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Hypoenergetic and Strongly Hypoenergetic k-Cyclic Graphs

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Abstract

The energy $E(G)$ of a graph G is defined as the sum of the absolute values of its eigenvalues. An n-vertex graph G is said to be hypoenergetic if $E(G) < n$ and strongly hypoenergetic if $E(G) < n - 1$. A connected graph with cyclomatic number k is called a k-cyclic graph. In this paper, we consider hypoenergetic and strongly hypoenergetic k-cyclic graphs. We first show that there exist hypoenergetic and strongly hypoenergetic k-cyclic graphs of order n and maximum degree Δ for all (suitable large) n and Δ. Then we show that for $\Delta \geq 4$ there exist hypoenergetic unicyclic, bicyclic and tricyclic graphs for all n except very few small values of n. For $\Delta \leq 3$ we show that $K_{2,3}$ is the unique hypoenergetic graph among all unicyclic and bicyclic graphs.

1 Introduction

Let G be a simple graph with n vertices and m edges. The *cyclomatic number* of a connected graph is defined as $c(G) = m - n + 1$. A graph G with $c(G) = k$ is called a *k-cyclic graph*. Denote by Δ the maximum degree of a graph. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the adjacency matrix $A(G)$ of G are said to be the eigenvalues of the

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graph G. The nullity of G, denoted by $n_0(G)$ (or simply n_0), is the multiplicity of zero in the eigenvalues of G. The *energy* of G is defined as

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For several classes of graphs it has been demonstrated that the energy exceeds the number of vertices (see, [8]). In 2007, Nikiforov [11] showed that for almost all graphs,

$$E = \left(\frac{4}{3\pi} + o(1)\right) n^{3/2}.$$

Thus the number of graphs satisfying the condition $E < n$ is relatively small. In [10], a *hypoenergetic* graph is defined to be a graph satisfying $E < n$. In [13], a *strongly hypoenergetic* graph is defined to be a graph satisfying $E < n - 1$. For hypoenergetic trees, Gutman et al. [9] obtained the following results.

Lemma 1.1. *[9] (a) There exist hypoenergetic trees of order n with maximum degree $\Delta \leq 3$ only for $n = 1, 3, 4, 7$ (a single such tree for each value of n, see Figure 1); (b) If $\Delta = 4$, then there exist hypoenergetic trees for all $n \geq 5$, such that $n \equiv k$ (mod 4), $k = 0, 1, 3$; (c) If $\Delta \geq 5$, then there exist hypoenergetic trees for all $n \geq \Delta + 1$.*

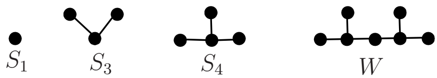

Figure 1: The hypoenergetic trees with maximum degree at most 3.

And the authors [9] proposed the following conjecture.

Conjecture 1.2. *[9] There exist hypoenergetic trees of order n with $\Delta = 4$ for any $n \equiv 2$ (mod 4), $n > 2$. Consequently, there exist hypoenergetic trees of order n with $\Delta = 4$ for all $n \geq 5$.*

We will give a very simple proof to this conjecture later, and therefore, Lemma 1.1 is extended to the following result.

Lemma 1.3. *(a) There exist hypoenergetic trees of order n with maximum degree $\Delta \leq 3$ only for $n = 1, 3, 4, 7$ (a single such tree for each value of n, see Figure 1); (b) If $\Delta \geq 4$, then there exist hypoenergetic trees for all $n \geq \Delta + 1$.*

For hypoenergetic unicyclic and bicyclic graphs, You and Liu [14] obtained the following results.

Lemma 1.4. *[14] (1) There exist hypoenergetic unicyclic graphs for all* $n \geq 7$*; (2) If* n *is even and* $\Delta \in [\frac{n}{2}, n-1]$ *or* n *is odd and* $\Delta \in [\frac{n+1}{2}, n-1]$*, then there exist hypoenergetic unicyclic graphs with maximum degree* Δ *for all* $n \geq 9$*.*

Lemma 1.5. *[14] (1) There exist hypoenergetic bicyclic graphs for all* $n \geq 8$*; (2) If* n *is even and* $\Delta \in [\frac{n}{2}+1, n-1]$ *or* n *is odd and* $\Delta \in [\frac{n+1}{2}, n-1]$*, then there exist hypoenergetic bicyclic graphs with maximum degree* Δ *for all* $n \geq 9$*.*

Recently, You, Liu and Gutman [15] considered hypoenergetic tricyclic and k-cyclic graphs, they obtained the following results.

Lemma 1.6. *[15] (1) There exist hypoenergetic tricyclic graphs for all* $n \geq 8$*; (2) If* n *is even and* $\Delta \in [\frac{n}{2}+1, n-1]$ *or* n *is odd and* $\Delta \in [\frac{n+3}{2}, n-1]$*, then there exist hypoenergetic tricyclic graphs with maximum degree* Δ *for all* $n \geq 10$*.*

Lemma 1.7. *[15] There exist hypoenergetic* k*-cyclic graphs for any* k*.*

In this paper, we consider hypoenergetic and strongly hypoenergetic k-cyclic graphs with order n and maximum degree Δ. In Section 2, we will show that for any given k three exist hypoenergetic and strongly hypoenergetic k-cyclic graphs of order n and maximum degree Δ for all (suitable large) n and Δ. In Sections 3-5, we consider hypoenergetic unicyclic, bicyclic and tricyclic graphs, respectively. We obtain that for $\Delta \geq 4$ there exist hypoenergetic unicyclic, bicyclic and tricyclic graphs for all n except very few small values of n. For $\Delta \leq 3$ we show that $K_{2,3}$ is the unique hypoenergetic graph among all unicyclic and bicyclic graphs. These results greatly extend the results in Lemmas 1.4-1.6.

2 Hypoenergetic and strongly hypoenergetic k-cyclic graphs

The following results are need in the sequel.

Lemma 2.1. *[7] Let* G *be a graph with* n *vertices and* m *edges. If the nullity of* G *is* n_0*, then* $E(G) \leq \sqrt{2m(n-n_0)}$*.*

Lemma 2.2. *[1] Suppose that G is a simple graph on n vertices without isolated vertex. Then*

(1) $n_0(G) = n - 2$ if and only if G is isomorphic to a complete bipartite graph K_{n_1,n_2}, where $n_1 + n_2 = n$, $n_1, n_2 > 0$.

(2) $n_0(G) = n - 3$ if and only if G is isomorphic to a complete tripartite graph K_{n_1,n_2,n_3}, where $n_1 + n_2 + n_3 = n$, $n_1, n_2, n_3 > 0$.

