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Distributional Chaos of Generalized Belusov–Zhabotinskii's Reaction Models

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Abstract

In this paper we study the dynamics of a family of coupled lattice dynamical systems (CLS), of the form $x_n^{m+1} = (1-\epsilon)f(x_n^m) + \epsilon/2[f(x_{n-1}^m) + f(x_{n+1}^m)]$, which generalize the model stated by K. Kaneko in [Phys. Rev. Lett., **65**, 1391-1394, 1990] for the Belusov–Zhabotinskii's chemical reaction. We present a definition of distributional chaos on a sequence (DCS) for CLS systems and we state two different sufficient conditions for having DCS.

1. INTRODUCTION

Discrete Dynamical Systems (DDS's), i.e., a couple composed by a space $\mathbb{X}$ (usually compact and metric) and a continuous self–map ψ on $\mathbb{X}$, are an active line of research in mathematics (see e.g., [1] or [5]) because they model many phenomena coming from biology, physics, chemistry, engineering and social sciences (see for example, [4], [12], [18] or [17]). In most cases in the formulation of such models ψ is a C^∞, an analytical or a polynomial map.

Coming from physical/chemical engineering applications, such a digital filtering, imaging and spatial vibrations of the elements which compose a given chemical product, a generalization of DDS's have recently appeared as an important subject of research, we mean the so called *Lattice Dynamical Systems* or *1d Spatiotemporal Discrete Systems* (LDS). In the next section we provide all the definitions. To show the importance of these type of systems, see for instance [2].

To analyze when one of this type of systems have a complicated dynamics or not by the observation of one topological dynamics property is an open problem.

The aim of the present paper is, by the introduction of the notion of *distributional chaos on a sequence* (DCS) for coupled lattice systems (CLS), to characterize the dynamical complexity of a coupled lattice family of systems which generalizes the model stated by K. Kaneko in [11] (for more details see for references therein) for studying the Belusov–Zhabotinskii chemical reaction. We present two different sufficient conditions for having DCS for this family of CLS. These results complete and generalize [6, 7] where Li-Yorke chaos and topological entropy are respectively studied for the Belusov–Zhabotinskii chemical reaction. Notation and definitions will be provided in next sections.

The statement of our main results is the following:

Theorem A. *Let f be a continuous self-map defined on a compact interval $[a,b]$. If f is Li–Yorke chaotic, then the CLS system defined by f in the form* (4) *is distributionally chaotic with respect to a sequence considering $[a,b]_\infty$ endowed with the metrics ρ_i, $i = 1, 2$, respectively.*

Theorem B. *Let f be a continuous self-map defined on a compact interval $[a,b]$. If f has positive topological entropy, then the CLS system defined by f in the form* (4) *has an uncountable distributionally scrambled set, composed by almost periodic points, with respect to a sequence considering $[a,b]_\infty$ endowed with the metrics ρ_i, $i = 1, 2$, respectively.*

2. LATTICE DYNAMICAL SYSTEMS

The state space of LDS (Lattice Dynamical System) is the set

$$\mathcal{X} = \{x : x = \{x_i\},\ x_i \in \mathbb{R}^d,\ i \in \mathbb{Z}^D,\ \|x_i\| < \infty\},$$

where $d \geq 1$ is the dimension of the range space of the map of state x_i, $D \geq 1$ is the dimension of the lattice and the l^2 norm $\|x\|_2 = (\Sigma_{i\in\mathbb{Z}^D} \mid x_i \mid^2)^{1/2}$ is usually taken ($\mid x_i \mid$ is the length of the vector x_i).

K. Kaneko [11] states the following 1d-LD CML (Coupled Map Lattice) (for more details see for references therein) which is related to the Belusov–Zhabotinskii reaction (see [12] and for experimental study of chemical turbulence by this method [10], [9], [8]):

$$x_n^{m+1} = (1-\epsilon)f(x_n^m) + \epsilon/2[f(x_{n-1}^m) - f(x_{n+1}^m)], \tag{1}$$

where m is discrete time index, n is lattice side index with system size L (i.e. $n = 1, 2, \ldots L$), ϵ is coupling constant and $f(x)$ is the *unimodal map* on the unite closed interval $I = [0,1]$, i.e. $f(0) = f(1) = 0$ and f has unique critical point c with $0 < c < 1$ such that $f(c) = 1$. For simplicity we will deal with so called "tent map", defined by

$$f(x) = \begin{cases} 2x, & x \in [0, 1/2), \\ 2 - 2x, & x \in [1/2, 1]. \end{cases} \tag{2}$$

In general, one of the following periodic boundary conditions of the system (1) is assumed:

1. $x_n^m = x_{n+L}^m$,
2. $x_n^m = x_n^{m+L}$,
3. $x_n^m = x_{n+L}^{m+L}$,

standardly, the first case of the boundary conditions is used.

