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Solutions of Some Unsolved Problems on Hypoenergetic Unicyclic, Bicyclic and Tricyclic Graphs†

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Abstract

The energy E of a graph is defined as the sum of the absolute values of its eigenvalues. A graph with n vertices is said to be hypoenergetic if $E < n$. Li and Ma [X. Li, H. Ma, Hypoenergetic and strongly hypoenergetic k-cyclic graphs, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 41–60] studied hypoenergetic k-cyclic graphs. They showed that there exist hypoenergetic unicyclic, bicyclic, and tricyclic graphs for all n and the maximum degree $\Delta \geq 4$, except in the following cases of $\Delta = 4$, for which they did not determine whether or not there exist hypoenergetic graphs: (i) $n = 13$ for unicyclic graphs; (ii) $n = 8, 10, 11, 12, 14, 15$ for bicyclic graphs and (iii) $n = 8, 9, 11, 12, 15$ for tricyclic graphs. In this paper, we complete the solution of these problems, and show that there are no hypoenergetic graphs for all these cases.

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Let G be a simple graph with n vertices, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of its adjacency matrix. The energy of a graph G is defined as $E(G) = |\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|$. For more information on the energy of graphs, we refer to [1]. Recently, it has been demonstrated [2] that the energy exceeds the number of vertices for several classes of graphs, and a result of Nikiforov [3] showed that the number of graphs satisfying the condition $E < n$ is relatively small. Thus it is feasible to find them. In [4] a hypoenergetic graph is defined to be a graph satisfying $E < n$. Some results on hypoenergetic graphs were studied in [4–9].

In [9] Li and Ma showed that there exist hypoenergetic unicyclic, bicyclic, and tricyclic graphs for all n and the maximum vertex degree $\Delta \geq 4$, except in the following cases of $\Delta = 4$, for which they do not determine whether or not there exist hypoenergetic graphs: (i) $n = 13$ for unicyclic graphs; (ii) $n = 8, 10, 11, 12, 14, 15$ for bicyclic graphs and (iii) $n = 8, 9, 11, 12, 15$ for tricyclic graphs. In this note, we complete the solution of these problems, and show that there are no hypoenergetic graphs for all these cases.

Lemma [5]. Let G be a graph with n vertices and m edges, possessing q quadrangles, and let $d_1, d_2, \ldots, d_n$ be its vertex degrees. If

$$\sqrt{\frac{8m^3}{\sum_{i=1}^{n} d_i^2 - 2m + 8q}} \geq n$$

then G is non-hypoenergetic.

Theorem. A k-cyclic graph G with n vertices and maximum vertex degree $\Delta = 4$, which satisfies one of the following conditions is non-hypoenergetic:

(I) $k = 1$ and $n = 13$;

(II) $k = 2$ and $n = 8, 10, 11, 12, 14, 15$;

(III) $k = 3$ and $n = 8, 9, 11, 12, 15$.

Proof. Let n_1, n_2, n_3, n_4 be the number of vertices with degrees $1, 2, 3, 4$ in G, respectively. $n_4 \geq 1$ since $\Delta = 4$. Let

$$N = \sqrt{\frac{8m^3}{\sum_{i=1}^{n} d_i^2 - 2m + 8q}}$$

where m is the number of edges, q is the number of quadrangles, and $d_1, d_2, \ldots, d_n$ are its vertex degrees.

By Lemma, we only need to prove that $N \geq n$ for the graph G.

(I) $k=1$. Note that $0 \leq q \leq 1$ in any unicyclic graph.

If $n=13$, then $m=n=13$ and

$$\begin{cases} n_1+n_2+n_3+n_4=13 \\ n_1+2n_2+3n_3+4n_4=26\ . \end{cases}$$

We have $n_2+2n_3+3n_4=13$, and $1 \leq n_4 \leq 4$. All the solutions of n_1, n_2, n_3, n_4 are shown in Table 1.