Lemma 2.3. *[4] Let v be a pendent vertex of a graph G and u be the vertex in G adjacent to v. Then $n_0(G) = n_0(G - u - v)$, where $G - u - v$ is the induced subgraph of G obtained by deleting u and v.*

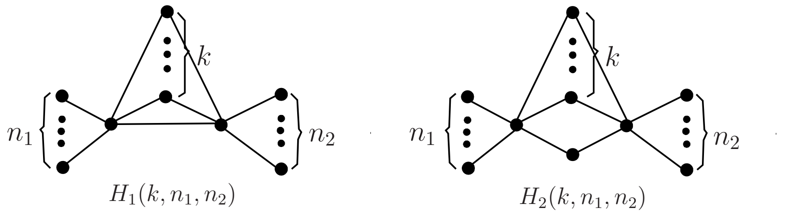

Figure 2: The Graphs $H_1(k, n_1, n_2)$ and $H_2(k, n_1, n_2)$.

Let $H_i(k, n_1, n_2)$ $(i = 1, 2)$ (or simply H_i) be the graph of order n given in Figure 2, where $k \geq 1$, $n_1 \geq 0, n_2 \geq 0$. Obviously, H_1 and H_2 are k-cyclic graphs, and $|V(H_1)| \geq k+2$, $|V(H_2)| \geq k+3$. If $n_1 = n_2 = 0$, then $n_0(H_1) = n-3$, $n_0(H_2) = n-2$ by Lemma 2.2; otherwise $n_0(H_1) = n_0(H_2) = n - 4$ by Lemma 2.3. Hence we have $n_0(H_1) \geq n - 4$ and $n_0(H_2) \geq n - 4$.

By Lemma 2.1, we have

$$E(H_i) \leq \sqrt{2m(n - n_0)} = \sqrt{2(n + k - 1)(n - n_0)} \leq \sqrt{8(n + k - 1)}.$$

If

$$\sqrt{8(n + k - 1)} < n, \tag{1}$$

then H_i is hypoenergetic. Inequality (1) can be transformed into $(n-4)^2 - 8k - 8 > 0$, which are obeyed by all $n > 4 + \sqrt{8(k + 1)}$. It is easy to check that

$$\max\{k + 1, 4 + \sqrt{8(k + 1)}\} = \begin{cases} 4 + \sqrt{8(k + 1)}, & \text{if } 1 \leq k \leq 13 \\ k + 1, & \text{if } k \geq 14 \end{cases}$$

and

$$\max\{k+2, 4+\sqrt{8(k+1)}\} = \begin{cases} 4+\sqrt{8(k+1)}, & \text{if } 1 \le k \le 12 \\ k+2, & \text{if } k \ge 13 \end{cases}.$$

Hence we have the following

Lemma 2.4. *(1) If* $n > \max\{k+1, 4+\sqrt{8(k+1)}\} = \begin{cases} 4+\sqrt{8(k+1)}, & \text{if } 1 \le k \le 13 \\ k+1, & \text{if } k \ge 14 \end{cases}$ *then* H_1 *is hypoenergetic.*

(2) If $n > \max\{k+2, 4+\sqrt{8(k+1)}\} = \begin{cases} 4+\sqrt{8(k+1)}, & \text{if } 1 \le k \le 12 \\ k+2, & \text{if } k \ge 13 \end{cases}$*, then* H_2 *is hypoenergetic.*

Notice that the inequality $\sqrt{8(k+1)} \le k+3$ holds for any $k \ge 1$, so we have the following

Theorem 2.5. *There exist hypoenergetic* k*-cyclic graphs for all* $n \ge k+8$*.*

If

$$\sqrt{8(n+k-1)} < n-1, \tag{2}$$

then H_i is strongly hypoenergetic. Inequality (2) can be transformed into $(n-5)^2 - 8k - 16 > 0$, which are obeyed by all $n > 5 + \sqrt{8(k+2)}$. It is easy to check that

$$\max\{k+1, 5+\sqrt{8(k+2)}\} = \begin{cases} 5+\sqrt{8(k+2)}, & \text{if } 1 \le k \le 15 \\ k+1, & \text{if } k \ge 16 \end{cases}$$

and

$$\max\{k+2, 5+\sqrt{8(k+2)}\} = \begin{cases} 5+\sqrt{8(k+2)}, & \text{if } 1 \le k \le 14 \\ k+2, & \text{if } k \ge 15 \end{cases}.$$

Hence we have the following

Lemma 2.6. *(1) If* $n > \max\{k+1, 5+\sqrt{8(k+2)}\} = \begin{cases} 5+\sqrt{8(k+2)}, & \text{if } 1 \le k \le 15 \\ k+1, & \text{if } k \ge 16 \end{cases}$ *then* H_1 *is strongly hypoenergetic.*

(2) If $n > \max\{k+2, 5+\sqrt{8(k+2)}\} = \begin{cases} 5+\sqrt{8(k+2)}, & \text{if } 1 \le k \le 14 \\ k+2, & \text{if } k \ge 15 \end{cases}$*, then* H_2 *is strongly hypoenergetic.*

In the following, we consider hypoenergetic and strongly hypoenergetic k-cyclic graphs with order n and maximum degree Δ.

Theorem 2.7. *(1) If $n-k$ is even and $\Delta \in [\frac{n+k}{2}, n-1]$ or $n-k$ is odd and $\Delta = n-1$, then there exist hypoenergetic k-cyclic graphs of order n with maximum degree Δ for all $n > \max\{k+1, 4+\sqrt{8(k+1)}\}$.*

(2) If $n-k$ is odd and $\Delta \in [\frac{n+k-1}{2}, n-2]$, then there exist hypoenergetic k-cyclic graphs of order n with maximum degree Δ for all $n > \max\{k+2, 4+\sqrt{8(k+1)}\}$.

Proof. Suppose $n-k$ is even and $\Delta \in [\frac{n+k}{2}, n-1]$ or $n-k$ is odd and $\Delta = n-1$. Let $G = H_1(k, \Delta-k-1, n-\Delta-1)$, then by Lemma 2.4, G is hypoenergetic when $n > \max\{k+1, 4+\sqrt{8(k+1)}\}$.

