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On Triregular Graphs whose Energy Exceeds the Number of Vertices

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Abstract

Let G be an n-vertex graph, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $E(G) = \sum_{i=1}^{n} |\lambda_i|$ is its energy and $M_k(G) = \sum_{i=1}^{n} (\lambda_i)^k$ its k-th spectral moment. A sufficient condition that the graph G be non-hypoenergetic (i. e., $E(G) \geq n$) is $\sqrt{[M_2(G)]^3/M_4(G)} \geq n$. In a recent paper [Majstorović, Klobučar, Gutman, *MATCH Commun. Math. Comput. Chem.* **62** (2009) 509–524] necessary and sufficient conditions for the validity of the latter inequality were determined for triregular graphs. These results are incorrect. We now present correct necessary and sufficient conditions for the validity of $\sqrt{[M_2(G)]^3/M_4(G)} \geq n$ in the case of triregular graphs.

1 Introduction

In this paper we are concerned with simple graphs. Let G be such a graph, let n and m be the numbers of its vertices and edges, respectively, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues [1, 2] (i. e., the eigenvalues of the adjacency matrix of G). Then the

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energy of G is [3, 4]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| \ .$$

The problem of characterizing (molecular) graphs for which the condition $E(G) > n$ is obeyed seems to be first time considered by England and Ruedenberg [5]. These authors, however, used a quantum–chemical language and way of reasoning, and the true mathematical meaning of their paper needed to be "translated" into a standard graph-theoretical terminology [6]. In the paper [6], and in three consecutive articles [7–9], several classes of graphs (some of proper chemical relevance) were shown to satisfy the condition $E \geq n$. In an earlier work [10], the validity of this inequality was confirmed also for regular graphs.

Graphs for which $E < n$ are referred to as *hypoenergetic* [11–16]. Therefore we may say that a graph for which $E \geq n$ is *non-hypoenergetic*. For a review of research on non-hypoenergetic graphs see [17].

The k-th spectral moment of a graph G is defined as

$$M_k(G) = \sum_{i=1}^{n} (\lambda_i)^k \ .$$

As well know in spectral graph theory, for a graph G with n vertices, m edges, q quadrangles, and vertex degrees $d_1, d_2, \ldots, d_n$,

$$M_2(G) = 2m \tag{1}$$

and

$$M_4(G) = 2\sum_{i=1}^{n} (d_i)^2 - 2m + 8q \ . \tag{2}$$

Rada and Tineo [18] obtained the following lower bound for graph energy (see also [19, 20]) :

$$E(G) \geq M_2(G) \sqrt{\frac{M_2(G)}{M_4(G)}} \ .$$

Bearing in mind (1) and (2), we directly arrive at:

Theorem 1.1. *Let G be a graph with n vertices and m edges, possessing q quadrangles, and let $d_1, d_2, \ldots, d_n$ be its vertex degrees. If the condition*

$$2m \sqrt{\frac{2m}{2\sum_{i=1}^{n} (d_i)^2 - 2m + 8q}} \geq n \tag{3}$$

is obeyed, then G is non-hypoenergetic.

Let x, a, and b be integers, $1 \leq x < a < b$. A graph is said to be (x, a, b)-triregular if the degrees of its vertices assume exactly three different values: x, a, and b.

In [9] the validity of the inequality (3) was investigated for triregular trees and connected triregular unicyclic and bicyclic graphs. Unfortunately, the results obtained there were incorrect. We now consider the very same problem and offer the flawless results.

For a connected (x, a, b)-triregular graph with n vertices and m edges we have

$$n_x + n_a + n_b = n \tag{4}$$

and

$$x\, n_x + a\, n_a + b\, n_b = 2m \tag{5}$$

where n_x is the number of vertices of degree x, n_a is the number of vertices of degree a, and n_b is the number of vertices of degree b. From inequalities (4) and (5) follows

$$n_a = \frac{n_x(x-b) + (bn-2m)}{b-a} \qquad ; \qquad n_b = \frac{n_x(a-x) - (an-2m)}{b-a} \ . \tag{6}$$