Table 1.

n_4	1	1	1	1	1	1	2	2	2	2	3	3	3	4
n_3	0	1	2	3	4	5	0	1	2	3	0	1	2	0
n_2	10	8	6	4	2	0	7	5	3	1	4	2	0	1
n_1	2	3	4	5	6	7	4	5	6	7	6	7	8	8
$\sum_{i=1}^{n} d_i^2$	58	60	62	64	66	68	64	66	68	70	70	72	74	76

It follows that $N \geq \sqrt{\frac{8m^3}{76-2m+8}} > 17 > n$.

(II) $k=2$. Note that $0 \leq q \leq 3$ in any bicyclic graph.

(i) If $n=8$, then $m=n+1=9$ and

$$\begin{cases} n_1+n_2+n_3+n_4=8 \\ n_1+2n_2+3n_3+4n_4=18\ . \end{cases}$$

We have $n_2+2n_3+3n_4=10$, and $1 \leq n_4 \leq 3$. All the solutions of n_1, n_2, n_3, n_4 are shown in Table 2.

Table 2.

n_4	1	1	1	1	2	2	2	3
n_3	0	1	2	3	0	1	2	0
n_2	7	5	3	1	4	2	0	1
n_1	0	0	2	3	2	3	4	4
$\sum_{i=1}^{n} d_i^2$	44	45	48	50	50	52	54	56

It follows that $N \geq \sqrt{\frac{8m^3}{56-2m+24}} > 9 > n$.

(ii) If $n=10$, then $m=n+1=11$ and

$$\begin{cases} n_1+n_2+n_3+n_4=10 \\ n_1+2n_2+3n_3+4n_4=22\ . \end{cases}$$

We have $n_2+2n_3+3n_4=12$, and $1\le n_4\le 4$. All the solutions of n_1,n_2,n_3,n_4 are shown in Table 3.

Table 3.

n_4	1	1	1	1	1	2	2	2	2	3	3	4
n_3	0	1	2	3	4	0	1	2	3	0	1	0
n_2	9	7	5	3	1	6	4	2	0	3	1	0
n_1	0	1	2	3	4	2	3	4	5	4	5	6
$\sum_{i=1}^n d_i^2$	52	54	56	58	60	58	60	62	64	64	66	70

It follows that $N\ge\sqrt{\frac{8m^3}{70-2m+24}}>12>n$.

(iii) If $n=11$, then $m=n+1=12$ and

$$\begin{cases} n_1+n_2+n_3+n_4=11 \\ n_1+2n_2+3n_3+4n_4=24\ . \end{cases}$$

We have $n_2+2n_3+3n_4=13$, and $1\le n_4\le 4$. All the solutions of n_1,n_2,n_3,n_4 are shown in Table 4.

Table 4.

n_4	1	1	1	1	1	1	2	2	2	2	3	3	3	4
n_3	0	1	2	3	4	5	0	1	2	3	0	1	2	0
n_2	5	8	6	4	2	0	7	5	3	1	4	2	0	1
n_1	5	1	2	3	4	5	2	3	4	5	4	5	6	6
$\sum_{i=1}^n d_i^2$	41	58	60	62	64	66	62	64	66	68	68	70	72	72

It follows that $N\ge\sqrt{\frac{8m^3}{72-2m+24}}>13>n$.

(iv) If $n=12$ and $k=2$, then $m=n+1=13$, $0\le q\le 3$, and

$$\begin{cases} n_1+n_2+n_3+n_4=12 \\ n_1+2n_2+3n_3+4n_4=26\ . \end{cases}$$

We have $n_2+2n_3+3n_4=14$, and $1\le n_4\le 4$. All the solutions of n_1,n_2,n_3,n_4 are shown in Table 5.

Table 5.

n_4	1	1	1	1	1	1	2	2	2	2	2	3	3	3	4	4
n_3	0	1	2	3	4	5	0	1	2	3	4	0	1	2	0	1
n_2	11	9	7	5	3	1	8	6	4	2	0	5	3	1	2	0
n_1	0	1	2	3	4	5	2	3	4	5	6	4	5	6	6	7
$\sum_{i=1}^n d_i^2$	60	62	64	66	68	70	66	68	70	72	74	72	74	76	78	80

It follows that $N\ge\sqrt{\frac{8m^3}{80-2m+24}}>15>n$.