Suppose $n-k$ is odd and $\Delta \in [\frac{n+k-1}{2}, n-2]$. Let $G = H_2(k, \Delta-k-1, n-\Delta-2)$, then by Lemma 2.4, G is hypoenergetic when $n > \max\{k+2, 4+\sqrt{8(k+1)}\}$. The proof is then complete. $\square$

By Lemma 2.6, similar to the proof of Theorem 2.7, we can obtain

Theorem 2.8. *(1) If $n-k$ is even and $\Delta \in [\frac{n+k}{2}, n-1]$ or $n-k$ is odd and $\Delta = n-1$, then there exist strongly hypoenergetic k-cyclic graphs of order n with maximum degree Δ for all $n > \max\{k+1, 5+\sqrt{8(k+2)}\}$.*

(2) If $n-k$ is odd and $\Delta \in [\frac{n+k-1}{2}, n-2]$, then there exist strongly hypoenergetic k-cyclic graphs of order n with maximum degree Δ for all $n > \max\{k+2, 5+\sqrt{8(k+2)}\}$.

In order to prove Conjecture 1.2 and extend the interval for Δ in Theorem 2.7, we need the following notations and preliminary results, which can be found in [13]. Let G and H be two graphs with disjoint vertex sets, and let $u \in V(G)$ and $v \in V(H)$. Construct a new graph $G \circ H$ from copies of G and H, by identifying the vertices u and v. Thus $|V(G \circ H)| = |V(G)| + |V(H)| - 1$. The graph $G \circ H$ is known as the *coalescence* of G and H with respect to u and v.

Lemma 2.9. *[13] Let G, H and $G \circ H$ be graphs as specified above. Then $E(G \circ H) \leq E(G) + E(H)$. Equality is attained if and only if either u is an isolated vertex of G or v is an isolated vertex of H or both.*

Lemma 2.10. *[13] Let G, H and $G \circ H$ be graphs as specified above. If G is strongly hypoenergetic and H is hypoenergetic (or vice versa), then $G \circ H$ is hypoenergetic.*

Proof of Conjecture 1.2. Suppose $n \equiv 2 \ (mod\ 4)$, $n > 2$. If $n = 6$, then by [3] (Table 2), there exists a unique tree T_6 of order 6 with $\Delta = 4$, and $E(T_6) = 5.818 < 6$, i.e., T_6 is hypoenergetic. Let S_5 be the 5-vertex star, then $\Delta(S_5) = 4$ and $E(S_5) = 4$. Let u be a leaf vertex in T_6 and v be a leaf vertex in S_5. Then by Lemma 2.9, for the coalescence $T_{10} = T_6 \circ S_5$ of T_6 and S_5 with respect to u and v, we have $E(T_{10}) < 10$. Obviously, T_{10} is a tree of order 10 with $\Delta = 4$. By consecutively doing the coalescence operations $(\cdots((T_6 \circ S_5) \circ S_5) \cdots) \circ S_5$, we can construct hypoenergetic trees with $\Delta = 4$ for any $n \geq 10$ such that $n \equiv 2$ $(mod\ 4)$. The proof is thus complete. $\square$

Theorem 2.11. *(1) If $n-k$ is even and $\max\{\frac{2k+1}{2}, \frac{5+k+\sqrt{8(k+2)}}{2}\} < \Delta \leq n-1$, then there exist hypoenergetic k-cyclic graphs of order n with maximum degree Δ for all $n > \max\{k+3, 7+\sqrt{8(k+2)}\}$.*

(2) If $n-k$ is odd and $\max\{\frac{2k+1}{2}, \frac{4+k+\sqrt{8(k+2)}}{2}\} < \Delta \leq n-1$, then there exist hypoenergetic k-cyclic graphs of order n with maximum degree Δ for all $n > \max\{k+4, 7+\sqrt{8(k+2)}\}$.

Proof. (1) Suppose $n-k$ is even. By Theorem 2.7, we may assume that $\Delta \leq \frac{n+k}{2} - 1$. Let $G = H_1(k, \Delta-k-1, \Delta-k-1)$, then $|V(G)| = 2\Delta - k \leq n-2$. Since $|V(G)| = 2\Delta - k > \max\{k+1, 5+\sqrt{8(k+2)}\}$, G is strongly hypoenergetic by Lemma 2.6.

Let H be a hypoenergetic tree of order $n-2\Delta+k+1$ with $\Delta = 4$ if $n-2\Delta+k+1 \geq 5$ and S_3 if $n-2\Delta+k+1 = 3$ (Such an H does exist by Lemma 1.3). Let u be a vertex of degree 2 in G, v a leaf vertex in H and $G \circ H$ be the coalescence of G and H with respect to u and v. Since $\Delta > \max\{\frac{2k+1}{2}, \frac{5+k+\sqrt{8(k+2)}}{2}\}$, we have $\Delta \geq 4$. Hence $G \circ H$ is a k-cyclic graph of order n with maximum degree Δ. By Lemma 2.10, $G \circ H$ is hypoenergetic.

(2) Suppose $n-k$ is odd. By Theorem 2.7, we may assume that $\Delta \leq \frac{n+k-1}{2} - 1$. Let $G = H_2(k, \Delta-k-1, \Delta-k-1)$, then $|V(G)| = 2\Delta - k + 1 \leq n-2$. Since $|V(G)| = 2\Delta - k + 1 > \max\{k+2, 5+\sqrt{8(k+2)}\}$, G is strongly hypoenergetic by Lemma 2.6.

Let H be a hypoenergetic tree of order $n-2\Delta+k$ with $\Delta = 4$ if $n-2\Delta+k \geq 5$ and S_3 if $n-2\Delta+k = 3$. Let u be a vertex of degree 2 in G, v a leaf vertex in H and $G \circ H$ be the coalescence of G and H with respect to u and v. Since $\Delta > \max\{\frac{2k+1}{2}, \frac{4+k+\sqrt{8(k+2)}}{2}\}$, we have $\Delta \geq 4$. Hence $G \circ H$ is a k-cyclic graph of order n with maximum degree Δ. By Lemma 2.10, $G \circ H$ is hypoenergetic. $\square$

Similar to the proof of Conjecture 1.2, we can obtain the following result, which provides a useful method to construct more hypoenergetic k-cyclic graphs.

Theorem 2.12. *If there exist a t-vertex hypoenergetic k-cyclic graph with $\Delta \geq 4$ and at least a vertex of degree at most $\Delta - 1$, then there exist hypoenergetic k-cyclic graphs with Δ for all $n \geq t$, such that $n \equiv t$ $(mod\ 4)$.*

3 Hypoenergetic unicyclic graphs

This section is devoted to finding more hypoenergetic unicyclic graphs, greatly extending the results in [14].