The equation (1) was studied by many authors, mostly experimentally or semi-analytically than analytically. The first paper with analytic results is [3], where it was proved that this system is Li–Yorke chaotic, [7] gives alternative and easier proof of it. Moreover in [6] the complexity via the notion of positive topological entropy is analyzed.

Example 1. *Consider, as an example the 2–element one–way coupled logistic lattice (OCLL, see [13]) $H : I^2 \to I^2$ written as*

$$\begin{array}{rcl} x_1^{m+1} & = & (1-\epsilon)f(x_1^m)+\epsilon f(x_2^m), \\ x_2^{m+1} & = & \epsilon f(x_1^m)+(1-\epsilon)f(x_2^m), \end{array} \tag{3}$$

where f is the tent map.

In this paper we consider the family of coupled lattice dynamical systems (CLS), of the form

$$x_n^{m+1} = (1-\epsilon)f(x_n^m)+\epsilon/2[f(x_{n-1}^m)+f(x_{n+1}^m)], \tag{4}$$

where f is a continuous interval map without any more restrictions. This family contains CML systems of the form (1).

3. FROM CLS TO CLASSICAL DDS

Consider the set of sequences of real numbers

$$R_\infty = \{(..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...) : a_n \in \mathbb{R}, n \in \mathbb{Z}\}.$$

Let $x_1 = (x_1^m)_{m\in\mathbb{Z}}$, $x_2 = (x_2^m)_{m\in\mathbb{Z}} \in R_\infty$, in R_∞ we consider the following two non-equivalent metrics:

$$\rho_1(x_1, x_2) = \sum_{n=-\infty}^{n=\infty} \frac{|x_n^1 - x_n^2|}{2^{|n|}} \tag{5}$$

and

$$\rho_2(x_1, x_2) = \sup\{\left|x_n^1 - x_n^2\right| : n \in \mathbb{Z}\}. \tag{6}$$

Note that (R_∞, ρ_i), $i = 1, 2$, is a complete metric space. We consider $[a, b]_\infty$ the subset of R_∞ composed by sequences with terms in the compact interval $[a, b]$ endowed with the restriction of ρ_i.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f : [a, b] \to [a, b]$ be a continuous self–map. Let $x = \{x_m^n : m \in \mathbb{N}_0, n \in \mathbb{Z}\}$ be a solution of the CLS system (4) with initial condition $\alpha = (\alpha_n = \alpha_n^0)_{n\in\mathbb{Z}}$ where $\alpha_n \in [a, b]$ for all $n \in \mathbb{Z}$.

Define for all $m \in \mathbb{N}_0$, $x_m = (..., x_{-1}^m, x_0^m, x_1^m, ...)$ and consider the self-map F_f defined on $[a,b]_\infty$ in the form

$$F_f(x_m) = (..., x_{-1}^{m+1}, x_0^{m+1}, x_1^{m+1}, ...) = x_{m+1} \tag{7}$$

where $x_0 = \alpha$ and $x_n^{m+1} = (1-\epsilon)f(x_n^m) + \epsilon/2[f(x_{n-1}^m) + f(x_{n+1}^m)]$, $m \in \mathbb{N}_0$.

Remark 2. *From the previous construction, for a given self-map f defined on a compact interval $[a,b]$, the CLS system* (1) *associated with f is equivalent to the classical dynamical system $([a,b]_\infty, F_f)$ where F_f is defined in* (7).

4. DEFINITIONS AND AUXILIARY RESULTS

Let us start introducing two of the most well–known notions of chaos for classical DDS. Consider f be a self–map defined on a compact interval $[a,b]$.

