-1,n_2,n_3,n_4,1)) \leq W(T(n;n_1+n_4+n_5-2,n_2,n_3,1,1)) \leq W(T(n;n_1+n_3+n_4+n_5-3,n_2,1,1,1)) \leq W(T(n;n-5,1,1,1,1)) < W(T_D)$. Thus, this lemma holds for $m = 5$.

If $m \geq 6$, by Lemma 2.1, Lemma 2.6 and the induction hypothesis it follows that $W(T) = W(T(n;n_1,n_2,...,n_m)) < W(T(n;n_1+n_m,n_2,...,n_{m-1})) \leq W(T(n;n-5,1,1,1,1)) < W(T_D)$.

This completes the proof of this lemma. ∎

Lemma 2.8 *Suppose $n \geq 28$, and T is a tree with only one branching point. If $T \notin \{T(n;n-3,1,1), T(n;n-4,2,1), T(n;n-5,3,1), T(n;n-4,1,1,1), T(n;n-6,4,1), T(n;n-5,2,2), T(n;n-7,5,1), T(n;n-5,2,1,1)\}$, then $W(T) < W(T_D)$.*

Proof. Suppose the degree of the unique branching point is m, then $T = T(n;n_1,...,n_m)$. Without loss of generality, assume $n_1 \geq \cdots \geq n_m$. If $m \geq 5$, then the conclusion follows from Lemma 2.7. We consider the next two cases.

Case 1. $m = 3$.

If $n_3 \geq 2$, since $T \neq T(n;n-5,2,2)$, then $n_1 \geq n_2 \geq 3$. By Lemma 2.1 and Lemma 2.6 it follows that $W(T(n;n_1,n_2,n_3)) \leq W(T(n;n_1+n_3-2,n_2,2)) \leq W(T(n;n-6,3,2)) < W(T_D)$.

If $n_3 = 1$, since $T \notin \{T(n;n-3,1,1), T(n;n-4,2,1), T(n;n-5,3,1), T(n;n-6,4,1), T(n;n-7,5,1)\}$, then $n_1 \geq n_2 \geq 6$. By Lemma 2.1 and Lemma 2.6 it follows that $W(T) \leq W(T(n;n-8,6,1)) < W(T_D)$.

Case 2. $m = 4$. Since $T \neq T(n;n-4,1,1,1)$, then $n_2 \geq 2$. Two subcases occur.

Subcase 1. $n_3 \geq 2$. Note that $n_1 \geq 3$ (by $n \geq 28$) and $n_2 \geq n_3 \geq 2$, by Lemma 2.1 and Lemma 2.6 it follows that $W(T(n;n_1,n_2,n_3,n_4)) \leq W(T(n;n_1+n_4-1,n_2,n_3,1)) < W(T(n;n_1+n_4-1,n_2+n_3-1,1,1)) \leq W(T(n;n-6,3,1,1)) < W(T_D)$.

Subcase 2. $n_3 = 1$. Then, $n_4 = 1$. Since $T \neq T(n;n-5,2,1,1)$, then $n_1 \geq n_2 \geq 3$, by Lemma 2.1 and Lemma 2.6 it follows that $W(T) = W(T(n;n_1,n_2,1,1)) \leq W(T(n;n-6,3,1,1)) < W(T_D)$.

This completes the proof of this lemma. ∎

Lemma 2.9 *Suppose $n \geq 28$, and $T = T(n;p_1,...,p_{r-1};q_1,...,q_{t-1})$. If $t \geq 5$, then $W(T) < W(T(n;n-5,1,1,1,1)) < W(T_D)$.*

Proof. By Lemma 2.3, Lemmas 2.6-2.7 it follows that $W(T) < W(T(n;q_1,...,q_{t-1},n-q_1-\cdots-q_{t-1}-1)) \leq W(T(n;n-5,1,1,1,1)) < W(T_D)$. ∎

Lemma 2.10 *Suppose $n \geq 28$, and $T = T(n;p_1,...,p_{r-1};q_1,...,q_{t-1})$. If $t = r = 4$, then $W(T) < W(T_D)$.*

Proof. By hypothesis, $T = T(n;p_1,p_2,p_3;q_1,q_2,q_3)$. Without loss of generality, suppose that $q_1+q_2+q_3 \geq p_1+p_2+p_3$. We consider the next cases.