Lemma 3.1. *[14] If $n \leq 6$, then there do not exist any hypoenergetic unicyclic graphs.*

Lemma 3.2. *If n is even and $\Delta \in [5, n-1]$ or n is odd and $\Delta \in [6, n-1]$, then there exist hypoenergetic unicyclic graphs of order n with maximum degree Δ for all $n \geq 9$.*

Proof. Notice that when $k = 1$, we have that $n > \max\{k+2, 4+\sqrt{8(k+1)}\}$ implies $n \geq 9$, $n > \max\{k+4, 7+\sqrt{8(k+2)}\}$ implies $n \geq 12$, $\Delta > \max\{\frac{2k+1}{2}, \frac{5+k+\sqrt{8(k+2)}}{2}\}$ implies $\Delta \geq 6$ and $\Delta > \max\{\frac{2k+1}{2}, \frac{4+k+\sqrt{8(k+2)}}{2}\}$ implies $\Delta \geq 5$. Hence the result follows from Theorem 2.7 for $9 \leq n \leq 11$ and from Theorem 2.11 for $n \geq 12$. $\square$

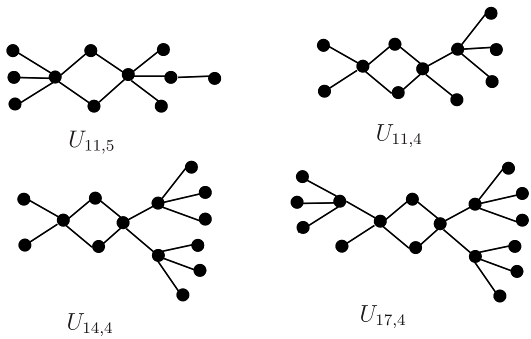

Figure 3: Graphs $U_{11,5}, U_{11,4}, U_{14,4}$ and $U_{17,4}$.

Table 1

n	Δ	$E(U_{n,\Delta})$	n	Δ	$E(U_{n,\Delta})$	n	Δ	$E(U_{n,\Delta})$
7	5	6.89898	8	6	7.39104	11	5	10.58501
7	6	6.64681	8	7	7.07326	14	4	13.90827
8	4	7.72741	9	5	8.24621	17	4	16.96885
8	5	7.65069	11	4	10.87716			

In the following, we consider the case $4 \leq \Delta \leq 7$. Let $U_{7,5} = H_2(1,3,0)$, $U_{7,6} = H_1(1,4,0)$, $U_{8,4} = H_2(1,2,2)$, $U_{8,5} = H_2(1,3,1)$, $U_{8,6} = H_2(1,4,0)$, $U_{8,7} = H_1(1,5,0)$ and $U_{9,5} = H_2(1,3,2)$. Let $U_{11,5}, U_{11,4}, U_{14,4}$ and $U_{17,4}$ be the graphs given in Figure 3. Obviously, these graphs $U_{n,l}$ are hypoenergetic unicyclic graphs of order n with

$\Delta = l$ by Table 1. Since $U_{8,4}, U_{11,4}, U_{14,4}, U_{17,4}$, $U_{9,5}$ and $U_{11,5}$ are hypoenergetic, by Theorem 2.12, we can obtain

Lemma 3.3. *(1) If $\Delta = 4$, then there exist hypoenergetic unicyclic graphs of order n for all $n = 8, 11, 12$ and $n \geq 14$; (2) If $\Delta = 5$, then there exist hypoenergetic unicyclic graphs of order n for all odd $n \geq 9$.*

Combining Lemmas 3.2, 3.3 and Table 1, we can obtain

Theorem 3.4. *If (a) $n = 8, 11, 12$ or $n \geq 14$ and $\Delta = 4$ or (b) $n \geq 7$ and $\Delta \in [5, n-1]$, then there exist hypoenergetic unicyclic graphs with order n and maximum degree Δ.*

When $n \leq 6$, by Lemma 3.1, there exist no hypoenergetic unicyclic graphs. By [2], there are 12 unicyclic graphs with $n = 7$ and $\Delta = 4$. In these graphs, the minimal energy is $E = 7.1153 > n = 7$, and the extremal graph is $H_2(1,1,2)$. We can also show that there are no hypoenergetic unicyclic graphs with $n = 9$ or 10 and $\Delta = 4$. Thus, $n = 13$ is the only case for which we can not determine whether or not there exist hypoenergetic unicyclic graphs of order $n = 13$ and $\Delta = 4$. But we can show that there are no hypoenergetic unicyclic graphs with $n = 13$, $\Delta = 4$ and girth $g \geq 7$. The details are tedious and hence omitted.

In the end of this section, we consider the remaining case $\Delta \leq 3$. The following results are needed.

Lemma 3.5. *[12] Let G be a graph of order n with at least n edges and with no isolated vertices. If G is quadrangle-free and $\Delta(G) \leq 3$, then $E(G) > n$.*

Lemma 3.6. *[6] If F is an edge cut of a simple graph G, then $E(G - F) \leq E(G)$, where $G - F$ is the subgraph obtained from G by deleting the edges in F.*

Lemma 3.7. *If there exists an edge cut F of a connected graph G such that $G - F$ has two components G_1 and G_2, and both G_1 and G_2 are non-hypoenergetic, then G is non-hypoenergetic.*

Proof. It follows from Lemma 3.6 that

$$E(G) \geq E(G - F) = E(G_1) + E(G_2) \geq |V(G_1)| + |V(G_2)| = n,$$

which completes the proof. $\square$

Theorem 3.8. *There does not exist any hypoenergetic unicyclic graph with $\Delta \leq 3$.*

Proof. Let G be an n-vertex unicyclic graph with $\Delta \leq 3$. We will show that G is non-hypoerergetic. If $n \leq 6$, then G is non-hypoenergetic by Lemma 3.1. If G is quadrangle-free, then G is non-hypoenergetic by Lemma 3.5. So in the following we assume that $n \geq 7$ and G contains a quadrangle $C = x_1x_2x_3x_4x_1$. We only need to consider the following four cases:

Case 1. There exists an edge e on C such that the end vertices of e are of degree 2.

Without loss of generality, we assume that $d(x_1) = d(x_4) = 2$. Let $F = \{x_1x_2, x_4x_3\}$, then $G - F$ has two components, say G_1 and G_2, where G_1 is the tree of order 2 with $x_1 \in V(G_1)$ and G_2 is a tree of order at least 5 since $n \geq 7$. Since $\Delta(G) \leq 3$, G_2 can not be isomorphic to W. Therefore G_1, G_2 are non-hypoenergetic by Lemma 1.1 (a). The result follows from Lemma 3.7.

Case 2. There exist exactly two nonadjacent vertices x_i and x_j on C such that $d(x_i) = d(x_j) = 2$.