For triregular graphs it must be $n_x \geq 1$, $n_a \geq 1$, and $n_b \geq 1$. Since $n_a \geq 1$ holds if and only if $n_x \leq [b(n-1) + a - 2m]/(b-x)$, whereas $n_b \geq 1$ holds if and only if $n_x \geq [a(n-1) + b - 2m]/(a-x)$, we have

$$n_x \leq \frac{b(n-1) + a - 2m}{b-x} \tag{7}$$

and

$$n_x \geq \frac{a(n-1) + b - 2m}{a-x} \ . \tag{8}$$

By d_i we denote the degree of i-th vertex. Then

$$\sum_{i=1}^{n} (d_i)^2 = x^2\, n_x + a^2\, n_a + b^2\, n_b$$

which combined with equalities (6) yields

$$\sum_{i=1}^{n} (d_i)^2 = n_x(a-x)(b-x) + 2m(a+b) - abn \ .$$

From this, inequality (3) becomes

$$\sqrt{\frac{4m^3}{n_x(a-x)(b-x)+m(2a+2b-1)-abn+4q}} \geq n$$

implying

$$n_x \leq \frac{4m^3+n^2[abn-4q-m(2a+2b-1)]}{n^2(a-x)(b-x)}\ . \tag{9}$$

Theorem 1.2. *Let G be connected (x,a,b)-triregular graph with n vertices and m edges. Let n_x be the number of vertices of degree x. Then inequality* (3) *holds if and only if*

$$n_x \leq \frac{4m^3+n^2[abn-4q-m(2a+2b-1)]}{n^2(a-x)(b-x)}\ .$$

2 Triregular trees

Let T be a triregular n-vertex tree with vertex degrees 1, a, and b, $1<a<b\leq n-2$. Then $n\geq 5$ and the number of edges is $m=n-1$. Now, by applying Theorem 1.2 we obtain that inequality (3) holds if and only if

$$n_1 \leq \frac{(5+ab-2a-2b)n^3+(2a+2b-13)n^2+12n-4}{n^2(a-1)(b-1)}\ . \tag{10}$$

On the other hand, by applying inequality (8) we get

$$n_1 \geq \frac{(a-2)(n-1)+b}{a-1}\ . \tag{11}$$

By combining (10) and (11), we obtain:

Theorem 2.1. *Let T be an n-vertex $(1,a,b)$-triregular tree, $1<a<b\leq n-2$. Let n_1 be the number of vertices of degree 1. Then inequality* (3) *holds if and only if $a=2$ and*

$$n_1 \leq \frac{n^3+(2b-9)n^2+12n-4}{n^2(b-1)}\ . \tag{12}$$

Proof. When $a=2$, inequality (10) reduces to (12). So the sufficiency is obvious.

Conversely, if inequality (3) holds, then inequality (10) holds by Theorem 1.2. From inequality (11), we get

$$\frac{(a-2)(n-1)+b}{a-1} \leq \frac{(5+ab-2a-2b)n^3+(2a+2b-13)n^2+12n-4}{n^2(a-1)(b-1)}$$

which is equivalent to

$$(a-3)n^3+(b^2-b-ab-a+11)n^2-12n+4\leq 0\ . \tag{13}$$

Denote by $f(a)$ the left–hand side of (13). For $a=3$, it must be $b\geq 4$ and then $f(3)=(b^2-4b+8)n^2-12n+4\geq 8n^2-12n+4>0$, a contradiction. When $a\geq 4$, we have that $n\geq a+b$ and

$$f(a)\geq n^3+[b^2-(a+1)b+11-a]n^2-12n+4\geq (11+b)n^2-12n+4>0$$

a contradiction. Hence it cannot be $a\geq 3$, i. e. it must be $a=2$. This leads to inequality (12). ∎

Furthermore we can determine the interval for b in Theorem 2.1 as follows.