Case 1. $q_1+q_2+q_3 \geq 5$. By Lemma 2.1 we have $W(T) < W(T(n; q_1, q_2, q_3, n-q_1-q_2-q_3-1)) \leq W(T(n; q_1+q_3-1, q_2, 1, n-q_1-q_2-q_3-1)) \leq W(T(n; q_1+q_2+q_3-2, 1, 1, n-q_1-q_2-q_3-1))$. Note that $q_1+q_2+q_3-2 \geq 3$ and $n-q_1-q_2-q_3-1 > p_1+p_2+p_3 \geq 3$, then $W(T(n; q_1+q_2+q_3-2, 1, 1, n-q_1-q_2-q_3-1)) \leq W(T(n; n-6, 3, 1, 1)) < W(T_D)$ follows from Lemma 2.1 and Lemma 2.6.

Case 2. $q_1+q_2+q_3 = 4$. This implies that $T = T(n; p_1, p_2, p_3; 2, 1, 1)$. By Lemma 2.3 and Lemma 2.6 it follows that $W(T) = W(T(n; p_1, p_2, p_3; 2, 1, 1)) < W(T(n; 1, 1; 2, 1, 1)) < W(T_D)$.

Case 3. $q_1+q_2+q_3 = 3$. Then, $p_1+p_2+p_3 = 3$. By Lemma 2.1 and Lemma 2.6 it follows that $W(T) = W(T(n; 1, 1, 1; 1, 1, 1)) < W(T(n; 2, 1; 1, 1, 1)) < W(T_D)$. ∎

Lemma 2.11 *Suppose $n \geq 28$, and $T = T(n; p_1, ..., p_{r-1}; q_1, ..., q_{t-1})$ with $t = 4$, $r = 3$. If $T \neq T(n; 1, 1; 1, 1, 1)$, then $W(T) < W(T_D)$.*

Proof. By hypothesis, $T = T(n; p_1, p_2; q_1, q_2, q_3)$. Two cases occur as follows.

Case 1. $q_1+q_2+q_3 \geq 5$. By Lemma 2.1, we have $W(T) < W(T(n; q_1, q_2, q_3, n-q_1-q_2-q_3-1)) \leq W(T(n; q_1+q_3-1, q_2, 1, n-q_1-q_2-q_3-1)) \leq W(T(n; q_1+q_2+q_3-2, 1, 1, n-q_1-q_2-q_3-1))$. Note that $n-q_1-q_2-q_3-1 \geq p_1+p_2+1 \geq 3$ and $q_1+q_2+q_3-2 \geq 3$, by Lemma 2.1 and Lemma 2.6 it follows that $W(T(n; q_1+q_2+q_3-2, 1, 1, n-q_1-q_2-q_3-1)) \leq W(T(n; n-6, 3, 1, 1)) < W(T_D)$.

Case 2. $q_1+q_2+q_3 = 4$. This implies that $T = T(n; p_1, p_2; 2, 1, 1)$. By Lemma 2.3 and Lemma 2.6 it follows that $W(T(n; p_1, p_2; 2, 1, 1)) \leq W(T(n; 1, 1; 2, 1, 1)) < W(T_D)$.

Case 3. $q_1+q_2+q_3 = 3$. Since $T \neq T(n; 1, 1; 1, 1, 1)$, then $3 \leq p_1+p_2 \leq n-5$. We divide the proof into four subcases.

Subcase 1. $3 \leq p_1+p_2 \leq 6$. By Lemma 2.1, it follows that $W(T) \leq W(T(n; p_2+p_1-1, 1; 1, 1, 1))$. Recall that $n \geq 28$, by Propositions 2.1 and Lemma 2.6 we have $W(T(n; p_2+p_1-1, 1; 1, 1, 1)) \leq W(T(n; 2, 1; 1, 1, 1)) < W(T_D)$.

Subcase 2. $7 \leq p_1+p_2 \leq n-7$. This implies that $n-p_1-p_2-1 \geq 6$. By Lemma 2.1 and Lemma 2.6 it follows that $W(T) < W(T(n; p_1, p_2, n-p_1-p_2-1)) \leq W(T(n; p_2+p_1-1, 1, n-p_1-p_2-1)) \leq W(T(n; n-8, 6, 1)) < W(T_D)$.

Subcase 3. $p_1+p_2 = n-6$. By Lemma 2.1 and Lemma 2.6 it follows that $W(T) \leq W(T(n; n-7, 1; 1, 1, 1)) < W(T_D)$.

Subcase 4. $p_1 + p_2 = n - 5$. By Lemma 2.1 and Lemma 2.6 it follows that $W(T) \leq W(T(n; n-6, 1; 1, 1, 1)) < W(T_D)$.