Without loss of generality, we assume that $d(x_2) = d(x_4) = 2$, $d(x_1) = d(x_3) = 3$. Let y_3 be the adjacent vertex outside C of x_3. Then $G - x_3y_3$ has two components, say G_1 and G_2, where G_1 is a unicyclic graph and G_2 is a tree. Notice that G_1 is non-hypoenergetic by Case 1. If $G_2 \ncong S_1, S_3, S_4, W$, then we are finished by Lemmas 1.1 (a) and 3.7. So we only need to consider the following four cases.

Subcase 2.1. $G_2 \cong S_1$.

Let $F = \{x_2x_3, x_3x_4\}$, then $G - F$ has two components, say G_1' and G_2', where G_1' is a tree of order at least 4 and G_2' is a tree of order 2. If $G_1' \ncong S_4, W$, then we are finished by Lemmas 1.1 (a) and 3.7. If $G_1' \cong S_4$, then $n = 6$, a contradiction. If $G_1' \cong W$, then G must be the graph as given in Figure 4 (a), by direct computing, we have $E(G) = 9.78866 > 9 = n$.

Subcase 2.2. $G_2 \cong S_3$.

Then G must have the structure as given in Figure 4 (b) or (c). In the former case, $G - y_3z$ has two components, say G_1' and G_2', where G_1' is a unicyclic graph and G_2' is a tree of order 2. It follows from Subcase 2.1 that G_1' is non-hypoenergetic. Therefore we are finished by Lemmas 1.1 (a) and 3.7. In the latter case, $G - \{x_1x_2, x_4x_3\}$ has

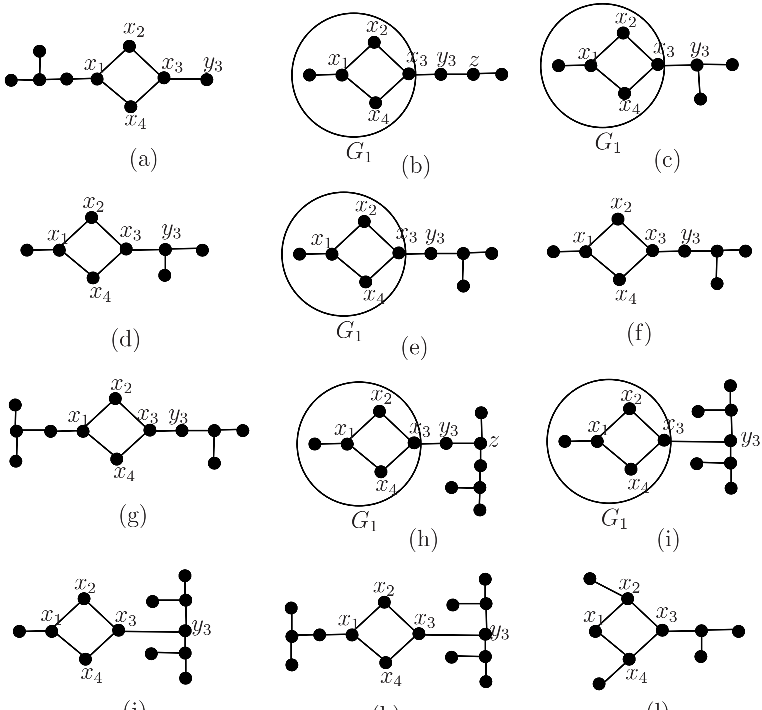

Figure 4: The graphs in the proof of Theorem 3.8.

two components, say G_1' and G_2', where G_1' is a tree of order at least 3 and G_2' is a tree of order 5. If $G_1' \ncong S_3, S_4, W$, then we are finished by Lemmas 1.1 (a) and 3.7. Since $\Delta(G) \leq 3$, G_1' can not be isomorphic to S_4 or W. If $G_1' \cong S_3$, then G must be the graph as given in Figure 4 (d), by direct computing, we have $E(G) = 8.81463 > 8 = n$.

Subcase 2.3. $G_2 \cong S_4$.

Then G must have the structure as given in Figure 4 (e). Let $F = \{x_2x_3, x_3x_4\}$, then $G - F$ has two components, say G_1' and G_2', where G_1' is a tree of order at least 4 and G_2' is a tree of order 5. If $G_1' \ncong S_4, W$, then we are finished by Lemmas 1.1 (a) and 3.7. If $G_1' \cong S_4$, then G must be the graph as given in Figure 4 (f), by direct computing, we have $E(G) = 9.78866 > 9 = n$. If $G_1' \cong W$, then G must be the graph as given in Figure 4 (g). Now, $G - \{x_1x_2, x_3x_4\}$ has two components, say G_1'' and G_2'', where G_i'' is a tree of order 6, $i = 1, 2$. Therefore we are finished by Lemmas 1.1 (a) and 3.7.

Subcase 2.4. $G_2 \cong W$.

Then G must have the structure as given in Figure 4 (h) or (i). In the former case, $G - y_3z$ has two components, say G_1' and G_2', where G_1' is a unicyclic graph and G_2' is a tree of order 6. It follows from Subcase 2.1 that G_1' is non-hypoenergetic. Therefore we are finished by Lemmas 1.1 (a) and 3.7. In the latter case, $G - \{x_2x_3, x_3x_4\}$ has two components, say G_1' and G_2', where G_1' is a tree of order at least 4 and G_2' is a tree of order 8. If $G_1' \ncong S_4, W$, then we are finished by Lemmas 1.1 (a) and 3.7. If $G_1' \cong S_4$, then G must be the graph as given in Figure 4 (j), by direct computing, we have $E(G) = 13.05749 > 12 = n$. If $G_1' \cong W$, then G must be the graph as given in Figure 4 (k). Now, $G - \{x_1x_2, x_3x_4\}$ has two components, say G_1'' and G_2'', where G_1'' is a tree of order 6 and G_2'' is a tree of order 9. Therefore we are finished by Lemmas 1.1 (a) and 3.7.

Case 3. There exists exactly one vertices x_i on C such that $d(x_i) = 2$.

Without loss of generality, we assume that $d(x_1) = 2$. Let $F = \{x_1x_4, x_2x_3\}$, then $G - F$ has two components, say G_1 and G_2, where G_1 is the tree of order at least 3 with $x_1 \in V(G_1)$ and G_2 is a tree of order at least 4. Since $\Delta(G) \leq 3$, G_1, G_2 can not be isomorphic to S_4 or W. So if $G_1 \ncong S_3$, then we are finished by Lemmas 1.1 (a) and 3.7. If $G_1 \cong S_3$, then $G - \{x_1x_2, x_2x_3\}$ has two components, say G_1' and G_2', where G_1' is the tree of order at least 5 with $x_1 \in V(G_1')$ and G_2' is a tree of order 2. If $G_1' \ncong W$, then we are finished by Lemmas 1.1 (a) and 3.7. If $G_1' \cong W$, then G must be the graph as given in Figure 4 (l), by direct computing, we have $E(G) = 9.80028 > 9 = n$.