Definition 3. *A pair of points $x, y \in [a,b]$ is called a* Li-Yorke pair *if*

1. $\limsup_{n\to\infty} |f^n(x), f^n(y)| > 0$
2. $\liminf_{n\to\infty} |f^n(x), f^n(y)| = 0.$

The map f is Li-Yorke chaotic if it has a Li-Yorke pair.

Remark 4. *In the setting of DDS defined on compact metric spaces different from compact intervals and the circle the definition of Li-Yorke chaos is given via the existence of an uncountable scrambled set, i.e., an uncountable set such that every two points are a Li-Yorke pair. In the case of interval and circle system the existence of one Li-Yorke pair implies the presence of an uncountable number of them (see [14]).*

Let $\{p_i\}_{i\in\mathbb{N}}$ be an increasing sequence of positive integers, let $x, y \in [a,b]$ and $t \in \mathbb{R}$. Let

$$\begin{aligned}
\phi_{xy}^{(n)}(t, \{p_i\}_{i\in\mathbb{N}}) &:= \frac{1}{n}\#\{i : |f^{p_i}(x) - f^{p_i}(y)| < t, 0 \le i < n\}, \\
\phi_{xy}(t, \{p_i\}_{i\in\mathbb{N}}) &:= \liminf_{n\to\infty} \phi_{xy}^{(n)}(t, \{p_i\}_{i\in\mathbb{N}}), \\
\phi_{xy}^*(t, \{p_i\}_{i\in\mathbb{N}}) &:= \limsup_{n\to\infty} \phi_{xy}^{(n)}(t, \{p_i\}_{i\in\mathbb{N}})
\end{aligned}$$

where $\#(A)$ denotes the cardinality of a set A. Using these notations distributional chaos with respect to a sequence is defined as follows:

Definition 5. *A pair of points* $(x, y) \in [a, b]^2$ *is called* distributionally chaotic with respect to a sequence $\{p_i\}_{i\in\mathbb{N}}$ *if* $\phi_{xy}(s, \{p_i\}_{i\in\mathbb{N}}) = 0$ *for some* $s > 0$ *and* $\phi^*_{xy}(t, \{p_i\}_{i\in\mathbb{N}}) = 1$ *for all* $t > 0$.

A set S *containing at least two points is called* distributionally scrambled with respect to $\{p_i\}_{i\in\mathbb{N}}$ *if any pair of distinct points of* S *is distributionally chaotic with respect to* $\{p_i\}_{i\in\mathbb{N}}$.

A map f *is* distributionally chaotic with respect to $\{p_i\}_{i\in\mathbb{N}}$, *if it has an uncountable set distributionally scrambled with respect to* $\{p_i\}_{i\in\mathbb{N}}$.

Definition 6. *A point* x *is called* almost periodic *of* f, *if for any* $\varepsilon > 0$ *there exists* $N > 0$ *such that for any* $q \geq 0$, *there exists* $r, q < r \leq q + N$, *holding* $|f^r(x) - x| < \varepsilon$. *By* $\mathrm{AP}(f)$ *we denote the set of all almost periodic points of* f.

The following results from Oprocha [16] and Liao et al. [15] will play a key role in the proof of our main results.

Lemma 1. *Let* f *be a continuous self-map on* $[a, b]$. *The map* f *is Li–Yorke chaotic iff there exists an increasing sequence* $\{p_i\}_{i\in\mathbb{N}}$ *such that* f *is distributionally chaotic repect to* $\{p_i\}_{i\in\mathbb{N}}$.

Lemma 2. *Let* f *be a continuous self-map on* $[a, b]$. *If* f *has positive topological entropy, then there exists an increasing sequence* $\{p_i\}_{i\in\mathbb{N}}$ *such that* f *has an uncountable distributionally scrambled set* T *with respect to* $\{p_i\}_{i\in\mathbb{N}}$. *Moreover, the set* T *is composed by almost periodic points.*

For details on the definition of topological entropy see [19].

5. PROOF OF THE MAIN RESULTS

Note that the definition of distributional chaos in a sequence $\{p_i\}_{i\in\mathbb{N}}$ for a continuous self-map f defined on an interval $[a, b]$ is equivalent to the existence of an uncountable subset $S \subset [a, b]$ such that for any $x, y \in S$, $x \neq y$,

- there exists $\delta > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\delta)}(|f^{p_i}(x) - f^{p_i}(y)|) = 0,$$

- for every $t > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,t)}(|f^{p_i}(x) - f^{p_i}(y)|) = 1,$$

where $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.

