

# All Connected Graphs with Maximum Degree at Most 3 whose Energies are Equal to the Number of Vertices

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## Abstract

The energy  $E(G)$  of a graph  $G$  is defined as the sum of the absolute values of its eigenvalues. Let  $S_2$  be the star of order 2 (or  $K_2$ ) and  $Q$  be the graph obtained from  $S_2$  by attaching two pendent edges to each of the end vertices of  $S_2$ . Majstorović et al. conjectured that  $S_2$ ,  $Q$  and the complete bipartite graphs  $K_{2,2}$  and  $K_{3,3}$  are the only connected graphs with maximum degree  $\Delta \leq 3$  whose energies are equal to the number of vertices. This paper is devoted to giving a confirmative proof to this conjecture.

## 1 Introduction

We use Bondy and Murty [2] for terminology and notations not defined here. Let  $G$  be a simple graph with  $n$  vertices and  $m$  edges. The *cyclomatic number* of a connected graph  $G$  is defined as  $c(G) = m - n + 1$ . A graph  $G$  with  $c(G) = k$  is called a  $k$ -*cyclic graph*. In particular, for  $c(G) = 0, 1$  or  $2$  we call  $G$  a tree, unicyclic or bicyclic graph, respectively. Denote by  $\Delta$  the maximum degree of a graph. The eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  of the adjacency matrix  $A(G)$  of  $G$  are said to be the eigenvalues of the

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graph  $G$ . The *energy* of  $G$  is defined as

$$E = E(G) = \sum_{i=1}^n |\lambda_i| .$$

For several classes of graphs it has been demonstrated that the energy exceeds the number of vertices (see, [6]). In 2007, Nikiforov [12] showed that for almost all graphs,

$$E = \left( \frac{4}{3\pi} + o(1) \right) n^{3/2} .$$

Thus the number of graphs  $G$  satisfying the condition  $E(G) < n$  is relatively small. In [8], a connected graph  $G$  of order  $n$  is called *hypoenergetic* if  $E(G) < n$ . For hypoenergetic graphs with  $\Delta \leq 3$ , we have the following previously known results.

**Lemma 1.1.** [7] *There exist only four hypoenergetic trees with  $\Delta \leq 3$ , depicted in Figure 1.*

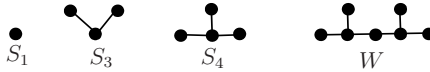


Figure 1: The hypoenergetic trees with maximum degree at most 3.

**Lemma 1.2.** [13] *Let  $G$  be a graph of order  $n$  with at least  $n$  edges and with no isolated vertices. If  $G$  is quadrangle-free and  $\Delta(G) \leq 3$ , then  $E(G) > n$ .*

The present authors first in [9] showed that complete bipartite graph  $K_{2,3}$  is the only hypoenergetic graph among all unicyclic and bicyclic graphs with  $\Delta \leq 3$ , and then recently they obtained the following general result:

**Lemma 1.3.** [10] *Complete bipartite graph  $K_{2,3}$  is the only hypoenergetic connected cycle-containing (or cyclic) graph with  $\Delta \leq 3$ .*

Therefore, all connected hypoenergetic graphs with maximum degree at most 3 have been characterized.

**Lemma 1.4.** [10]  *$S_1, S_3, S_4, W$  and  $K_{2,3}$  are the only hypoenergetic connected graphs with  $\Delta \leq 3$ .*

In [11] Majstorović et al. proposed the following conjecture, which is the second half of their Conjecture 3.7.

**Conjecture 1.5.** [11] *There are exactly four connected graphs  $G$  with order  $n$  and  $\Delta \leq 3$  for which the equality  $E(G) = n$  holds, which are depicted in Figure 2.*

In this paper, we will prove this conjecture.

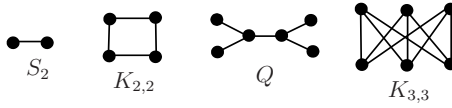


Figure 2: All connected graphs with maximum degree at most 3 and  $E = n$ .

## 2 Main results

The following results are needed in the sequel.

**Lemma 2.1.** [5] *If  $F$  is an edge cut of a graph  $G$ , then  $E(G - F) \leq E(G)$ , where  $G - F$  is the subgraph obtained from  $G$  by deleting the edges in  $F$ .*

**Lemma 2.2.** [5] *Let  $F = [S, V \setminus S]$  be an edge cut of a graph  $G$  with vertex set  $V$ , where  $S$  is a nonempty proper subset of  $V$ . Suppose that  $F$  is not empty and all edges in  $F$  are incident to one and only one vertex in  $S$ , i. e., the edges in  $F$  form a star. Then  $E(G - F) < E(G)$ .*

**Lemma 2.3.** [1] *The energy of a graph can not be an odd integer.*

In the following we first show that Conjecture 1.5 holds for trees, unicyclic and bicyclic graphs, respectively. Then we show that Conjecture 1.5 holds in general.

Let  $F$  be an edge cut of a connected graph  $G$ . If  $G - F$  has exactly two components  $G_1$  and  $G_2$ , then we denote  $G - F = G_1 + G_2$  for convenience. The following lemma is needed.

**Lemma 2.4.** *Let  $F$  be an edge cut of a connected graph  $G$  of order  $n$  such that  $G - F = G_1 + G_2$ . If  $E(G_1) \geq |V(G_1)|$ ,  $E(G_2) \geq |V(G_2)|$  and either at least one of the above inequalities is strict or the edges in  $F$  form a star or both, then  $E(G) > n$ .*

*Proof.* If  $E(G_1) > |V(G_1)|$  or  $E(G_2) > |V(G_2)|$ , then by Lemma 2.1, we have

$$E(G) \geq E(G - F) = E(G_1) + E(G_2) > |V(G_1)| + |V(G_2)| = n.$$

Otherwise by Lemma 2.2, we have

$$E(G) > E(G - F) = E(G_1) + E(G_2) \geq |V(G_1)| + |V(G_2)| = n$$

which completes the proof. ■

The result Lemma 2.4 is easy but useful in our proofs.

**Theorem 2.5.**  $S_2$  and  $Q$  are the only two trees  $T$  with order  $n$  and  $\Delta \leq 3$  for which the equality  $E(T) = n$  holds.

*Proof.* Let  $T$  be a tree with  $n$  vertices and  $\Delta \leq 3$ . From Table 2 of [3], we know that  $S_2$  and  $Q$  are the only two trees with  $\Delta \leq 3$  and  $n \leq 10$  for which the equality  $E = n$  holds. By Lemma 2.3, we may assume that  $n \geq 12$  is even. We will prove that  $E(T) > n$ .

We divide the trees with  $\Delta \leq 3$  into two classes: **Class 1** contains the trees  $T$  that have an edge  $e$ , such that  $T - e = T_1 + T_2$  and  $T_1, T_2 \not\cong S_1, S_3, S_4, W$ . **Class 2** contains the trees  $T$  in which there exists no edge  $e$ , such that  $T - e = T_1 + T_2$  and  $T_1, T_2 \not\cong S_1, S_3, S_4, W$ , i. e., for any edge  $e$  of  $T$  at least one of components of  $T - e$  is isomorphic to a tree in  $\{S_1, S_3, S_4, W\}$ .