Case 4. $d(x_1) = d(x_2) = d(x_3) = d(x_4) = 3$.

Let $F = \{x_1x_4, x_2x_3\}$, then $G - F$ has two components, say G_1 and G_2, where G_1 and G_2 are trees of order at least 4 and it is easy to check that G_1, G_2 can not be isomorphic to S_4 or W. Therefore we are finished by Lemmas 1.1 (a) and 3.7. The proof is thus complete. $\square$

4 Hypoenergetic bicyclic graphs

This section is devoted to finding more hypoenergetic bicyclic graphs, also greatly extending corresponding results in [14].

Lemma 4.1. *[14] If $n = 4, 6, 7$, then there do not exist any hypoenergetic bicyclic graphs.*

Lemma 4.2. *If n is even and $\Delta \in [7, n-1]$ or n is odd and $\Delta \in [6, n-1]$, then there exist hypoenergetic bicyclic graphs of order n with maximum degree Δ for all $n \geq 9$.*

Proof. Notice that when $k = 2$, we have that $n > \max\{k+2, 4+\sqrt{8(k+1)}\}$ implies $n \geq 9$, $n > \max\{k+4, 7+\sqrt{8(k+2)}\}$ implies $n \geq 13$, $\Delta > \max\{\frac{2k+1}{2}, \frac{5+k+\sqrt{8(k+2)}}{2}\}$ implies $\Delta \geq 7$ and $\Delta > \max\{\frac{2k+1}{2}, \frac{4+k+\sqrt{8(k+2)}}{2}\}$ implies $\Delta \geq 6$. Hence the result follows from Theorem 2.7 for $9 \leq n \leq 12$ and from Theorem 2.11 for $n \geq 13$. □

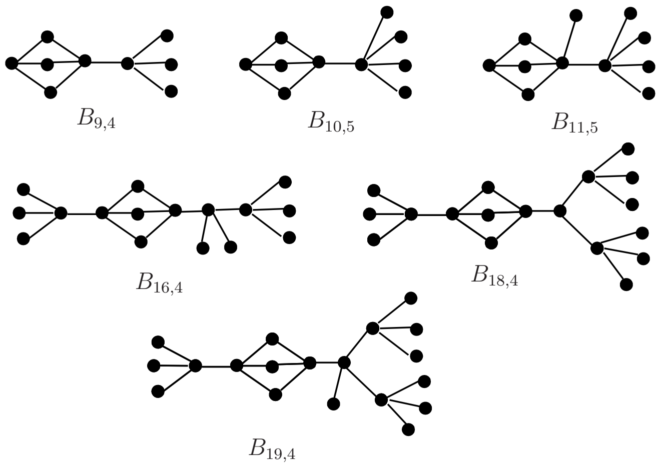

Figure 5: Graphs $B_{9,4}, B_{10,5}, B_{11,5}, B_{16,4}, B_{18,4}$ and $B_{19,4}$.

Table 2

n	Δ	$E(B_{n,\Delta})$	n	Δ	$E(B_{n,\Delta})$	n	Δ	$E(B_{n,\Delta})$
8	5	7.90778	9	5	8.48528	16	4	15.77861
8	6	7.74597	10	5	9.25036	18	4	17.94188
8	7	7.68165	10	6	8.98112	19	4	18.87354
9	4	8.75560	11	5	10.74799			

In the following, we consider the case $4 \leq \Delta \leq 7$. Let $B_{8,5} = H_2(2,1,2)$, $B_{8,6} = H_2(2,0,3)$, $B_{8,7} = H_1(2,0,4)$, $B_{9,5} = H_2(2,2,2)$ and $B_{10,6} = H_2(2,2,3)$. Let $B_{10,5}$, $B_{11,5}$, $B_{9,4}$, $B_{16,4}$, $B_{18,4}$ and $B_{19,4}$ be the graphs given in Figure 5. Obviously, these graphs $B_{n,l}$ are hypoenergetic bicyclic graphs of order n with $\Delta = l$ by Table 2. By Theorem 2.12, we can obtain

Lemma 4.3. *(1) If $\Delta = 4$, then there exist hypoenergetic bicyclic graphs of order n for all $n = 9, 13$ and $n \geq 16$; (2) If $\Delta = 5$, then there exist hypoenergetic bicyclic graphs of order n for all $n \geq 8$; (3) If $\Delta = 6$, then there exist hypoenergetic bicyclic graphs of order n for all even $n \geq 8$.*

Combining Lemmas 4.2, 4.3 and Table 2, we can obtain

Theorem 4.4. *If (a) $n = 9, 13$ or $n \geq 16$ and $\Delta = 4$ or (b) $n \geq 8$ and $\Delta \in [5, n-1]$, then there exist hypoenergetic bicyclic graphs with order n and maximum degree Δ.*

When $n = 4, 6, 7$, by Lemma 4.1, there exist no hypoenergetic bicyclic graphs. By [3] (Table 1), there are two bicyclic graphs with $n = 5$ and $\Delta = 4$, and the minimal energy is $E = 6.04090 > n = 5$, and the extremal graph is $H_1(2,0,1)$. Thus, for $\Delta = 4$, $n = 8, 10, 11, 12, 14, 15$ are the only few cases for which we can not determine whether or not there exist hypoenergetic bicyclic graphs. One can employ a computer to determine them easily.

In the end of this section, we consider the remaining case $\Delta \leq 3$.

Theorem 4.5. *Complete bipartite graph $K_{2,3}$ is the only hypoenergetic bicyclic graphs with $\Delta \leq 3$.*

Proof. Let G be an n-vertex bicyclic graphs with $\Delta \leq 3$. If $n = 4, 6, 7$, then G is non-hypoenergetic by Lemma 4.1. If $n = 5$, by [3] (Table 1), there are three bicyclic graphs with $\Delta \leq 3$, and $K_{2,3}$ is the only hypoenergetic graph with $E(K_{2,3}) = 4.8990$. If G is quadrangle-free, then G is non-hypoenergetic by Lemma 3.5. So in the following we assume that G contains a quadrangle, $G \ncong K_{2,3}$ and $n \geq 8$. We will show that G is non-hypoenergetic.