By combining the above arguments, the result follows. ∎

Lemma 2.12 *Suppose $n \geq 28$, and $T = T(n; p_1, ..., p_{r-1}; q_1, ..., q_{t-1})$ with $t = 3$, $r = 3$. If $T \notin \{T(n; 1, 1; 1, 1), T(n; 1, 1; 2, 1), T(n; 1, 1; n-5, 1), T(n; 1, 1; 3, 1), T(n; 2, 1; 2, 1), T(n; 1, 1; n-6, 1), T(n; 1, 1; 4, 1), T(n; 2, 1; 3, 1), T(n; 1, 1; 2, 2)\}$, then $W(T) < W(T_D)$.*

Proof. By hypothesis, $T = T(n; p_1, p_2; q_1, q_2)$. Without loss of generality, suppose $q_1 + q_2 \geq p_1 + p_2$. Since $T \notin \{T(n; 1, 1; 1, 1), T(n; 1, 1; 2, 1), T(n; 2, 1; 2, 1)\}$, then $4 \leq q_1 + q_2 \leq n - 4$. We consider the next cases.

Case 1. $q_1 + q_2 = 4$. Since $T \neq T(n; 1, 1; 3, 1)$ and $T \neq T(n; 1, 1; 2, 2)$, then $3 \leq p_1 + p_2 \leq 4$. Two subcases occur as follows.

Subcase 1. $p_1 + p_2 = 3$. Since $T \neq T(n; 2, 1; 3, 1)$, then $T = T(n; 2, 1; 2, 2)$. By Lemma 2.6, $W(T(n; 2, 1; 2, 2)) < W(T_D)$.

Subcase 2. $p_1 + p_2 = 4$. By Lemma 2.1 and Lemma 2.6, we have $W(T) \leq W(T(n; p_1, p_2; 3, 1)) \leq W(T(n; 3, 1; 3, 1)) < W(T_D)$.

Case 2. $q_1 + q_2 = 5$. Two subcases occur as follows.

Subcase 1. $q_1 = 3$, $q_2 = 2$. By Lemma 2.3 and Lemma 2.6, we have $W(T) = W(T(n; p_1, p_2; 3, 2)) \leq W(T(n; 1, 1; 3, 2)) < W(T_D)$.

Subcase 2. $q_1 = 4$, $q_2 = 1$. Since $T \neq T(n; 1, 1; 4, 1)$, then $3 \leq p_1 + p_2 \leq 5$. By Lemma 2.1, Proposition 2.1 and Lemma 2.6, we have $W(T) = W(T(n; p_1, p_2; 4, 1)) \leq W(T(n; p_1 + p_2 - 1, 1; 4, 1)) \leq W(T(n; 2, 1; 4, 1)) < W(T_D)$.

Case 3. $q_1 + q_2 = 6$. By Lemma 2.1, Lemma 2.3 and Lemma 2.6, it follows that $W(T) \leq W(T(n; p_1, p_2; 5, 1)) \leq W(T(n; 1, 1; 5, 1)) < W(T_D)$.

Case 4. $7 \leq q_1 + q_2 \leq n - 7$. Then, $n - q_1 - q_2 - 1 \geq 6$. By Lemma 2.1 and Lemma 2.6 it follows that $W(T) < W(T(n; q_1, q_2, n - q_1 - q_2 - 1)) \leq W(T(n; q_1 + q_2 - 1, 1, n - q_1 - q_2 - 1)) \leq W(T(n; n-8, 6, 1)) < W(T_D)$.

Case 5. $q_1 + q_2 = n - 6$. By Lemma 2.1, Lemma 2.3 and Lemma 2.6 it follows that $W(T) \leq W(T(n; p_1, p_2; n-7, 1)) \leq W(T(n; 1, 1; n-7, 1)) < W(T_D)$.

Case 6. $q_1 + q_2 = n - 5$. Two subcases occur as follows.

Subcase 1. $q_2 = 1$. Then, $q_1 = n - 6$. Since $T \neq T(n; 1, 1; n-6, 1)$, then $T = T(n; 2, 1; n-6, 1)$. By Lemma 2.6, we have $W(T) = W(T(n; 2, 1; n-6, 1)) < W(T_D)$.

Subcase 2. $q_2 \geq 2$. By Lemma 2.1, Lemma 2.3 and Lemma 2.6 it follows that $W(T) \leq W(T(n; p_1, p_2; n-7, 2)) \leq W(T(n; 1, 1; n-7, 2)) < W(T_D)$.

