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Wiener Index of Line Graphs *

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Abstract

The Wiener index $W(G)$ of a connected graph G is the sum of distances of all pairs of vertices in G. We show that for any connected graph G with $\delta(G) \geq 2$, $W(L(G)) \geq W(G)$, with equality if and only if G is a cycle, where $L(G)$ is the line graph of G. We also present lower and upper bounds for $W(L(G))$ in terms of a graph invariant called Gutman index of G. As an immediate consequence, a relation between Gutman index and Wiener index of a tree is derived.

1 Introduction

All graphs considered here are finite, undirected and simple. Let $G = (V(G), E(G))$ be a graph. $|V(G)|$ and $|E(G)|$ are called the *order* and the *size* of G, respectively. Let v be a vertex of G. The degree of v, denoted by $d_G(v)$, is the number of edges incident with v in G. The minimum degree $\delta(G)$ of G is $\min\{d(v) : v \in V(G)\}$. The distance $d_G(u, v)$ of two vertices $u, v \in V(G)$ is the length of shortest path connecting u and v in G. If there is no confusion, we simply use $d(v)$ and $d(u, v)$ instead of $d_G(v)$ and $d_G(u, v)$, respectively.

The line graph $L(G)$ of G, is the graph with $E(G)$ as its vertex set, in which two vertices are joined by an edge if they have a common vertex in G. The Wiener index, as defined in the abstract, is extensively studied in both theoretical and practical aspects, see the reviews [6, 10], and some recent papers [1, 3, 11, 15–17]. But, little are known about the Wiener index of line graphs. For some recent results on it, we refer to [4] and [5].

One of the fundamental problems for a graph invariant is to determine its extremal (maximum and minimum) values among certain classes of graphs.

Entringer et al. [7] proved that among all trees of order n, the Wiener index is maximized

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by the path P_n, and minimized by the star $K_{1,n-1}$. Combining the above with the fact that adding a new edge to a connected graph decreases the Wiener index, one can readily conclude that among all connected graphs, P_n has the maximum value of Wiener index and the complete graph K_n has the minimum Wiener index.

Buckley [2] found the following relation between the Wiener index of a tree and that of its line graph, which makes the problem trivial to determine the extremal values of the Wiener index among line graphs of all trees of given order.

Theorem 1.1. *(Buckley [2]). Let T be a tree of order n. Then*

$$W(L(T)) = W(T) - \binom{n}{2} .$$

Unfortunately, there is not a similar relation between the Wiener index of a connected unicyclic graph and that of its line graph. But, it was shown that

Theorem 1.2. *(Gutman and Pavlović [9]). If G is a connected unicyclic graph of order n, then $W(L(G)) \le W(G)$, with equality if and only if $G \cong C_n$.*

So, this gives rise to another natural question. When does a graph G have $W(L(G)) > W(G)$, $W(L(G)) = W(G)$ or $W(L(G)) < W(G)$? Indeed, each of three cases may occur in connected bicyclic graphs. In this note, we will prove that

Theorem 1.3. *Let G be a connected graph of order n with $\delta(G) \ge 2$. Then $W(L(G)) \ge W(G)$, with equality if and only if $G \cong C_n$.*

We give the proof of Theorem 1.3 in Section 3.

2 New bounds on Wiener index of line graphs

As we haven seen before, if H is a connected spanning subgraph of a graph G, then $W(H) > W(G)$. However, one can easily find some such pairs of graphs for each of three possibilities $W(L(H)) > W(L(G))$, $W(L(H)) = W(L(G))$ and $W(L(H)) < W(L(G))$. It is easy to prove that for any connected graph G of order n, $W(L(G)) \ge \binom{n-1}{2}$, with equality if and only if $G \cong K_{1,n-1}$, see [4].

It turns out to be a challenging problem to determine the maximum value of Wiener index among line graphs of all connected graphs of order n. In [4], the authors proved that for any connected graph of order n, $W(L(G)) \le \frac{4}{5^5}n^5 + O(n^{9/2})$, and the coefficient of n^5 is best possible. Their proof is based on two facts on a graph invariant, called Gutman index, which was introduced in [8]. For a connected graph G, the Gutman index $Gut(G)$ of G is $\sum_{\{u,v\}\subseteq V(G)} d(u)d(v)d(u,v)$. It is shown that for any connected graph G of order n, $|W(L(G)) - \frac{1}{4}Gut(G)| \leq \frac{n^4}{8}$ and $Gut(G) \leq \frac{4}{5^5}n^5 + O(n^{9/2})$ in [4].

Let us recall the other graph invariant called edge-Wiener index, which was introduced recently in [13, 14]. For a connected graph G, define the distance $D(f,g)$ of two edges $f = uv$, $g = xy$ as the length of a shortest path between a vertex of f and a vertex of g. Namely, $D(f,g) = \min\{d(u,x), d(u,y), d(v,x), d(v,y)\}$. Then $W_e(G) = \sum_{\{f,g\}\subseteq E(G)} D(f,g)$ is called the edge-Wiener index of G. In the next theorem, we give lower and upper bounds for the edge-Wiener index of a graph in terms of its Gutman index and size.