If the cycles in G are disjoint, then it is clear that there exists a path P connecting the two cycles in G. Obviously, for any edge e on P, $G - e$ has two components which are unicyclic graphs. Thus G is non-hypoenergetic by Lemma 3.7 and Theorem 3.8.

Otherwise, the cycles in G have two or more common vertices. Then we can assume that G contains a subgraph as given in Figure 6 (a), where P_1, P_2, P_3 are paths in G.

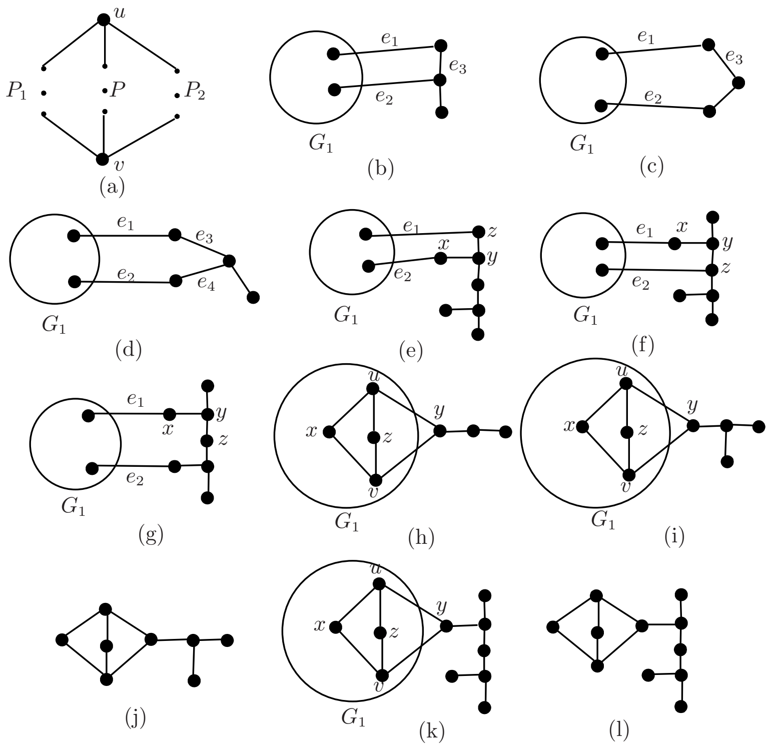

Figure 6: The graphs in the proof of Theorem 4.5.

We distinguish the following three cases:

Case 1. At least one of P_1, P_2 and P_3, say P_2 has length not less than 3.

Let e_1 and e_2 be the edges on P_2 incident with u and v, respectively. Then $G - \{e_1, e_2\}$ has two components, say G_1 and G_2, where G_1 is a unicyclic graph and G_2 is a tree of order at least 2. It follows from Theorem 3.8 that G_1 is non-hypoenergetic. If $G_2 \ncong S_3, S_4, W$, then we are finished by Lemmas 1.1 (a) and 3.7. So we only need to consider the following three cases:

Subcase 1.1. $G_2 \cong S_3$.

Then G must have the structure as given in Figure 6 (b) or (c). In either case, $G - \{e_2, e_3\}$ has two components, say G_1' and G_2', where G_1' is a unicyclic graph and G_2' is a tree of order 2. By Theorem 3.8, G_1' is non-hypoenergetic. Therefore the result follows from Lemmas 1.1 (a) and 3.7.

Subcase 1.2. $G_2 \cong S_4$.

Then G must have the structure as given in Figure 6 (d). Obviously, $G - \{e_3, e_4\}$ has two components, say G_1' and G_2', where G_1' is a unicyclic graph and G_2' is a tree of order 2. Therefore the result follows from Theorem 3.8, Lemmas 1.1 (a) and 3.7.

Subcase 1.3. $G_2 \cong W$.

Then G must have the structure as given in Figure 6 (e), (f) or (g). Obviously, $G - \{xy, yz\}$ has two components, say G_1' and G_2', where G_1' is a unicyclic graph and G_2' is a tree of order 5 or 2. Therefore the result follows from Theorem 3.8, Lemmas 1.1 (a) and 3.7.

Case 2. All the paths P_1, P_2 and P_3 have length 2.

We assume that $P_1 = uxv$, $P = uzv$ and $P_2 = uyv$. Let $F = \{uy, vy\}$, then $G - F$ has two components, say G_1 and G_2, where G_1 is a unicyclic graph and G_2 is a tree. It follows from Theorem 3.8 that G_1 is non-hypoenergetic. If $G_2 \ncong S_1, S_3, S_4, W$, then we are finished by Lemmas 1.1 (a) and 3.7. So we only need to consider the following four cases.

Subcase 2.1. $G_2 \cong S_1$.

Let $F' = \{uy, zv, xv\}$, then $G - F'$ has two components, say G_1' and G_2', where G_2' is the tree of order 2 with $y \in V(G_2')$, G_1' is a tree of order at least 6 since $n \geq 8$. Since $\Delta(G) \leq 3$, G_1' can not be isomorphic to W. Therefore G_1', G_2' are non-hypoenergetic by Lemma 1.1 (a). The result follows from Lemma 3.7.

Subcase 2.2. $G_2 \cong S_3$.

Then G must have the structure as given in Figure 6 (h). Let $F' = \{uy, zv, xv\}$, then $G - F'$ has two components, say G_1' and G_2', where G_2' is the path of order 4 with $y \in V(G_2')$, G_1' is a tree of order at least 4 since $n \geq 8$. Since $\Delta(G) \leq 3$, G_1' can not be isomorphic to S_4 or W. Therefore G_1', G_2' are non-hypoenergetic by Lemma 1.1 (a). The result follows from Lemma 3.7.

Subcase 2.3. $G_2 \cong S_4$.

Then G must have the structure as given in Figure 6 (i). Let $F' = \{uy, zv, xv\}$, then $G - F'$ has two components, say G_1' and G_2', where G_2' is the tree of order 5 with $y \in V(G_2')$, G_1' is a tree of order at least 3. Since $\Delta(G) \leq 3$, G_1' can not be isomorphic to S_4 or W. If $G_1' \ncong S_3$, then we are finished by Lemmas 1.1 (a) and 3.7. If $G_1' \cong S_3$, then G must be the graph as given in Figure 6 (j), by direct computing, we have $E(G) = 8.24621 > 8 = n$.

Subcase 2.4. $G_2 \cong W$.