Theorem 2.2. *Let T be an n-vertex $(1,a,b)$-triregular tree, $1<a<b\leq n-2$. Let n_1 be the number of vertices of degree 1. Then inequality (3) holds if and only if $n\geq 8$, $a=2$,*

$$3\leq b\leq \frac{3}{2}+\sqrt{n-\frac{27}{4}+\frac{12}{n}-\frac{4}{n^2}}$$

and

$$n_1\leq \frac{n^3+(2b-9)n^2+12n-4}{n^2(b-1)}\ .$$

Proof. Clearly, the sufficiency follows from Theorem 2.1. Now we suppose that inequality (3) holds. Then by Theorem 2.1 we have $a=2$ and

$$n_1\leq \frac{n^3+(2b-9)n^2+12n-4}{n^2(b-1)}\ .$$

From inequality (11) for $a=2$, we get $n_1\geq b$. Therefore

$$b\leq \frac{n^3+(2b-9)n^2+12n-4}{n^2(b-1)}$$

which can be transformed into

$$n^3+(3b-9-b^2)n^2+12n-4\geq 0\ . \tag{14}$$

Denote $f(b)=-n^2b^2+3n^2b+n^3-9n^2+12n-4$. Then $f(b)$ is a quadratic polynomial in the variable b and inequality (14) is equivalent to $f(b)\geq 0$. Since the discriminant $D = (3n^2)^2 - 4(-n^2)(n^3 - 9n^2 + 12n - 4) = 4n^5 - 27n^4 + 48n - 16 > 0$ and the two roots of $f(b) = 0$ are

$$b_{1,2} = \frac{-3n^2 \mp \sqrt{4n^5 - 27n^4 + 48n - 16}}{-2n^2} = \frac{3}{2} \pm \sqrt{n - \frac{27}{4} + \frac{12}{n} - \frac{4}{n^2}}$$

we see that $f(b) \geq 0$ if and only if

$$\frac{3}{2} - \sqrt{n - \frac{27}{4} + \frac{12}{n} - \frac{4}{n^2}} \leq b \leq \frac{3}{2} + \sqrt{n - \frac{27}{4} + \frac{12}{n} - \frac{4}{n^2}} .$$

Bearing in mind the fact that $b \geq 3$, we have

$$3 \leq b \leq \frac{3}{2} + \sqrt{n - \frac{27}{4} + \frac{12}{n} - \frac{4}{n^2}}$$

which holds for $n \geq 8$. The proof is thus complete. ∎

Let T be an n-vertex $(1, 2, b)$-triregular tree. Then by applying equality (6) we have $n_2 = [n_1(1 - b) + (b - 2)n + 2]/(b - 2)$, or equivalently, $n_1 = [(n - n_2)(b - 2) + 2]/(b - 1)$. Hence inequality (12) can be rewritten as

$$\frac{(n - n_2)(b - 2) + 2}{b - 1} \leq \frac{n^3 + (2b - 9)n^2 + 12n - 4}{n^2(b - 1)}$$

which is equivalent to

$$n_2 \geq \frac{(b - 3)n^3 + (11 - 2b)n^2 - 12n + 4}{(b - 2)n^2} . \tag{15}$$

Similarly, by applying equality (6) we have $n_b = (n_1 - 2)/(b - 2)$ and inequality (12) can be rewritten as

$$(b - 2)n_b + 2 \leq \frac{n^3 + (2b - 9)n^2 + 12n - 4}{n^2(b - 1)}$$

which is equivalent to

$$n_b \leq \frac{n^3 - 7n^2 + 12 - 4}{(b - 1)(b - 2)n^2} . \tag{16}$$

Hence we have the following:

Remark 2.3. *Inequality* (12) *in Theorems 2.1 and 2.2 can be replaced by one of the inequalities* (15) *and* (16), *where* n_2 *and* n_b *are the numbers of vertices of degree* 2 *and* b, *respectively.*

From inequality (16) we see that when b is large n_b should have small value so that inequality (3) holds. By using inequality (15), we can prove the following:

Corollary 2.4. *Let T be an n-vertex $(1,2,3)$-triregular tree. Let n_2 be the number of vertices of degree 2. Then inequality (3) holds if and only if $n_2 \geq 5$, $n \geq 8$, or $n_2 = 4$, $n = 8$ or 10.*

Proof. By Theorem 2.2 and Remark 2.3, inequality (3) holds if and only if $n \geq 8$ and inequality (15) holds. Now, inequality (15) becomes $n_2 \geq (5n^2 - 12n + 4)/n^2$, or equivalently,

$$(5 - n_2)n^2 - 12n + 4 \leq 0 \; . \tag{17}$$

Since $n \geq 8$, it is clear that (17) holds if and only if $n_2 \geq 5$ or $n_2 = 4$ and $8 \leq n \leq 11$. From equality (6) we have $n_3 = n_1 - 2$. Hence when $n_2 = 4$, n is even and so $n = 8$ or 10. The proof is thus complete. ∎