Theorem 2.1. *Let G be a connected graph of size m. Then we have*

$$\frac{1}{4}(Gut(G) - m) - \binom{m}{2} \leq W_e(G) \leq \frac{1}{4}(Gut(G) - m)\ .$$

Moreover, the lower bound is attained if and only if G is a tree.

Proof. Let $f = uv$ and $g = xy$ be any two edges of G. By the definition of $D(f,g)$, it is obvious that

$$\frac{1}{4}(d(u,x) + d(u,y) + d(v,x) + d(v,y)) \geq D(f,g) \geq \frac{1}{4}(d(u,x) + d(u,y) + d(v,x) + d(v,y)) - 1\ .$$

Then

$$\begin{aligned}
W_e(G) &= \sum_{\{f,g\}\subseteq E(G)} D(f,g) \\
&\leqslant \sum_{\{f,g\}\subseteq E(G)} \frac{1}{4}\left(d(u,x) + d(u,y) + d(v,x) + d(v,y)\right) \\
&= \frac{1}{4}\left(\sum_{uv\in E(G)} (d(u)d(v) - 1)d(u,v) + \sum_{uv\notin E(G)} d(u)d(v)d(u,v)\right) \\
&= \frac{1}{4}\left(\sum_{\{u,v\}\subseteq V(G)} d(u)d(v)d(u,v) - \sum_{uv\subseteq E(G)} d(u,v)\right) \\
&= \frac{1}{4}(Gut(G) - m)\ .
\end{aligned}$$

By the similar way, we obtain $W_e(G) \geq \frac{1}{4}(Gut(G) - m) - \binom{m}{2}$.

If G is a tree then clearly, $D(f,g) = \frac{1}{4}(d(u,x) + d(u,y) + d(v,x) + d(v,y)) - 1$ for any two edges f and g of G, where $f = uv$ and $g = xy$. It immediately implies that

$$\frac{1}{4}(Gut(G) - m) - \binom{m}{2} = W_e(G) \ .$$

Now suppose that G is not a tree and $\frac{1}{4}(Gut(G) - m) - \binom{m}{2} = W_e(G)$. Let C be a shortest cycle of G. We choose two edges f and g of C such that the distance $D(f,g)$ is the largest among all pairs of edges of C in $L(G)$. It is trivial to see that $D(f,g) > \frac{1}{4}(d(u,x) + d(u,y) + d(v,x) + d(v,y)) - 1$. This results in $\frac{1}{4}(Gut(G) - m) - \binom{m}{2} < W_e(G)$, a contradiction. Thus G must be a tree. $\square$

As $d_{L(G)}(f,g) = D(f,g) + 1$ for any two edges $f, g \in E(G)$, it follows from the definition of $W_e(G)$ and $W(L(G))$, that $W(L(G)) = W_e(G) + \binom{m}{2}$, where $m = |E(G)|$. So, we have

Corollary 2.2. *Let G be a connected graph of size m. Then we have*

$$\frac{1}{4}(Gut(G) - m) \leq W(L(G)) \leq \frac{1}{4}(Gut(G) - m) + \binom{m}{2} \ .$$

Moreover, the lower bound is attained if and only if G is a tree.

Since $m \leq \binom{n}{2}$ for any graph G of order n, the result of Corollary 2.2 improves the bound $|W(L(G)) - \frac{1}{4}Gut(G)| \leq \frac{n^4}{8}$ obtained in [4], which we mentioned earlier.

Corollary 2.3. *(Gutman [8]). For a tree T of order n, we have*

$$Gut(T) = 4W(T) - (2n-1)(n-1) \ .$$

Proof. Let $m = |E(T)|$. Then $m = n-1$. By Corollary 2.2, we have $W(L(T)) = \frac{1}{4}(Gut(T) - m)$. Together with the result of Theorem 1.1 that $W(L(T)) = W(T) - \binom{n}{2}$, we have $W(T) - \binom{n}{2} = \frac{1}{4}(Gut(G) - m)$. So, one can easily transform it to our desired form. $\square$

3 A lower bound of Gutman index

Theorem 3.1. *Let G be a connected graph of size m with $\delta(G) \geq 2$. If G is not a cycle, then $Gut(G) > 4W(G) + m$.*

Proof. Assume that G is a graph as given in the hypothesis of the theorem. First, suppose that there are no adjacent vertices of degree two in G. It follows that for any pair of adjacent vertices $u, v \in V(G)$, $d(u)d(v) \geq 2 \times 3 = 6$. So,

$$\begin{aligned} Gut(G) - 4W(G) &= \sum_{\{u,v\} \subseteq V(G)} ((d(u)d(v) - 4)d(u,v) \\ &\geq \sum_{uv \in E(G)} (2 \times 3 - 4)d(u,v) \\ &= \sum_{uv \in E(G)} 2 = 2m > m\ . \end{aligned}$$

Now we prove the general case by induction on the size m of G. Since $\delta(G) \geq 2$ and G is not a cycle, our induction basis is $m = 5$. It is clear that $K_4 - e$, the graph obtained from K_4 by deleting an edge, is the only graph G with size $m = 5$, which meets our condition. Since there are no adjacent vertices of degree two in $K_4 - e$, the result follows. Next, assume that $m > 5$, and that $Gut(H) > 4W(H) + l$ for any graph H of size $l < m$, provided that H is connected, $\delta(H) \geq 2$ and is not a cycle. By the result of the previous paragraph, it remains to consider the case when there exist adjacent vertices of degree two in G.