Then G must have the structure as given in Figure 6 (k). Let $F' = \{uy, zv, xv\}$, then $G - F'$ has two components, say G_1' and G_2', where G_2' is the tree of order 8 with $y \in V(G_2')$, G_1' is a tree of order at least 3. Since $\Delta(G) \leq 3$, G_1' can not be isomorphic to S_4 or W. If $G_1' \ncong S_3$, then we are finished by Lemmas 1.1 (a) and 3.7. If $G_1' \cong S_3$, then G must be the graph as given in Figure 6 (l), by direct computing, we have $E(G) = 11.60185 > 11 = n$.

Case 3. One of the paths P_1, P_2 and P_3 has length 1, and the other two paths have length 2.

Without loss of generality, we assume that $P_1 = uxv$, $P = uv$ and $P_2 = uyv$. Then similar to the proof of Case 2, we can show that G is non-hypoenergetic. The proof is thus complete. $\square$

5 Hypoenergetic tricyclic graphs

This section is devoted to finding more hypoenergetic tricyclic graphs, greatly extending corresponding results in [15].

Lemma 5.1. *[15] If $n = 4, 5, 7$, then there do not exist any hypoenergetic tricyclic graphs.*

Lemma 5.2. *If n is even and $\Delta \in [7, n-1]$ or n is odd and $\Delta \in [8, n-1]$, then there exist hypoenergetic tricyclic graphs of order n with maximum degree Δ for all $n \geq 10$.*

Proof. Notice that when $k = 3$, we have that $n > \max\{k+2, 4+\sqrt{8(k+1)}\}$ implies $n \geq 10$, $n > \max\{k+4, 7+\sqrt{8(k+2)}\}$ implies $n \geq 14$, $\Delta > \max\{\frac{2k+1}{2}, \frac{5+k+\sqrt{8(k+2)}}{2}\}$ implies $\Delta \geq 8$ and $\Delta > \max\{\frac{2k+1}{2}, \frac{4+k+\sqrt{8(k+2)}}{2}\}$ implies $\Delta \geq 7$. Hence the result follows from Theorem 2.7 for $10 \leq n \leq 13$ and from Theorem 2.11 for $n \geq 14$. $\square$

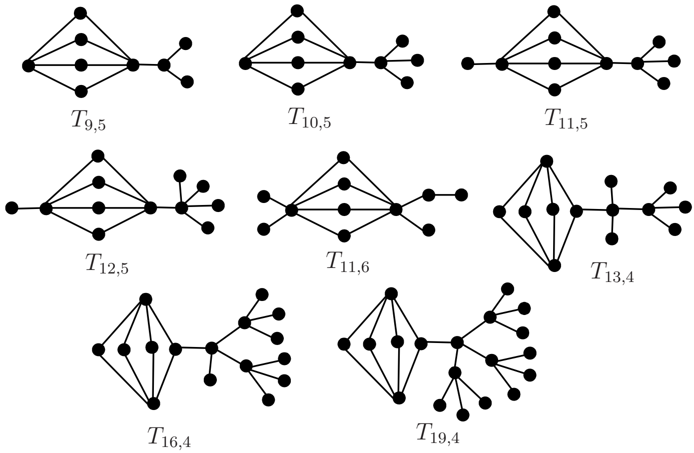

Figure 7: Graphs $T_{9,5}, T_{10,5}, T_{11,5}, T_{12,5}, T_{11,6}, T_{13,4}, T_{16,4}$ and $T_{19,4}$.

Table 3

n	Δ	$E(T_{n,\Delta})$	n	Δ	$E(T_{n,\Delta})$	n	Δ	$E(T_{n,\Delta})$
6	4	5.65685	9	8	8.50189	11	7	9.63287
8	6	7.91375	10	5	9.50432	12	5	11.50305
9	5	8.93180	10	6	9.15298	13	4	12.78001
9	6	8.59845	11	5	10.00000	16	4	15.90909
9	7	8.46834	11	6	10.94832	19	4	18.88809

In the following, we consider hypoenergetic tricyclic graphs with $4 \le \Delta \le 8$. Let $T_{6,4} = H_2(3,0,0)$, $T_{8,6} = H_2(3,0,2)$, $T_{9,6} = H_2(3,1,2)$, $T_{9,7} = H_2(3,0,3)$, $T_{9,8} = H_1(3,0,4)$, $T_{10,6} = H_2(3,2,2)$ and $T_{11,7} = H_2(3,2,3)$. Let $T_{9,5}, T_{10,5}, T_{11,5}, T_{12,5}, T_{11,6}$, $T_{13,4}, T_{16,4}$ and $T_{19,4}$ be the graphs given in Figure 7. Obviously, these graphs $T_{n,l}$ are hypoenergetic tricyclic graphs of order n with $\Delta = l$ by Table 3. By Theorem 2.12, we can obtain

Lemma 5.3. *(1) If $\Delta = 4$, then there exist hypoenergetic tricyclic graphs of order n for all $n = 6, 10, 13, 14$ and $n \ge 16$; (2) If $\Delta = 5$, then there exist hypoenergetic tricyclic graphs of order n for all $n \ge 9$; (3) If $\Delta = 6$, then there exist hypoenergetic tricyclic graphs of order n for all $n \ge 8$; (4) If $\Delta = 7$, then there exist hypoenergetic tricyclic graphs of order n for all odd $n \ge 9$.*

Combining Lemmas 5.2, 5.3 and Table 3, we can obtain

Theorem 5.4. *If (a) $n = 6, 10, 13, 14$ or $n \geq 16$ and $\Delta = 4$ or (b) $n \geq 8$ and $\Delta = 6$ or (c) $n \geq 9$ and $\Delta = 5$ or 7 or (d) $n \geq 9$ and $\Delta \in [8, n-1]$, then there exist hypoenergetic tricyclic graphs with order n and maximum degree Δ.*

When $n = 4, 5, 7$, by Lemma 5.1, there exist no hypoenergetic tricyclic graphs. By [5] (Table 1), there are four tricyclic graphs with $n = 6$ and $\Delta = 5$. In these graphs, the minimal energy is $E = 6.89260 > n = 6$, and the extremal graph is $H_1(3, 0, 1)$. When $n = 8$ and $\Delta = 7$, it is easy to check that there are five tricyclic graphs, and the minimal energy is $E = 8.04552 > n = 8$, the extremal graph is $H_1(3, 0, 3)$. We also can obtain that the minimal energy among all tricyclic graphs with $n = 8$ and $\Delta = 5$ is $E = 8 = n$, and the extremal graph is $H_2(3, 1, 1)$. Thus, for $\Delta = 4$, $n = 8, 9, 11, 12, 15$ are the only few cases for which we can not determine whether or not there exist hypoenergetic tricyclic graphs. One can employ a computer to determine them.

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