**Case 1.**  $T$  belongs to Class 1. Then there exists an edge  $e$  such that  $T - e = T_1 + T_2$  and  $T_1, T_2 \not\cong S_1, S_3, S_4, W$ . Hence by Lemmas 1.1 and 2.2, we have  $E(T) > E(T - e) = E(T_1) + E(T_2) \geq |V(T_1)| + |V(T_2)| = n$ , which completes the proof.

**Case 2.**  $T$  belongs to Class 2. Consider the center of  $T$ . There are two subcases: either  $T$  has a (unique) center edge  $e$  or a (unique) center vertex  $v$ .

**Subcase 2.1.**  $T$  has a center edge  $e$ . The two fragments attached to  $e$  will be denoted by  $T_1$  and  $T_2$ , i. e.,  $T - e = T_1 + T_2$ .

Without loss of generality, we assume that  $T_1$  is isomorphic to a tree in  $\{S_1, S_3, S_4, W\}$ .

If  $T_1$  is isomorphic to a tree in  $\{S_1, S_3, S_4\}$ , then it is easy to see that  $n \leq 11$ , which is a contradiction.

If  $T_1 \cong W$  and it is attached to the center edge  $e$  through the vertex of degree 2, then it is easy to see that  $T$  must be the tree as given in Figure 3 (a) or (b). By direct computing, we have that  $E(T) = 12.61708 > 12 = n$  in the former case while  $E(T) = 14.91128 > 14 = n$  in the latter case. If  $T_1 \cong W$  and it is attached to the center edge  $e$  through a pendent vertex, see Figure 3 (c). Since  $T$  belongs to Class 2, deleting the edge  $f$ , we then have that  $T_2 \cup e$  is isomorphic to a tree in  $\{S_1, S_3, S_4, W\}$ , which contradicts to the fact that  $e$  is the center edge of  $T$ .

**Subcase 2.2.**  $T$  has a center vertex  $v$ . If  $v$  is of degree 2, then the two fragments attached to it will be denoted by  $T_1$  and  $T_2$ . If  $v$  is of degree 3, then the three

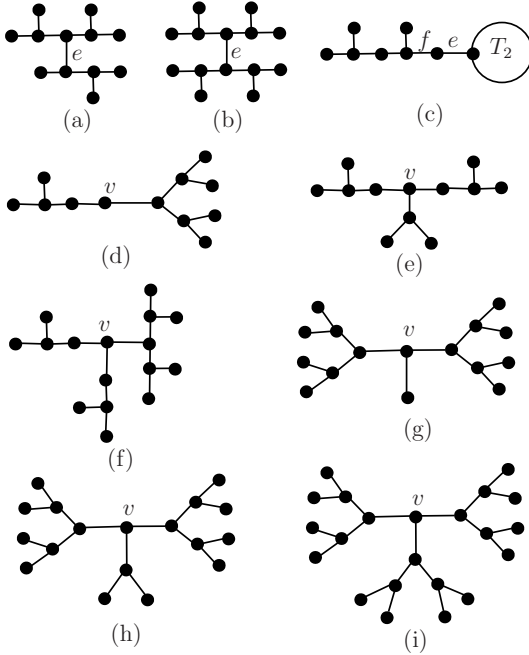


Figure 3: The graphs in the proof of Theorem 2.5.

fragments attached to it will be denoted by  $T_1$ ,  $T_2$  and  $T_3$ .

Let  $v_i$  be the adjacent vertex of  $v$  in  $T_i$ . Denote  $T - vv_1 = T_1 + T'_2$ . Since  $T$  belongs to Class 2, either  $T_1$  or  $T'_2$  is isomorphic to a tree in  $\{S_1, S_3, S_4, W\}$ .

**Subsubcase 2.2.1.**  $T'_2$  is isomorphic to a tree in  $\{S_1, S_3, S_4, W\}$ .

Clearly  $T'_2 \not\cong S_1$ . If  $T'_2 \cong S_3$  or  $S_4$ , then it is easy to see that  $n \leq 7$ , which is a contradiction. If  $T'_2 \cong W$  and  $v$  is of degree 3, then it is easy to see that  $n \leq 10$ , which is a contradiction. If  $T'_2 \cong W$  and  $v$  is of degree 2, i. e.,  $N(v) = \{v_1, v_2\}$ . Consider  $T - vv_2$ , since  $T$  belongs to Class 2, we have that  $T_1 \cup vv_1$  is isomorphic to a tree in  $\{S_1, S_3, S_4, W\}$ . By the fact that  $v$  is the center of  $T$ , we have that  $T_1 \cup vv_1 \cong W$ , and so  $n = 13$ , which is a contradiction.

**Subsubcase 2.2.2.**  $T_1$  is isomorphic to a tree in  $\{S_1, S_3, S_4, W\}$ .

If  $T_1 \cong S_1$ , then it is easy to see that  $n \leq 4$ , which is a contradiction.

If  $T_1 \cong S_3$  and  $v_1$  is of degree 2 in  $T_1$ , then it is easy to see that  $n \leq 10$ , which is

a contradiction. If  $T_1 \cong S_3$  and  $v_1$  is a pendent vertex in  $T_1$ , denote by  $u$  the unique adjacent vertex of  $v_1$  in  $T_1$ . Since  $T$  belongs to Class 2, deleting the edge  $uv_1$ , we then have that  $T'_2 \cup vv_1$  is isomorphic to a tree in  $\{S_1, S_3, S_4, W\}$ , and so  $n \leq 9$ , which is a contradiction.

If  $T_1 \cong S_4$  or  $T_1 \cong W$  and  $v_1$  is of degree 2 in  $T_1$ , then by the facts that  $T$  belongs to Class 2,  $v$  is the center of  $T$  and  $n$  is even, it is not hard to obtain that  $T_2, T_3$  must be isomorphic to a tree in  $\{S_1, S_3, S_4, W\}$ , and at least one of  $T_2$  and  $T_3$  is isomorphic to a tree in  $\{S_4, W\}$ , and if  $T_2$  ( $T_3$ , respectively) is isomorphic to  $W$ , then  $v_2$  ( $v_3$ , respectively) is of degree 2 in  $T_2$  ( $T_3$ , respectively). Hence there are 6 such trees, as given in Figure 3 (d), (e), (f), (g), (h) and (i). The energy of these trees are 12.72729 ( $> 12 = n$ ), 12.65406 ( $> 12 = n$ ), 16.81987 ( $> 16 = n$ ), 16.77215 ( $> 16 = n$ ), 19.18674 ( $> 18 = n$ ) and 23.38426 ( $> 22 = n$ ), respectively.