3 Triregular unicyclic graphs

For an n-vertex unicyclic (x,a,b)-triregular graph, it must be $x = 1$, $m = n$, $n \geq 4$ and the number of quadrangles q is either 0 or 1. Now, by applying Theorem 1.2 we obtain that inequality (3) holds if and only if

$$n_1 \leq \frac{(5 + ab - 2a - 2b)n - 4q}{(a-1)(b-1)} \; . \tag{18}$$

On the other hand, by applying inequality (8) we get

$$n_1 \geq \frac{(a-2)n + b - a}{a - 1} \; . \tag{19}$$

By combining (18) and (19), we obtain:

Theorem 3.1. *Let G be an n-vertex unicyclic $(1,a,b)$-triregular graph, $1 < a < b \leq n-1$. Let n_1 be the number of its vertices of degree 1, and q the number of its quadrangles, where $q = 0$ or 1. Then inequality (3) holds if and only if $a = 2$ and*

$$n_1 \leq \frac{n - 4q}{b - 1} \; . \tag{20}$$

Proof. Note first that when $a = 2$, then inequality (18) reduces to (20). So the sufficiency is obvious.

Conversely, if inequality (3) holds, then inequality (18) also holds. Combining (18) and (19) we get

$$\frac{(a-2)n+b-a}{a-1} \le \frac{(5+ab-2a-2b)n-4q}{(a-1)(b-1)}$$

which is equivalent to

$$(a-3)n \le -(b-a)(b-1)-4q \ .$$

Since $-(b-a)(b-1)-4q < 0$, then $(a-3)n < 0$, which is satisfied only if $a = 2$. So we get $a = 2$, implying inequality (20). ∎

Furthermore we can determine the interval for b in Theorem 3.1 as follows.

Theorem 3.2. *Let G be an n-vertex unicyclic $(1,a,b)$-triregular graph, $1 < a < b \le n-1$. Let n_1 be the number of its vertices of degree 1, and q be the number of its quadrangles, where $q = 0$ or 1. Then inequality* (3) *holds if and only if $a = 2$,*

$$3 \le b \le \frac{3+\sqrt{4n+1-16q}}{2} \qquad \textit{and} \qquad n_1 \le \frac{n-4q}{b-1} \ .$$

Proof. The sufficiency follows from Theorem 3.1.

In order to verify the necessity, we suppose that inequality (3) holds. Then by Theorem 3.1, $a = 2$ and $n_1 \le (n-4q)/(b-1)$. From inequality (19) for $a = 2$, we get $n_1 \ge b-2$. Therefore

$$b-2 \le \frac{n-4q}{b-1}$$

which can be transformed into

$$b^2-3b+2+4q-n \le 0 \ . \tag{21}$$

Denote $f(b) = b^2-3b+2+4q-n$. Then inequality (21) is equivalent to $f(b) \le 0$. For $f(b) = 0$, the discriminant $D = 4n+1-16q > 0$ for $n \ge 4$ and the respective roots are $(3 \pm \sqrt{4n+1-16q})/2$. So $f(b) \le 0$ if and only if

$$\frac{3-\sqrt{4n+1-16q}}{2} \le b \le \frac{3+\sqrt{4n+1-16q}}{2} \ .$$

Since $b \geq 3$, we finally get

$$3 \leq b \leq \frac{3 + \sqrt{4n + 1 - 16q}}{2}$$

which completes the proof. ∎

Let G be an n-vertex unicyclic $(1,2,b)$-triregular graph. Then by applying equality (6) we have $n_2 = [n_1(1-b) + (b-2)n]/(b-2)$, or equivalently, $n_1 = [(n - n_2)(b-2)]/(b-1)$. Hence inequality (20) can be rewritten as

$$\frac{(n-n_2)(b-2)}{b-1} \leq \frac{n-4q}{b-1}$$

which is equivalent to

$$n_2 \geq \frac{(b-3)n + 4q}{b-2} \,. \tag{22}$$

Similarly, by applying equality (6) we have $n_b = n_1/(b-2)$ and inequality (20) can be rewritten as

$$(b-2)n_b \leq \frac{n-4q}{b-1}$$

i. e.,

$$n_b \leq \frac{n-4q}{(b-1)(b-2)} \,. \tag{23}$$

Hence we have the following:

Remark 3.3. *Inequality* (20) *in Theorems 3.1 and 3.2 can be replaced by one of the inequalities* (22) *and* (23)*, where n_2 and n_b are the numbers of vertices of degree* 2 *and b, respectively.*

When $b = 3$, inequality (22) becomes $n_2 \geq 4q$. Hence by Theorem 3.2 and Remark 3.3, we have:

Corollary 3.4. *Let G be an n-vertex unicyclic* $(1,2,3)$*-triregular graph, n_2 be the number of its vertices of degree* 2*, and q be the number of its quadrangles, where $q = 0$ or* 1*. Then inequality* (3) *holds if and only if $n_2 \geq 4q$.*

4 Triregular Bicyclic graphs

For an n-vertex bicyclic (x,a,b)-triregular graph, it must have $x = 1$, $m = n+1$, $n \geq 5$, and the number of quadrangles q is 0, 1, 2, or 3. Now, by applying Theorem 1.2 we obtain that inequality (3) holds if and only if

$$n_1 \leq \frac{(5+ab-2a-2b)n^3 + (13-2a-2b-4q)n^2 + 12n + 4}{n^2(a-1)(b-1)} \,. \tag{24}$$

On the other hand, by applying inequality (8) we get

$$n_1 \geq \frac{(a-2)n + b - a - 2}{a-1} \,. \tag{25}$$

By combining (24) and (25), we obtain:

Theorem 4.1. *Let G be an n-vertex bicyclic $(1,a,b)$-triregular graph, $1 < a < b \leq n-1$. Let n_1 be the number of its vertices of degree 1, and q be the number of its quadrangles, where $q = 0, 1, 2$, or 3. Then inequality (3) holds if and only if $a = 2$ and*

$$n_1 \leq \frac{n^3 + (9-2b-4q)n^2 + 12n + 4}{n^2(b-1)} \tag{26}$$

or $a = 3$, $b = 4$, $q = 0$, and

$$n_1 \leq \frac{3n^3 - n^2 + 12n + 4}{6n^2} \,. \tag{27}$$

Proof. Note first that for $a = 2$, the inequality (24) reduces to (26). In addition, for $a = 3$, $b = 4$, and $q = 0$, the inequality (24) reduces to (27). So the sufficiency is obvious.

Conversely, if inequality (3) holds, then inequality (24) also holds. Combining (24) and (25) we get

$$\frac{(a-2)n + b - a - 2}{a-1} \leq \frac{(5+ab-2a-2b)n^3 + (13-2a-2b-4q)n^2 + 12n + 4}{n^2(a-1)(b-1)}$$

which is equivalent to

$$(a-3)n^3 + (b^2 - b - ab + 3a - 11 + 4q)n^2 - 12n - 4 \leq 0 \,. \tag{28}$$

Let $f(a) = (a-3)n^3 + (b^2 - b - ab + 3a - 11 + 4q)n^2 - 12n - 4$. Then inequality (28) is equivalent to $f(a) \leq 0$.

Case 1. $a \geq 4$. Then $f(a) = (a-3)n^3 + (b^2 - (a+1)b + 3a - 11 + 4q)n^2 - 12n - 4 \geq n^3 + n^2 - 12n - 4 > 0$, a contradiction.

Case 2. $a = 3$. Then $f(3) = (b^2 - 4b - 2 + 4q)n^2 - 12n - 4 = [(b-2)^2 - 6 + 4q]n^2 - 12n - 4$.

When $b \geq 5$, then $f(3) \geq 3n^2 - 12n - 4 > 0$, for $n \geq 5$, a contradiction.

When $b = 4$, then $f(3) = (-2 + 4q)n^2 - 12n - 4$. If $q = 0$, then $f(3) = -2n^2 - 12n - 4 \leq 0$. If $q = 1$, then $f(3) = 2n^2 - 12n - 4$. Since $a = 3$ and $n \geq 7$, it follows that $f(3) > 0$, a contradiction. If $q = 2$, then $f(3) = 6n^2 - 12n - 4 > 0$ and, obviously, if $q = 3$, then also $f(3) > 0$, a contradiction.