(v) If $n = 14$, then $m = n + 1 = 15$ and

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = 14 \\ n_1 + 2n_2 + 3n_3 + 4n_4 = 30\ . \end{cases}$$

We have $n_2 + 2n_3 + 3n_4 = 16$, and $1 \leq n_4 \leq 5$. All the solutions of n_1, n_2, n_3, n_4 are shown in Table 6.

Table 6.

n_4	1	1	1	1	1	1	1	2	2	2	2	2	2
n_3	0	1	2	3	4	5	6	0	1	2	3	4	5
n_2	13	11	9	7	5	3	1	10	8	6	4	2	0
n_1	0	1	2	3	4	5	6	2	3	4	5	6	7
$\sum_{i=1}^n d_i^2$	68	70	72	74	76	78	80	74	76	78	80	82	84
n_4	3	3	3	3	4	4	4	5					
n_3	0	1	2	3	0	1	2	0					
n_2	7	5	3	1	4	2	0	1					
n_1	4	5	6	7	6	7	8	8					
$\sum_{i=1}^n d_i^2$	80	82	84	86	86	88	90	92					

It follows that $N \geq \sqrt{\frac{8m^3}{92-2m+24}} > 17 > n$.

(vi) If $n = 15$, then $m = n + 1 = 16$ and

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = 15 \\ n_1 + 2n_2 + 3n_3 + 4n_4 = 32\ . \end{cases}$$

We have $n_2 + 2n_3 + 3n_4 = 15$, and $1 \leq n_4 \leq 5$. All the solutions of n_1, n_2, n_3, n_4 are shown in Table 7.

Table 7.

n_4	1	1	1	1	1	1	1	1	2	2	2	2	2	2
n_3	0	1	2	3	4	5	6	7	0	1	2	3	4	5
n_2	14	12	10	8	6	4	2	0	11	9	7	5	3	1
n_1	0	1	2	3	4	5	6	7	2	3	4	5	6	7
$\sum_{i=1}^n d_i^2$	72	74	76	78	80	82	84	86	78	80	82	84	86	88
n_4	3	3	3	3	3	4	4	4	5	5				
n_3	0	1	2	3	4	0	1	2	0	1				
n_2	8	6	4	2	0	5	3	1	2	0				
n_1	4	5	6	7	8	6	7	8	8	9				
$\sum_{i=1}^n d_i^2$	84	86	88	90	92	90	92	94	96	98				

It follows that $N \geq \sqrt{\frac{8m^3}{98-2m+24}} > 19 > n$.

(III) $k = 3$. Note that $0 \leq q \leq 6$ in any tricyclic graph.

(i)If $n = 8$, then $m = n + 2 = 10$ and

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = 8 \\ n_1 + 2n_2 + 3n_3 + 4n_4 = 20\ . \end{cases}$$

We have $n_2+2n_3+3n_4=12$, and $1\le n_4\le 4$. All the solutions of n_1,n_2,n_3,n_4 are shown in Table 8.

Table 8.

n_4	1	1	1	2	2	2	2	3	3	4
n_3	2	3	4	0	1	2	3	0	1	0
n_2	5	3	1	6	4	2	0	3	1	0
n_1	0	1	2	0	1	2	3	2	3	4
$\sum_{i=1}^n d_i^2$	54	56	58	56	58	60	62	62	64	68

It follows that $N\ge\sqrt{\frac{8m^3}{68-2m+48}}>9>n$.

(ii) If $n=9$, then $m=n+2=11$ and

$$\begin{cases} n_1+n_2+n_3+n_4=9 \\ n_1+2n_2+3n_3+4n_4=22\ . \end{cases}$$

We have $n_2+2n_3+3n_4=13$, and $1\le n_4\le 4$. All the solutions of n_1,n_2,n_3,n_4 are shown in Table 9.