If  $T_1 \cong W$  and  $v_1$  is a pendent vertex in  $T_1$ , denote by  $u$  the unique adjacent vertex of  $v_1$  in  $T_1$ . Since  $T$  belongs to Class 2, deleting the edge  $uv_1$ , we then have that  $T'_2 \cup vv_1$  is isomorphic to a tree in  $\{S_1, S_3, S_4, W\}$ , which contradicts to the fact that  $v$  is the center vertex of  $T$ . The proof is thus complete.  $\blacksquare$

From Table 1 of [3], we know that  $K_{2,2}$  is the only connected graph of order 4 with  $\Delta \leq 3$  and  $E = 4$ . From Tables 1 and 2 of [4], we know that  $K_{3,3}$  is the only connected cycle-containing graph of order 6 with  $\Delta \leq 3$  and  $E = 6$ .

**Theorem 2.6.**  *$K_{2,2}$  is the only unicyclic graph with  $\Delta \leq 3$  for which the equality  $E = n$  holds.*

*Proof.* Let  $G \not\cong K_{2,2}$  be a unicyclic graph of order  $n$  with  $\Delta \leq 3$ . It is sufficient to show that  $E(G) > n$ . By Lemmas 1.2 and 2.3, we can assume that  $n \geq 8$  is even and  $G$  contains a quadrangle  $C = x_1x_2x_3x_4x_1$ . We distinguish the following four cases:

**Case 1.** There exists an edge  $e$  on  $C$  such that the end vertices of  $e$  are of degree 2.

Without loss of generality, we assume that  $d(x_1) = d(x_4) = 2$ .

Let  $F = \{x_1x_2, x_4x_3\}$ , then  $G - F = G_1 + G_2$ , where  $G_1 \cong S_2$  and  $G_2$  is a tree of order at least 6 since  $n \geq 8$ . Since  $\Delta(G) \leq 3$ ,  $G_2$  can not be isomorphic to  $W$  or  $Q$ . Therefore we have  $E(G_1) = |V(G_1)|$  and  $E(G_2) > |V(G_2)|$  by Lemma 1.1 and Theorem 2.5. It follows from Lemma 2.4 that  $E(G) > n$ .

**Case 2.** There exist exactly two nonadjacent vertices  $x_i$  and  $x_j$  on  $C$  such that  $d(x_i) = d(x_j) = 2$ .

Without loss of generality, we assume that  $d(x_2) = d(x_4) = 2$ ,  $d(x_1) = d(x_3) = 3$ . Let  $y_3$  be the adjacent vertex of  $x_3$  outside  $C$ . Then  $G - x_3y_3 = G_1 + G_2$ , where  $G_1$  is a unicyclic graph and  $G_2$  is a tree. Notice that  $E(G_1) \geq |V(G_1)|$  by Lemma 1.3. If  $G_2 \not\cong S_1, S_3, S_4, W$ , then we have  $E(G_2) \geq |V(G_2)|$  by Lemma 1.1 and so  $E(G) > E(G - x_3y_3) \geq n$  by Lemma 2.4. Therefore we only need to consider the following four subcases.

**Subcase 2.1.**  $G_2 \cong S_1$ . Let  $F = \{x_2x_3, x_3x_4\}$ , then  $G - F = G'_1 + G'_2$ , where  $G'_2 \cong S_2$  and  $G'_1$  is a tree of order at least 6 since  $n \geq 8$ . If  $G'_1 \cong W$ , then  $n = 9$ , which is a contradiction. Otherwise, it follows from Lemmas 1.1 and 2.4 that  $E(G) > n$ .

**Subcase 2.2.**  $G_2 \cong S_3$ . Then  $G$  must have the structure as given in Figure 4 (a) or (b). In the former case,  $G - y_3z = G'_1 + G'_2$ , where  $G'_1$  is a unicyclic graph and  $G'_2 \cong S_2$ . It follows from Lemmas 1.4 and 2.4 that  $E(G) > n$ . In the latter case,  $G - \{x_1x_2, x_4x_3\} = G'_1 + G'_2$ , where  $G'_2$  is the tree of order 5 containing  $x_3$  and  $G'_1$  is a tree of order at least 3. By Lemma 1.1 and Theorem 2.5, we have  $E(G'_2) > |V(G'_2)|$ . If  $G'_1 \not\cong S_3, S_4, W$ , then we have  $E(G) > n$  by Lemmas 1.1 and 2.4. Since  $\Delta(G) \leq 3$ ,  $G'_1$  can not be isomorphic to  $S_4$  or  $W$ . If  $G'_1 \cong S_3$ , then  $G$  must be the graph as given in Figure 4 (c). By choosing the edge cut  $\{x_1x_2, x_1x_4\}$ , we can similarly obtain that  $E(G) > n$ .

**Subcase 2.3.**  $G_2 \cong S_4$ . Then  $G$  must have the structure as given in Figure 4 (d). Let  $F = \{x_2x_3, x_3x_4\}$ , then  $G - F = G'_1 + G'_2$ , where  $G'_2$  is the tree of order 5 containing  $x_3$  and  $G'_1$  is a tree of order at least 4. By Lemma 1.1 and Theorem 2.5, we have  $E(G'_2) > |V(G'_2)|$ . If  $G'_1 \not\cong S_4, W$ , then we have  $E(G) > n$  by Lemmas 1.1 and 2.4. If  $G'_1 \cong S_4$ , then  $n = 9$ , which is a contradiction. If  $G'_1 \cong W$ , then  $G$  must be the graph as given in Figure 4 (e). By choosing the edge cut  $\{x_1x_2, x_3x_4\}$ , we can similarly obtain that  $E(G) > n$ .

**Subcase 2.4.**  $G_2 \cong W$ . Then  $G$  must have the structure as given in Figure 4 (f) or (g). In the former case,  $G - y_3z = G'_1 + G'_2$ , where  $G'_1$  is a unicyclic graph and  $G'_2$  is a tree of order 6. It follows from Lemmas 1.4 and 2.4 that  $E(G) > n$ . In the latter case,  $G - \{x_2x_3, x_3x_4\} = G'_1 + G'_2$ , where  $G'_2$  is the tree of order 8 containing

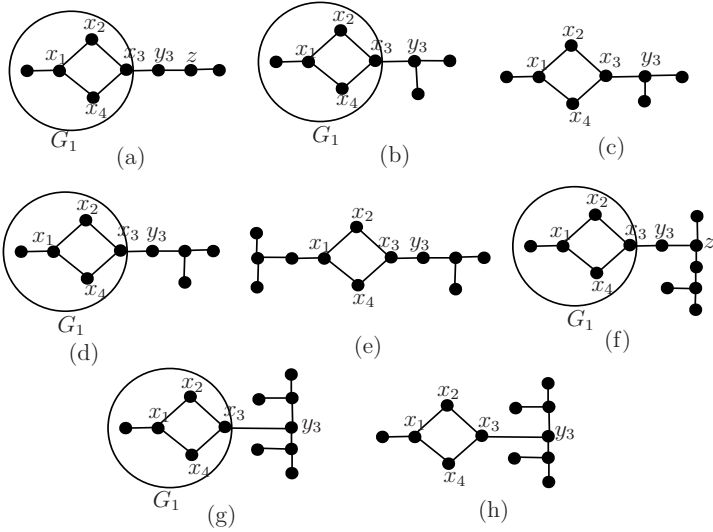


Figure 4: The graphs in the proof of Theorem 2.6.