Proof of Theorem A. Since the map f is Li–Yorke chaotic, by Lemma 1 there exists an increasing sequence $\{p_i\}_{i\in\mathbb{N}}$ such that f is distributionally chaotic with repect to $\{p_i\}_{i\in\mathbb{N}}$. Let $S \subset [a,b]$ be the uncountable set distributionally scrambled with respect to $\{p_i\}_{i\in\mathbb{N}}$ for f. Let $E \subset [a,b]_\infty$ be the uncountable set such that each element of it is a constant sequence equal to an element of S. Let $x = \{x_n = a\}_{n\in\mathbb{N}}$ and $y = \{y_n = b\}_{n\in\mathbb{N}}$ be two differnt elements of E. Then, there exists $\delta > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\delta)}(\rho_1(F^{p_i}(x), F^{p_i}(y))) =$$

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\delta)}(\sum_{n=-\infty}^{\infty} \frac{|f^{p_i}(a) - f^{p_i}(b)|}{2^{|n|}}) =$$

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\delta)}(3\,|f^{p_i}(a) - f^{p_i}(b)|) = 0.$$

and for every $t > 0$ is

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\frac{t}{3})}(\rho_1(F^{p_i}(x), F^{p_i}(y))) =$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\frac{t}{3})}(\sum_{n=-\infty}^{\infty} \frac{|f^{p_i}(a) - f^{p_i}(b)|}{2^{|n|}}) =$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\frac{t}{3})}(3\,|f^{p_i}(a) - f^{p_i}(b)|) = 1.$$

In a similar way for the distance ρ_2 we have that there exists $\delta^* > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\delta^*)}(\rho_2(F^{p_i}(x), F^{p_i}(y))) =$$

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\delta^*)}(\sup|f^{p_i}(a) - f^{p_i}(b)|) =$$

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,\delta^*)}(|f^{p_i}(a) - f^{p_i}(b)|) = 0,$$

and for every $t > 0$ is held

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \chi_{[0,t)}(\rho_2(F^{p_i}(x), F^{p_i}(y))) =$$
$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \chi_{[0,t)}(\sup|f^{p_i}(a) - f^{p_i}(b)|) =$$
$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \chi_{[0,t)}(|f^{p_i}(a) - f^{p_i}(b)|) = 1.$$

Thus, F is distributionally chaotic with respect to $\{p_i\}_{i\in\mathbb{N}}$ respectively using in $[a,b]_\infty$ the metrics ρ_1 and ρ_2 ending the proof. ∎

Proof of Theorem B. Since f has positive topological entropy by Lemma 2 there exists an increasing sequence $\{p_i\}_{i\in\mathbb{N}}$ such that f is distributionally chaotic with repect to $\{p_i\}_{i\in\mathbb{N}}$. Let $S \subset [a,b]$ be the uncountable set distributionally scrambled with respect to $\{p_i\}_{i\in\mathbb{N}}$ for f composed by almost periodic points. Let $E \subset [a,b]_\infty$ be the uncountable set such that each element of it is a constant sequence equal to an element of S. The proof of Theorem A states that E is an uncontable distributionally scrambled set for F with respect to $\{p_i\}_{i\in\mathbb{N}}$. Now, we shall see that E is composed by almost periodic points of F respectively for the metrics ρ_1 and ρ_2. Indeed, let $\alpha = \{x_n = x^*\}_{n\in\mathbb{N}} \in E$ where $x \in \mathrm{AP}(f)$. Then, for any $\varepsilon > 0$ there exists $N > 0$ such that for any $q \geq 0$, there exists $r, q < r \leq q + N$, holding $|f^r(x^*) - x^*| < \varepsilon$. In this setting,

$$\rho_1(F^r(x_0), x_0) = \sum_{n=-\infty}^{\infty} \frac{|f^r(x^*) - x^*|}{2^{|n|}} < 3\varepsilon$$

and

$$\rho_2(F^r(x_0), x_0) = \sup|f^r(x^*) - x^*| \leq \frac{\varepsilon}{3},$$

proving that $E \subset \mathrm{AP}(F)$ ending the proof. ∎

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