Case 1. There are two adjacent vertices, say x and y, of degree two in G such that $N(x) \cap N(y) = \emptyset$.

Contract the edge xy of G, and denote the resulting new vertex and graph by z and H, respectively. Clearly, H is connected, $\delta(H) \geq 2$ and is not a cycle, but has size $m - 1$. By the induction hypothesis, $Gut(H) > 4W(H) + (m - 1)$. Note also that $d_H(z) = 2$ and for any vertex $w \in V(G) \setminus \{x, y\}$, $d_H(w) = d_G(w)$. Moreover, $d_H(z, w) \leq \min\{d_G(x, w), d_G(y, w)\}$ and $d_H(u, v) \leq d_G(u, v)$ for any two vertices $u, v \in V(G) \setminus \{x, y\}$. Because G is not a cycle, there exists a vertex of degree at least three. Therefore,

$$\begin{aligned} Gut(G) - 4W(G) &= \sum_{\{u,v\} \subseteq V(G)} ((d_G(u)d_G(v) - 4)d_G(u,v) \\ &\geq Gut(H) - 4W(H) + \sum_{w \in V(G) \setminus \{x\}} ((d_G(x)d_G(w) - 4)d_G(x,w) \\ &> (m - 1) + (2 \times 3 - 4) \times 1 > m\ . \end{aligned}$$

Case 2. Any two adjacent vertices x and y of degree two satisfy $N(x) \cap N(y) \neq \emptyset$.

So, those edges with both end vertices of degree two are independent in G. Let X be these pairs of adjacent vertices u and v of G, such that $d(u) = d(v) = 2$, Y the set of remaining pairs of adjacent vertices, and let Z be the set of pairs of nonadjacent vertices of G. Hence

$|X| + |Y| = m$, $|X| + |Y| + |Z| = \frac{1}{2}n(n-1)$, $|X| \le n/2$. Furthermore, since G is not a cycle and $\delta(G) \ge 2$, $|X| \le n/2 < m/2$. Thus, we have

$$\begin{aligned} Gut(G) &= \sum_{u,v \in V(G)} d(u)d(v)d(u,v) \\ &= \sum_{u,v \in X \cup Z} d(u)d(v)d(u,v) + \sum_{u,v \in Y} d(u)d(v)d(u,v) \\ &\ge \sum_{u,v \in X \cup Z} 4d(u,v) + \sum_{u,v \in Y} 6d(u,v) \\ &\ge 4W(G) + 2(m - |X|) \\ &> 4W(G) + 2\left(m - \frac{m}{2}\right) = 4W(G) + m \ . \end{aligned}$$

The proof is completed. □

So, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. If $G \cong C_m$, then $L(G) \cong C_m \cong G$, and $W(L(G)) = W(G)$. If G is a connected graph with $\delta(G) \ge 2$ and is not a cycle, then by Theorems 2.1 and 3.1, $W(L(G)) \ge \frac{1}{4}(Gut(G) - m) > W(G)$. □

4 Concluding remarks

We saw that $W(L(G))$, $W_e(G)$ and $Gut(G)$ are three closely related graph invariants. The minimum values of $W(L(G))$ and $W_e(G)$ are determined among all connected graphs of order n, and the corresponding extremal graphs are known as well. But, the exact maximum values of $W(L(G))$ and $W_e(G)$ remain open, see [4] and [14] for the detail. For the Gutman index, we would like to propose the following open question.

Problem 4.1. What are the extremal values of Gutman index and the corresponding extremal graphs among all connected graphs of given order, or given size, or both?

In the proof of Theorem 2.1, for two edges $f = uv$ and $g = xy$ of G, we define a new kind of distance $D'(f,g)$ as the average distance between a vertex of f and a vertex of g. Namely, $D'(f,g) = \frac{1}{4}(d(u,x) + d(u,y) + d(v,x) + d(v,y))$. Let us denote the sum $\sum_{\{f,g\} \subseteq V(G)} D'(f,g)$ as $W'_e(G)$. As we have shown in Theorem 2.1, for any connected graph G, $W'_e(G) = \frac{1}{4}(Gut(G) - m)$ and $W'_e(G) - \binom{m}{2} \le W_e(G) \le W'_e(G)$, where $m = |E(G)|$. Moreover, by Corollary 2.2, we get

$W'_e(G) \leq W(L(G)) \leq W'_e(G) + \binom{m}{2}$, with equality in left hand side if and only if G is a tree. So, it might be of independent interest to study $W'_e(G)$.

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