$x_3$  and  $G'_1$  is a tree of order at least 4. If  $G'_1 \not\cong S_4, W$ , then we have  $E(G) > n$  by Lemmas 1.1 and 2.4. If  $G'_1 \cong S_4$ , then  $G$  must be the graph as given in Figure 4 (h). By choosing the edge cut  $\{x_1x_2, x_1x_4\}$ , we can similarly obtain that  $E(G) > n$ . If  $G'_1 \cong W$ , then  $n = 15$ , which is a contradiction.

**Case 3.** There exists exactly one vertices  $x_i$  on  $C$  such that  $d(x_i) = 2$ .

Without loss of generality, we assume that  $d(x_1) = 2$ . Let  $F = \{x_1x_4, x_2x_3\}$ , then  $G - F = G_1 + G_2$ , where  $G_1$  is the tree of order at least 3 containing  $x_1$  and  $G_2$  is a tree of order at least 4. Since  $\Delta(G) \leq 3$ ,  $G_1, G_2$  can not be isomorphic to  $S_4, W$  or  $Q$ . If  $G_1 \not\cong S_3$ , then we have  $E(G) > n$  by Lemmas 1.1, 2.4 and Theorem 2.5. If  $G_1 \cong S_3$ , then  $G - \{x_1x_2, x_2x_3\} = G'_1 + G'_2$ , where  $G'_1$  is the tree of order at least 5 containing  $x_1$  and  $G'_2 \cong S_2$ . If  $G'_1 \not\cong W$ , then we have  $E(G) > n$  by Lemmas 1.1 and 2.4. If  $G'_1 \cong W$ , then  $n = 9$ , which is a contradiction.

**Case 4.**  $d(x_1) = d(x_2) = d(x_3) = d(x_4) = 3$ .

Let  $F = \{x_1x_4, x_2x_3\}$ , then  $G - F = G_1 + G_2$ , where  $G_1$  and  $G_2$  are trees of order at least 4 and it is easy to see that  $G_1, G_2$  can not be isomorphic to  $S_4, W$  or  $Q$ . So it follows from Lemmas 1.1, 2.4 and Theorem 2.5 that  $E(G) > n$ . The proof is thus



complete. ■

**Theorem 2.7.** *There does not exist any bicyclic graph with  $\Delta \leq 3$  for which the equality  $E = n$  holds.*

*Proof.* Let  $G$  be a bicyclic graph of order  $n$  with  $\Delta \leq 3$ . We know that  $E(G) \neq n$  for  $n = 4$  or  $6$ . By Lemmas 1.2 and 2.3, we may assume that  $n \geq 8$  is even and  $G$  contains a quadrangle. Then we will show that  $E(G) > n$ .

If the cycles in  $G$  are disjoint, then it is clear that there exists a path  $P$  connecting the two cycles in  $G$ . For any edge  $e$  on  $P$ , we have  $G - e = G_1 + G_2$ , where  $G_1$  and  $G_2$  are unicyclic graphs. By Lemma 1.3, we have  $E(G_1) \geq |V(G_1)|$  and  $E(G_2) \geq |V(G_2)|$ . Therefore we have  $E(G) > n$  by Lemma 2.4. Otherwise, the cycles in  $G$  have two or more common vertices. Then we can assume that  $G$  contains a subgraph as given in Figure 5 (a), where  $P_1, P_2, P_3$  are paths in  $G$ . We distinguish the following three cases:

**Case 1.** At least one of  $P_1, P_2$  and  $P_3$ , say  $P_2$  has length not less than 3.

Let  $e_1$  and  $e_2$  be the edges on  $P_2$  incident with  $u$  and  $v$ , respectively. Then  $G - \{e_1, e_2\} = G_1 + G_2$ , where  $G_1$  is a unicyclic graph and  $G_2$  is a tree of order at least 2. It follows from Lemma 1.3 that  $E(G_1) \geq |V(G_1)|$ . If  $G_2 \not\cong S_3, S_4, W, S_2, Q$ , then we have  $E(G_2) > |V(G_2)|$  by Lemma 1.1 and Theorem 2.5, and so  $E(G) > n$  by Lemma 2.4. Hence we only need to consider the following five subcases.

**Subcase 1.1.**  $G_2 \cong S_3$ . Then  $G$  must have the structure as given in Figure 5 (b) or (c). In either case,  $G - \{e_2, e_3\} = G'_1 + G'_2$ , where  $G'_1$  is a unicyclic graph and  $G'_2 \cong S_2$ . Obviously,  $G'_1 \not\cong K_{2,2}$ . Then  $E(G'_1) > |V(G'_1)|$  by Lemma 1.3 and Theorems 2.6. Since  $E(G'_2) = |V(G'_2)|$ , we have  $E(G) > n$  by Lemma 2.4.

**Subcase 1.2.**  $G_2 \cong S_4$ . Then  $G$  must have the structure as given in Figure 5 (d). Obviously,  $G - \{e_3, e_4\} = G'_1 + G'_2$ , where  $G'_1$  is a unicyclic graph which is not isomorphic to  $K_{2,2}$  and  $G'_2 \cong S_2$ . Similar to the proof of Subcase 1.1, we have  $E(G) > n$ .

**Subcase 1.3.**  $G_2 \cong W$ . Then  $G$  must have the structure as given in Figure 5 (e), (f) or (g). Obviously,  $G - \{xy, yz\} = G'_1 + G'_2$ , where  $G'_1$  is a unicyclic graph which is not isomorphic to  $K_{2,2}$  and  $G'_2$  is a tree of order 5 or 2. Similarly, we can obtain that  $E(G) > n$ .