Hence we get $a = 2$ and consequently inequality (26) or $a = 3$, $b = 4$, $q = 0$ and consequently inequality (27). ∎

Theorem 4.2. *Let G be an n-vertex bicyclic $(1, a, b)$-triregular graph, $1 < a < b \leq n - 1$. Let n_1, n_b be the numbers of its vertices of degree 1 and b , respectively, and q be the number of its quadrangles, where $q = 0, 1, 2$ or 3 . Then inequality (3) holds if and only if*

(i) $a = 2$, $3 \leq b \leq \dfrac{3}{2} + \sqrt{n + \dfrac{29}{4} - 4q + \dfrac{12}{n} + \dfrac{4}{n^2}}$ *and*

$$n_1 \leq \frac{n^3 + (9 - 2b - 4q)n^2 + 12n + 4}{n^2(b-1)} \quad , \text{ or}$$

(ii) $a = 3$, $b = 4$, $q = 0$, $n_b = 1$, *and* $n \geq 9$, *or*

(iii) $a = 3$, $b = 4$, $q = 0$, $n_b = 2$, *and* $9 \leq n \leq 12$.

Proof. Suppose that inequality (3) holds. Then by Theorem 4.1, we distinguish the following two cases:

Case 1. $a = 2$ and $n_1 \leq \dfrac{n^3 + (9 - 2b - 4q)n^2 + 12n + 4}{n^2(b-1)}$.

From inequality (25) for $a = 2$, we get $n_1 \geq b - 4$. Therefore

$$b - 4 \leq \frac{n^3 + (9 - 2b - 4q)n^2 + 12n + 4}{n^2(b-1)}$$

which can be transformed into

$$n^2b^2 - 3n^2b - n^3 + (4q - 5)n^2 - 12n - 4 \leq 0 \ . \tag{29}$$

Denote $f(b) = n^2b^2 - 3n^2b - n^3 + (4q-5)n^2 - 12n - 4$. Then inequality (29) is equivalent to $f(b) \leq 0$. For $f(b) = 0$, the discriminant $D = 4n^5 + (29-16q)n^4 + 48n^3 + 16n^2 > 0$ for $n \geq 5$ and the respective roots are

$$\frac{3n^2 \pm \sqrt{4n^5 + (29-16q)n^4 + 48n^3 + 16n^2}}{2n^2} \,.$$

Consequently, $f(b) \leq 0$ holds if and only if

$$\frac{3}{2} - \sqrt{n + \frac{29}{4} - 4q + \frac{12}{n} + \frac{4}{n^2}} \leq b \leq \frac{3}{2} + \sqrt{n + \frac{29}{4} - 4q + \frac{12}{n} + \frac{4}{n^2}} \,.$$

Since $b \geq 3$, we finally get

$$3 \leq b \leq \frac{3}{2} + \sqrt{n + \frac{29}{4} - 4q + \frac{12}{n} + \frac{4}{n^2}} \,.$$

Case 2. $a = 3$, $b = 4$, $q = 0$, and $n_1 \leq \dfrac{3n^3 - n^2 + 12n + 4}{6n^2}$.

For $a = 3$, $b = 4$, and $q = 0$, it must be $n \geq 9$. From equality (6) for $x = 1$, $a = 3$, $b = 4$, and $m = n+1$, we obtain $n_4 = 2n_1 - n + 2$, i. e., $n_1 = (n_4 + n - 2)/2$. Therefore

$$n_1 \leq \frac{3n^3 - n^2 + 12n + 4}{6n^2}$$

if and only if

$$\frac{n_4 + n - 2}{2} \leq \frac{n}{2} - \frac{1}{6} + \frac{2}{n} + \frac{2}{3n^2}$$

if and only if

$$n_4 \leq \frac{5}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

if and only if $n \geq 13$ and $n_4 = 1$, or $9 \leq n \leq 12$, $n_4 \leq 2$. The necessity is thus complete.