Case 7. $q_1 + q_2 = n - 4$. This implies that $p_1 = p_2 = 1$. Since $T \neq T(n; 1, 1; n-5, 1)$, then $q_1 \geq q_2 \geq 2$. By Lemma 2.1 and Lemma 2.6, we have $W(T) = W(T(n; 1, 1; q_1, q_2)) \leq W(T(n; 1, 1; n-6, 2)) < W(T_D)$.

This completes the proof. ∎

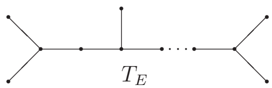

Fig. 5. The tree T_E.

Lemma 2.13 *Suppose $n \geq 28$, and T is a tree of order n with exactly three branching points. If $T \neq T_D$, then $W(T) < W(T_D)$.*

Proof. Let u_1, u_2, u_3 be the three branching points of T. Let u_1 be an in-branching point and u_2, u_3 be two out-branching points. Now suppose $d(u_1) = m$, T_1, ..., T_m be the components of $T - u_1$ and let them be paths except T_{m-1}, T_m, where the order of T_i is n_i for $1 \leq i \leq m$. By the definition of T it follows that $u_2 \in V(T_{m-1})$ and $u_3 \in V(T_m)$, which implies that $n_{m-1} \geq 3$ and $n_m \geq 3$. Without loss of generality, we suppose $n_{m-1} \geq n_m$. The next cases should be taken into account.

Case 1. $n_1 + n_2 + \cdots + n_{m-2} \geq 2$. Note that $n_{m-1} \geq n_m \geq 3$, by Lemma 2.1 and Lemma 2.6 it follows that $W(T) < W(T(n; n_{m-1}, n_m, n_1 + \cdots + n_{m-2})) \leq W(T(n; n-6, 3, 2)) < W(T_D)$.

Case 2. $n_1 + n_2 + \cdots + n_{m-2} = 1$. Then, $m = 3$ and $n_1 = 1$. Four subcases occur.

Subcase 1. $n_3 \geq 6$. Thus, $n_2 \geq 6$. By Lemma 2.1 and Lemma 2.6 it follows that $W(T) < W(T(n; n_2, n_3, 1)) \leq W(T(n; n-8, 6, 1)) < W(T_D)$.

Subcase 2. $n_3 = 5$. Then, $n_2 = n - n_1 - n_3 - 1 = n - 7$. By Lemma 2.3 and Lemma 2.6 it follows that $W(T) < W(T(n; 1, 1; n-7, 1)) < W(T_D)$.

Subcase 3. $n_3 = 4$. Then, $n_2 = n - n_1 - n_3 - 1 = n - 6$. By Lemma 2.3 it follows that $W(T) \leq W(T_E)$, where T_E is shown in Fig. 5. By Lemma 2.4, we have $W(T_E) = C_{n+1}^3 - 6n + 30$. Thus, $W(T_E) < W(T_D)$.

Subcase 4. $n_3 = 3$. Since $T \neq T_D$, by Lemma 2.3 it follows that $W(T) < W(T_D)$.

The result follows by combining the above arguments. ∎

Lemma 2.14 *Suppose $n \geq 28$, and T is a tree of order n with k branching points. If $k \geq 3$ and $T \neq T_D$, then $W(T) < W(T_D)$.*

Proof. We prove the lemma by induction on k. By Lemma 2.13, it is true for $k = 3$.

Let $k \geq 4$, and T be a tree of order n with k branching points. Then T must have an out-branching point, and by Lemma 2.3, $W(T) < W(T_C)$, where T_C has $k-1$ branching points. Thus, $W(T_C) \leq W(T_D)$ follows from the induction hypothesis and Lemma 2.13. This completes the proof. ∎

The proof of Theorem 1.1. If T has exactly one branching point, then the result follows from Lemma 2.8. If T has exactly two branching points, by symmetry we may suppose that T $=T(n; p_1, ..., p_{r-1}; q_1, ..., q_{t-1})$, where $r \leq t$, $p_1 \geq \cdots \geq p_{r-1}$ and $q_1 \geq \cdots \geq q_{t-1}$. Then result follows from Lemmas 2.9-2.12. If T has exactly k ($k \geq 3$) branching points, then result follows from Lemmas 2.13-2.14.

By combining the above arguments, Theorem 1.1 follows. ∎

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