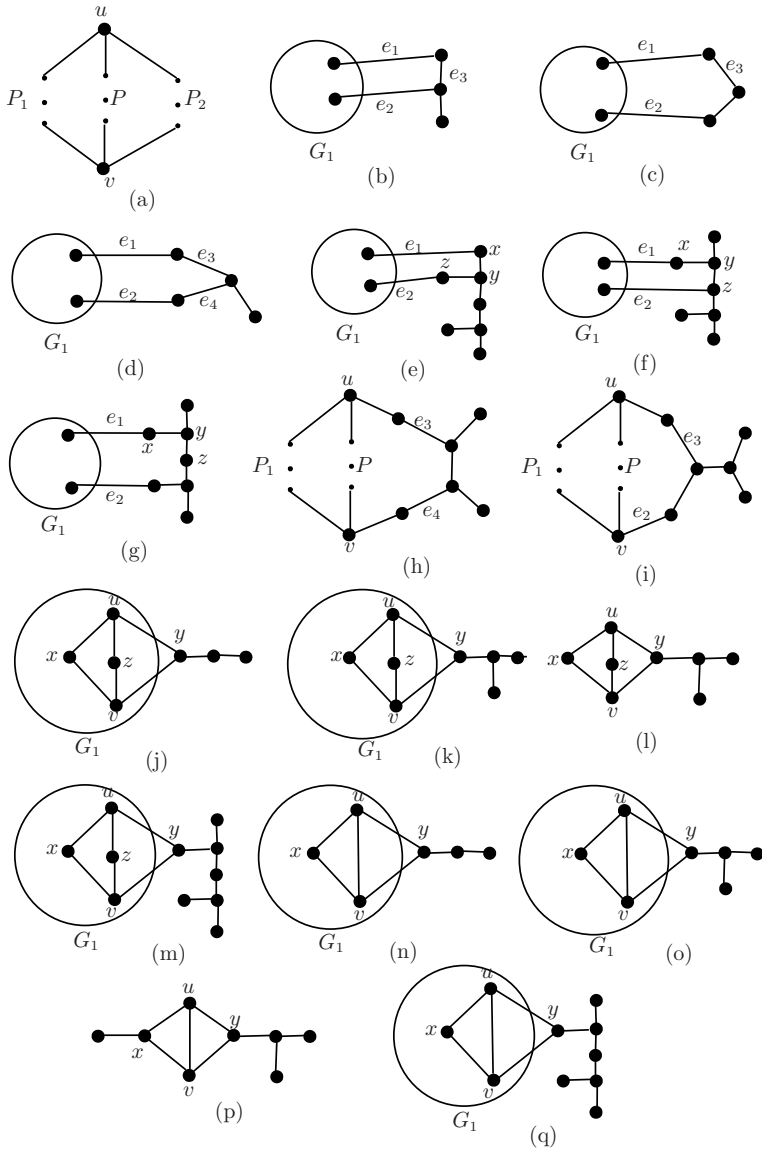


Figure 5: The graphs in the proof of Theorem 2.7.

**Subcase 1.4.**  $G_2 \cong S_2$ . Since  $G_1$  is a unicyclic graph, if  $G_1 \not\cong K_{2,2}$ , then we can similarly obtain that  $E(G) > n$ . If  $G_1 \cong K_{2,2}$ , then  $n = 6$ , which is a contradiction.

**Subcase 1.5.**  $G_2 \cong Q$ . Then  $G$  must have the structure as given in Figure 5 (h) or (i). In the former case,  $G - \{e_3, e_4\} = G'_1 + G'_2$ , where  $G'_2$  is a path of order 4 and  $G'_1$  is a unicyclic graph which is not isomorphic to  $K_{2,2}$ . Similarly, we can obtain that  $E(G) > n$ . In the latter case,  $G - \{e_2, e_3\} = G'_1 + G'_2$ , where  $G'_2$  is a tree of order 5 and  $G'_1$  is a unicyclic graph which is not isomorphic to  $K_{2,2}$ . Similarly, we can obtain that  $E(G) > n$ .

**Case 2.** All the paths  $P_1, P_2$  and  $P_3$  have length 2.

We assume that  $P_1 = u xv$ ,  $P_2 = u zv$  and  $P_3 = u y v$ . Let  $F = \{u y, v y\}$ , then  $G - F = G_1 + G_2$ , where  $G_1$  is a unicyclic graph and  $G_2$  is a tree. It follows from Lemma 1.3 that  $E(G_1) \geq |V(G_1)|$ . If  $G_2 \not\cong S_1, S_3, S_4, W$ , then we have  $E(G_2) \geq |V(G_2)|$  by Lemma 1.1 and so  $E(G) > n$  by Lemma 2.4. Hence we only need to consider the following four subcases.

**Subcase 2.1.**  $G_2 \cong S_1$ . Let  $F' = \{u y, z v, x v\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_2 \cong S_2$  and  $G'_1$  is a tree of order at least 6 since  $n \geq 8$ . It is easy to see that  $G'_1$  can not be isomorphic to  $Q$  or  $W$ . Therefore we have  $E(G'_1) > |V(G'_1)|$  and  $E(G'_2) = |V(G'_2)|$  by Lemma 1.1 and Theorem 2.5. It follows from Lemma 2.4 that  $E(G) > n$ .

**Subcase 2.2.**  $G_2 \cong S_3$ . Then  $G$  must have the structure as given in Figure 5 (j). Let  $F' = \{u y, z v, x v\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_2$  is the path of order 4 containing  $y$  and  $G'_1$  is a tree of order at least 4 since  $n \geq 8$ . Clearly,  $G'_1$  can not be isomorphic to  $S_4, Q$  or  $W$ . Similar to the proof of Subcase 2.1, we have  $E(G) > n$ .

**Subcase 2.3.**  $G_2 \cong S_4$ . Then  $G$  must have the structure as given in Figure 5 (k). Let  $F' = \{u y, z v, x v\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_2$  is the tree of order 5 containing  $y$  and  $G'_1$  is a tree of order at least 3. Clearly,  $G'_1$  can not be isomorphic to  $S_4$  or  $W$ . If  $G'_1 \not\cong S_3$ , then we can similarly obtain that  $E(G) > n$ . If  $G'_1 \cong S_3$ , then  $G$  must be the graph as given in Figure 5 (l). By choosing the edge cut  $\{u y, u z, x v\}$ , we can also obtain that  $E(G) > n$ .

**Subcase 2.4.**  $G_2 \cong W$ . Then  $G$  must have the structure as given in Figure 5 (m). Let  $F' = \{u y, z v, x v\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_2$  is the tree of order 8

containing  $y$  and  $G'_1$  is a tree of order at least 3. Clearly,  $G'_1$  can not be isomorphic to  $S_4$  or  $W$ . If  $G'_1 \cong S_3$ , then  $n = 11$ , which is a contradiction. If  $G'_1 \not\cong S_3$ , then we can similarly obtain that  $E(G) > n$ .

**Case 3.** One of the paths  $P_1, P_2$  and  $P_3$  has length 1, and the other two paths have length 2.

Without loss of generality, we assume that  $P = uv$ ,  $P_1 = u xv$  and  $P_2 = u y v$ . Let  $F = \{u y, v y\}$ , then  $G - F = G_1 + G_2$ , where  $G_1$  is a unicyclic graph and  $G_2$  is a tree. Similarly, if  $G_2 \not\cong S_1, S_3, S_4, W$ , then we have  $E(G) > n$ . Hence we also need to consider the following four subcases.

**Subcase 3.1.**  $G_2 \cong S_1$ . Let  $F' = \{u y, u v, x v\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_2 \cong S_2$  and  $G'_1$  is a tree of order at least 6 since  $n \geq 8$ . Since  $\Delta(G) \leq 3$ ,  $G'_1$  can not be isomorphic to  $Q$  or  $W$ . Similar to the proof of Subcase 2.1, we have  $E(G) > n$ .

**Subcase 3.2.**  $G_2 \cong S_3$ . Then  $G$  must have the structure as given in Figure 5 (n). Let  $F' = \{u y, u v, x v\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_2$  is the path of order 4 containing  $y$  and  $G'_1$  is a tree of order at least 4 since  $n \geq 8$ . Clearly,  $G'_1$  can not be isomorphic to  $S_4$  or  $W$ . Similarly, we have  $E(G) > n$ .