Table 9.

n_4	1	1	1	1	2	2	2	2	3	3	3	4
n_3	2	3	4	5	0	1	2	3	0	1	2	0
n_2	6	4	2	0	7	5	3	1	4	2	0	1
n_1	0	1	2	3	0	1	2	3	2	3	4	7
$\sum_{i=1}^n d_i^2$	58	60	62	64	60	62	64	60	66	68	70	71

It follows that $N\ge\sqrt{\frac{8m^3}{71-2m+48}}>10>n$.

(iii) If $n=11$, then $m=n+2=13$ and

$$\begin{cases} n_1+n_2+n_3+n_4=11 \\ n_1+2n_2+3n_3+4n_4=26\ . \end{cases}$$

We have $n_2+2n_3+3n_4=15$, and $1\le n_4\le 5$. All the solutions of n_1,n_2,n_3,n_4 are shown in Table 10.

Table 10.

n_4	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	5
n_3	2	3	4	5	6	0	1	2	3	4	0	1	2	3	0	1	0
n_2	8	6	4	2	0	9	7	5	3	1	6	4	2	0	3	1	0
n_1	0	1	2	3	4	0	1	2	3	4	2	3	4	5	4	5	6
$\sum_{i=1}^n d_i^2$	66	68	70	72	74	68	70	72	74	76	74	76	78	80	80	82	86

It follows that $N\ge\sqrt{\frac{8m^3}{86-2m+48}}>12>n$.

(iv) If $n = 12$, then $m = n + 2 = 14$ and

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = 12 \\ n_1 + 2n_2 + 3n_3 + 4n_4 = 28 \ . \end{cases}$$

We have $n_2 + 2n_3 + 3n_4 = 16$, and $1 \leq n_4 \leq 5$. All the solutions of n_1, n_2, n_3, n_4 are shown in Table 11.

Table 11.

n_4	1	1	1	1	1	2	2	2	2	2	2
n_3	2	3	4	5	6	0	1	2	3	4	5
n_2	9	7	5	3	1	10	8	6	4	2	0
n_1	0	1	2	3	4	0	1	2	3	4	5
$\sum_{i=1}^{n} d_i^2$	70	72	74	76	78	72	74	76	78	80	82

n_4	3	3	3	3	4	4	4	5
n_3	0	1	2	3	0	1	2	0
n_2	7	5	3	1	4	2	0	1
n_1	2	3	4	5	4	5	6	6
$\sum_{i=1}^{n} d_i^2$	78	80	82	84	84	86	88	90

It follows that $N \geq \sqrt{\frac{8m^3}{90-2m+48}} > 14 > n$.

(v) If $n = 15$, then $m = n + 2 = 17$ and

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = 15 \\ n_1 + 2n_2 + 3n_3 + 4n_4 = 34 \ . \end{cases}$$

We have $n_2 + 2n_3 + 3n_4 = 19$, and $1 \leq n_4 \leq 6$. All the solutions of n_1, n_2, n_3, n_4 are shown in Table 12.

Table 12.

n_4	1	1	1	1	1	1	1	2	2	2	2	2	2
n_3	2	3	4	5	6	7	8	1	2	3	4	5	6
n_2	12	10	8	6	4	2	0	11	9	7	5	3	1
n_1	0	1	2	3	4	5	6	1	2	3	4	5	6
$\sum_{i=1}^{n} d_i^2$	82	84	86	88	90	92	94	86	88	90	92	94	96

n_4	3	3	3	3	3	3	4	4	4	4	5	5	5	6
n_3	1	2	3	4	5	6	0	1	2	3	0	1	2	0
n_2	10	8	6	4	2	0	7	5	3	1	4	2	0	1
n_1	1	2	3	4	5	6	4	5	6	7	6	7	8	8
$\sum_{i=1}^{n} d_i^2$	98	100	102	104	106	108	96	98	100	102	102	104	106	108

It follows that $N \geq \sqrt{\frac{8m^3}{108-2m+48}} > 17 > n$.

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