**Subcase 3.3.**  $G_2 \cong S_4$ . Then  $G$  must have the structure as given in Figure 5 (o). Let  $F' = \{u y, u v, x v\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_2$  is the tree of order 5 containing  $y$  and  $G'_1$  is a tree of order at least 3. Clearly,  $G'_1$  can not be isomorphic to  $S_4$  or  $W$ . If  $G'_1 \not\cong S_3$ , then we can similarly obtain that  $E(G) > n$ . If  $G'_1 \cong S_3$ , then  $G$  must be the graph as given in Figure 5 (p). By choosing the edge cut  $\{x u, x v\}$ , we can similarly obtain that  $E(G) > n$ .

**Subcase 3.4.**  $G_2 \cong W$ . Then  $G$  must have the structure as given in Figure 5 (q). Let  $F' = \{u y, u v, x v\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_2$  is the tree of order 8 containing  $y$  and  $G'_1$  is a tree of order at least 2. Clearly,  $G'_1$  can not be isomorphic to  $S_4$  or  $W$ . If  $G'_1 \cong S_3$ , then  $n = 11$ , which is a contradiction. If  $G'_1 \not\cong S_3$ , then we can similarly obtain that  $E(G) > n$ . The proof is thus complete. ■

**Proof of Conjecture 1.5:** Let  $G$  be a connected graph of order  $n$  with  $\Delta \leq 3$ . Clearly, if  $G$  is isomorphic to a graph in  $\{S_2, Q, K_{2,2}, K_{3,3}\}$ , then  $E(G) = n$ . We will prove that  $E(G) \neq n$  if  $G \not\cong S_2, Q, K_{2,2}$  or  $K_{3,3}$  by induction on the cyclomatic number  $c(G)$ . It follows from Theorems 2.5, 2.6 and 2.7 that the result holds for

$c(G) \leq 2$ . Let  $k \geq 3$  be an integer. We assume that the result holds for  $c(G) < k$ . Now let  $G$  be a graph with  $c(G) = k \geq 3$ . We will show that  $E(G) \neq n$ .

By Lemma 2.3, the result holds if  $n$  is odd. By the fact that  $K_{3,3}$  is the only connected cycle-containing graph of order 6 with  $\Delta \leq 3$  and  $E = 6$ , we know that the result holds for  $n \leq 6$ . So in the following we assume that  $n \geq 8$  is even. In our proof we will repeatedly make use of the following claim:

**Claim 1.** *Let  $F$  be an edge cut of  $G$  such that  $G - F = G_1 + G_2$  with  $c(G_1), c(G_2) < k$ . If  $G_1, G_2 \not\cong S_1, S_3, S_4, W$  or  $K_{2,3}$  and either the edges in  $F$  form a star or at least one of  $G_1$  and  $G_2$  is not isomorphic to  $S_2, Q$  or  $K_{2,2}$ , then we are done.*

*Proof.* By Lemma 1.4, we have  $E(G_1) \geq |V(G_1)|$  and  $E(G_2) \geq |V(G_2)|$ . Clearly,  $G_1, G_2 \not\cong K_{3,3}$ . If  $G_i \not\cong S_2, Q$  or  $K_{2,2}$ , then by induction hypothesis, we have  $E(G_i) \neq |V(G_i)|$ . Therefore we have  $E(G) > n$  by Lemma 2.4. ■

In what follows, we use  $\hat{G}$  to denote the graph obtained from  $G$  by repeatedly deleting the pendent vertices. Clearly,  $c(\hat{G}) = c(G)$ . Denote by  $\kappa'(\hat{G})$  the edge connectivity of  $\hat{G}$ . Since  $\Delta(\hat{G}) \leq 3$ , we have  $1 \leq \kappa'(\hat{G}) \leq 3$ . Therefore we only need to consider the following three cases.

**Case 1.**  $\kappa'(\hat{G}) = 1$ .

Let  $e$  be a cut edge of  $\hat{G}$ . Then  $\hat{G} - e$  has exactly two components, say,  $H_1$  and  $H_2$ . It is clear that  $c(H_1) \geq 1$ ,  $c(H_2) \geq 1$  and  $c(H_1) + c(H_2) = k$ . Consequently,  $G - e$  has exactly two components  $G_1$  and  $G_2$  with  $c(G_1) \geq 1$ ,  $c(G_2) \geq 1$  and  $c(G_1) + c(G_2) = k$ , where  $H_i$  is a subgraph of  $G_i$  for  $i = 1, 2$ . If neither  $G_1$  nor  $G_2$  is isomorphic to  $K_{2,3}$ , then we are done by Claim 1. Otherwise, without loss of generality, we assume that  $G_1 \cong K_{2,3}$ . Then  $G$  must have the structure as given in Figure 6 (a). Now, let  $F = \{e_1, e_2\}$ . Then  $G - F = G'_1 + G'_2$ , where  $G'_1 \cong K_{2,2}$  and  $G'_2 = G_2 \cup e$ . Therefore we have that  $c(G'_2) = k - 2 \geq 1$  and  $G'_2 \not\cong K_{2,2}, K_{2,3}$ , and so we are done by Claim 1.

**Case 2.**  $\kappa'(\hat{G}) = 2$ .

Let  $F = \{e_1, e_2\}$  be an edge cut of  $\hat{G}$ . Then  $\hat{G} - F$  has exactly two components, say,  $H_1$  and  $H_2$ . Clearly,  $c(H_1) + c(H_2) = k - 1 \geq 2$ .

**Subcase 2.1.**  $c(H_1) \geq 1$  and  $c(H_2) \geq 1$ . Therefore,  $G - F$  has exactly two components  $G_1$  and  $G_2$  with  $c(G_1) \geq 1$ ,  $c(G_2) \geq 1$  and  $c(G_1) + c(G_2) = k - 1$ , where  $H_i$

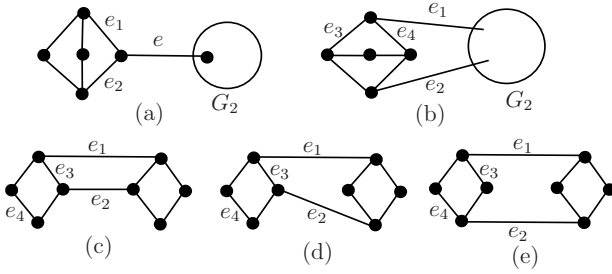


Figure 6: The graphs in the proof of Case 1 and Subcase 2.1 of Conjecture 1.5.

is a subgraph of  $G_i$  for  $i = 1, 2$ . If  $G_1, G_2 \not\cong K_{2,3}$  and at least one of  $G_1$  and  $G_2$  is not isomorphic to  $K_{2,2}$ , then we are done by Claim 1. If at least one of  $G_1$  and  $G_2$  is isomorphic to  $K_{2,3}$ , say  $G_1 \cong K_{2,3}$ . Then  $G$  must have the structure as given in Figure 6 (b). Now, let  $F' = \{e_2, e_3, e_4\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_1 \cong K_{2,2}$  and  $G'_2 = G_2 \cup e_1$ . Therefore we have that  $c(G'_2) = k - 3$  and  $G'_2 \not\cong K_{2,2}, K_{2,3}$ , and so we are done by Claim 1. If  $G_1, G_2 \cong K_{2,2}$ , then  $G$  must be the graph as given in Figure 6 (c), (d) or (e). Let  $F' = \{e_1, e_3, e_4\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_1 \cong S_2$  and  $G'_2 \not\cong K_{2,2}$  is a unicyclic graph. Hence we are done by Claim 1.

**Subcase 2.2.** One of  $H_1$  and  $H_2$ , say  $H_2$  is a tree. Therefore,  $G - F$  has exactly two components  $G_1$  and  $G_2$  with  $c(G_1) = k - 1$  and  $c(G_2) = 0$ , where  $H_i$  is a subgraph of  $G_i$  for  $i = 1, 2$ . Since  $k - 1 \geq 2$ ,  $G_1 \not\cong S_2, Q, K_{2,2}$ . If  $G_1 \not\cong K_{2,3}$  and  $G_2 \not\cong S_1, S_3, S_4, W$ , then we are done by Claim 1. So we assume that this is not true. We only need to consider the following five subsubcases.

**Subsubcase 2.2.1.**  $G_2 \cong S_1$ . Let  $V(G_2) = \{x\}$ ,  $e_1 = xx_1$  and  $e_2 = xx_2$ . It is clear that  $d_{G_1}(x_2) = 1$  or  $2$ . If  $d_{G_1}(x_2) = 1$ , let  $N_{G_1}(x_2) = \{y_1\}$  (see Figure 7 (a), where  $y_1$  may be equal to  $x_1$ ). Let  $F' = \{e_1, x_2y_1\}$ . Then  $G - F' = G'_1 + G'_2$ , where  $G'_1$  is a graph obtained from  $G_1$  by deleting a pendent vertex and  $G'_2 \cong S_2$ . Therefore,  $c(G'_1) = k - 1 \geq 2$ . If  $G'_1 \not\cong K_{2,3}$ , then we are done by Claim 1. Otherwise,  $n = 7$ , which is a contradiction.

If  $d_{G_1}(x_2) = 2$ , let  $N_{G_1}(x_2) = \{y_1, y_2\}$  (see Figure 7 (b), where one of  $y_1$  and  $y_2$  may be equal to  $x_1$ ). Let  $F' = \{e_1, x_2y_1, x_2y_2\}$ . Then  $G - F' = G'_1 + G'_2$ , where  $G'_1$  is a graph obtained from  $G_1$  by deleting a vertex of degree 2 and  $G'_2 \cong S_2$ . Therefore,

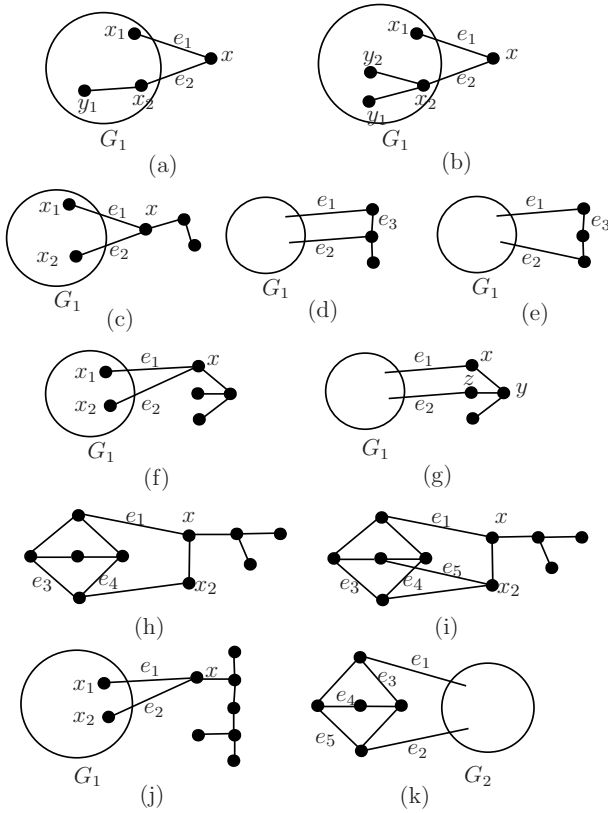


Figure 7: The graphs in the proof of Subcase 2.2 of Conjecture 1.5.

$c(G'_1) = k - 2 \geq 1$ . If  $G'_1 \not\cong K_{2,2}, K_{2,3}$ , then we are done by Claim 1. Otherwise,  $n = 6$  or  $7$ , which is a contradiction.

**Subsubcase 2.2.2.**  $G_2 \cong S_3$ . If  $e_1, e_2$  are incident with a common vertex in  $G_2$ , then  $G$  must have the structure as given in Figure 7 (c). Similar to the proof of Subsubcase 2.2.1, we can obtain that there exists an edge cut  $F'$  such that  $G - F' = G'_1 + G'_2$  satisfying that  $c(G'_1) = k - 1$  if  $d_{G_1}(x_2) = 1$  or  $c(G'_1) = k - 2$  if  $d_{G_1}(x_2) = 2$  and  $G'_2$  is a path of order 4. If  $G'_1 \not\cong K_{2,3}$ , then we are done by Claim 1. Otherwise  $n = 9$ , which is a contradiction.

If  $e_1, e_2$  are incident with two different vertices in  $G_2$ , then  $G$  must have the

structure as given in Figure 7 (d) or (e). Let  $F' = \{e_2, e_3\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_1 = G_1 \cup e_1$  and  $G'_2 \cong S_2$ . Therefore we have that  $c(G'_1) = k - 1 \geq 2$  and  $G'_2 \not\cong K_{2,3}$ , and so we are done by Claim 1.

**Subsubcase 2.2.3.**  $G_2 \cong S_4$ . If  $e_1, e_2$  are incident with a common vertex in  $G_2$ , then  $G$  must have the structure as given in Figure 7 (f). Similar to the proof of Subsubcase 2.2.1, we can obtain that there exists an edge cut  $F'$  such that  $G - F' = G'_1 + G'_2$  satisfying that  $c(G'_1) = k - 1$  if  $d_{G_1}(x_2) = 1$  or  $c(G'_1) = k - 2$  if  $d_{G_1}(x_2) = 2$  and  $G'_2$  is a tree of order 5. If  $G'_1 \not\cong K_{2,3}$ , then we are done by Claim 1. Otherwise  $G$  must be the graph as given in Figure 7 (h) or (i). In the former case let  $F'' = \{e_1, e_3, e_4\}$  while in the latter case let  $F'' = \{e_1, e_3, e_4, e_5\}$ . Then  $G - F'' = G''_1 + G''_2$ , where  $G''_1 \cong K_{2,2}$ ,  $G''_2$  is a tree of order 6 and  $G''_2 \not\cong Q$ . Therefore we are done by Claim 1.

If  $e_1, e_2$  are incident with two different vertices in  $G_2$ , then  $G$  must have the structure as given in Figure 7 (g). Let  $F' = \{xy, yz\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_1 = G_1 \cup \{e_1, e_2\}$  and  $G'_2 \cong S_2$ . Therefore we have that  $c(G'_1) = k - 1 \geq 2$  and  $G'_2 \not\cong K_{2,3}$ , and so we are done by Claim 1.

**Subsubcase 2.2.4.**  $G_2 \cong W$ . If  $e_1, e_2$  are incident with a common vertex in  $G_2$ , then  $G$  must have the structure as given in Figure 7 (j). Similar to the proof of Subsubcase 2.2.1, we can obtain that there exists an edge cut  $F'$  such that  $G - F' = G'_1 + G'_2$  satisfying that  $c(G'_1) = k - 1$  if  $d_{G_1}(x_2) = 1$  or  $c(G'_1) = k - 2$  if  $d_{G_1}(x_2) = 2$  and  $G'_2$  is a tree of order 8. If  $G'_1 \not\cong K_{2,3}$ , then we are done by Claim 1. Otherwise,  $n = 13$ , which is a contradiction.

If  $e_1, e_2$  are incident with two different vertices in  $G_2$ , then  $G$  must have the structure as given in Figure 5 (e), (f) or (g) ( $e_1, e_2$  may be incident with a common vertex in  $G_1$ ). Let  $F' = \{xy, yz\}$ , then  $G - F' = G'_1 + G'_2$ , where  $G'_2$  is the tree of order 5 or 2 containing  $y$ . Clearly,  $c(G'_1) = k - 1 \geq 2$  and  $G'_1 \not\cong K_{2,3}$ . Therefore we are done by Claim 1.

**Subsubcase 2.2.5.**  $G_1 \cong K_{2,3}$  and  $G_2 \not\cong S_1, S_3, S_4, W$ . It is easy to see that  $G$  must have the structure as given in Figure 7 (k). Let  $F' = \{e_1, e_3, e_4, e_5\}$ . Then  $G - F' = G'_1 + G'_2$ , where  $G'_1 \cong S_2$  and  $G'_2$  is a tree of order at least 6 since  $n \geq 8$ . It is easy to see that  $G'_2$  can not be isomorphic to  $W$  or  $Q$ . Therefore we are done by Claim 1.



**Case 3.**  $\kappa'(\hat{G}) = 3$ .

Noticing that  $\Delta(\hat{G}) \leq 3$  and  $\Delta(G) \leq 3$ , we obtain that  $G = \hat{G}$  is a connected 3-regular graph. Hence we have  $n + k - 1 = m = \frac{3}{2}n$ , i. e.,  $n = 2k - 2$ . Since  $n \geq 8$ , we have  $k \geq 5$ .

Let  $F = \{e_1, e_2, e_3\}$  be an edge cut of  $G$ . Then  $G - F$  has exactly two components, say,  $G_1$  and  $G_2$ . Clearly,  $c(G_1) + c(G_2) = k - 2 \geq 3$ . Let  $c(G_1) \geq c(G_2)$ . If  $c(G_2) \geq 3$ , then we are done by Claim 1. Hence we only need to consider the following three subcases.

**Subcase 3.1.**  $c(G_2) = 0$  and  $c(G_1) = k - 2$ . Let  $|V(G_2)| = n_2$ . Then we have  $3n_2 = \sum_{v \in V(G_2)} d_G(v) = 2(n_2 - 1) + 3 = 2n_2 + 1$ . Therefore,  $n_2 = 1$ , i. e.,  $G_2 = S_1$ . Let  $V(G_2) = \{x\}$ ,  $e_1 = xx_1$ ,  $e_2 = xx_2$  and  $e_3 = xx_3$ . Let  $N_{G_1}(x_2) = \{y_1, y_2\}$  (see Figure 8 (a)). Let  $F' = \{e_1, e_3, x_2y_1, x_2y_2\}$ . Then  $G - F' = G'_1 + G'_2$ , where  $G'_2 \cong S_2$  and  $G'_1$  is a graph obtained from  $G_1$  by deleting a vertex of degree 2. Therefore,  $c(G'_1) = k - 3 \geq 2$ . If  $G'_1 \not\cong K_{2,3}$ , then we are done by Claim 1. If  $G'_1 \cong K_{2,3}$ , then  $n = 7$ , which is a contradiction.

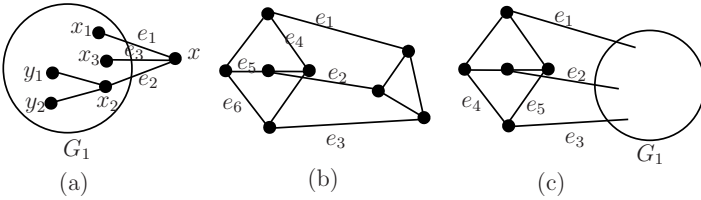


Figure 8: The graphs in the proof of Case 3 of Conjecture 1.5.

**Subcase 3.2.**  $c(G_2) = 1$  and  $c(G_1) = k - 3$ . Let  $|V(G_2)| = n_2$ . Then we have  $3n_2 = \sum_{v \in V(G_2)} d_G(v) = 2n_2 + 3$ . Therefore,  $n_2 = 3$ , i. e.,  $G_2$  is a triangle. If  $G_1 \not\cong K_{2,3}$ , then we are done by Claim 1. If  $G_1 \cong K_{2,3}$ , then  $G$  must be the graph as given in Figure 8 (b). Let  $F' = \{e_1, e_4, e_5, e_6\}$ . Then  $G - F' = G'_1 + G'_2$ , where  $G'_1 \cong S_2$  and  $G'_2$  is a bicyclic graph which is not isomorphic to  $K_{2,3}$ . Then we are done by Claim 1.

**Subcase 3.3.**  $c(G_2) = 2$  and  $c(G_1) = k - 4$ . Let  $|V(G_2)| = n_2$ . Then we have  $3n_2 = \sum_{v \in V(G_2)} d_G(v) = 2(n_2 + 1) + 3 = 2n_2 + 5$ . Therefore,  $n_2 = 5$ . If neither  $G_1$  nor  $G_2$  is isomorphic to  $K_{2,3}$ , then we are done by Claim 1. Otherwise, we assume

that  $G_2 \cong K_{2,3}$  (similar for  $G_1 \cong K_{2,3}$ ). Then  $G$  must have the structure as given in Figure 8 (c). Let  $F' = \{e_1, e_2, e_4, e_5\}$ . Then  $G - F' = G'_1 + G'_2$ , where  $G'_2 \cong K_{2,2}$  and  $G'_1$  is a  $(k - 4)$ -cyclic graph which is not isomorphic to  $K_{2,3}$ . Then we are done by Claim 1. The proof is thus complete. ■

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