Conversely, the sufficiency easily follows from Theorem 4.1. ∎

Let G be an n-vertex bicyclic $(1, 2, b)$-triregular graph. Then by applying equality (6) we obtain $n_2 = [n_1(1-b) + (b-2)n - 2]/(b-2)$, or, the same in another form, $n_1 = [(n - n_2)(b-2) - 2]/(b-1)$. Hence (26) can be rewritten as

$$\frac{(n - n_2)(b-2) - 2}{b-1} \leq \frac{n^3 + (9 - 2b - 4q)n^2 + 12n + 4}{n^2(b-1)}$$

which is equivalent to

$$n_2 \geq \frac{(b-3)n^3 + (2b+4q-11)n^2 - 12n - 4}{(b-2)n^2} \ . \tag{30}$$

Similarly, by applying equality (6) we have $n_b = (n_1+2)/(b-2)$ and inequality (26) can be rewritten as

$$(b-2)n_b - 2 \leq \frac{n^3 + (9-2b-4q)n^2 + 12n + 4}{n^2(b-1)}$$

from which we readily obtain

$$n_b \leq \frac{n^3 + (7-4q)n^2 + 12 + 4}{(b-1)(b-2)n^2} \ . \tag{31}$$

Hence we have the following:

Remark 4.3. *Inequality* (26) *in Theorems 4.1 and 4.2 can be replaced by one of the inequalities* (30) *and* (31)*, where* n_2 *and* n_b *are the numbers of vertices of degree* 2 *and* b *, respectively.*

Corollary 4.4. *Let* G *be an* n*-vertex bicyclic* $(1,2,3)$*-triregular graph,* n_2 *be the number of its vertices of degree* 2 *, and* q *be the number of its quadrangles, where* $q = 0, 1, 2$ *, or* 3 *. Then the inequality* (3) *holds if and only if*
(i) $q = 0$ *, or*
(ii) $q = 1$ *, or*
(iii) $q = 2$ *and* $n_2 \geq 3$ *, or*
(iv) $q = 2$ *,* $n_2 = 2$ *and* $n = 8, 10$ *, or* 12 *, or*
(v) $q = 3$ *and* $n_2 \geq 7$ *, or*
(vi) $q = 3$ *,* $n_2 = 6$ *, and* $n = 10$ *or* 12 *.*

Proof. By Theorem 4.2 and Remark 4.3, inequality (3) holds if and only if inequality (30) holds. Now, inequality (30) becomes $n_2 \geq [(4q-5)n^2 - 12n - 4]/n^2$, or equivalently,

$$(n_2 + 5 - 4q)n^2 + 12n + 4 \geq 0 \ . \tag{32}$$

It is clear that (32) is obeyed if and only if $q = 0$, or $q = 1$, or $q = 2$ and $(n_2 - 3)n^2 + 12n + 4 \geq 0$, or $q = 3$ and $(n_2 - 7)n^2 + 12n + 4 \geq 0$.

If $q = 2$, then $n \geq 7$. So $(n_2 - 3)n^2 + 12n + 4 \geq 0$ if and only if $n_2 \geq 3$, or $n_2 = 2$ and $7 \leq n \leq 12$. From equality (6) we have $n_3 = n_1 + 2$. Hence when $n_2 = 2$, then n is even and therefore $n = 8, 10$, or 12.

If $q = 3$, then $n \geq 6$. Since $n_3 = n_1 + 2$, if $n_2 = 5$, then $n \geq 9$; if $n_2 = 6$, then $n \geq 10$. So $(n_2 - 7)n^2 + 12n + 4 \geq 0$ if and only if $n_2 \geq 7$, or $n_2 = 6$ and $10 \leq n \leq 12$. Similarly, when $n_2 = 6$, n is even and so $n = 10$ or 12.

The proof is thus complete. ∎

5 On the Error in the Paper [9]

In the paper [9], and later also in the review [17], the inequality (10) was obtained for triregular trees. Then, in view of the evident relation $n_1 \geq a + b - 2$, it was concluded that

$$\frac{(5 + ab - 2a - 2b)n^3 + (2a + 2b - 13)n^2 + 12n - 4}{n^2(a-1)(b-1)} \geq a + b - 2 \tag{33}$$

which is correct. Next, it was concluded that (33) is a necessary and sufficient condition for the validity of inequality (3), which is incorrect. Then all conclusions drawn from the analysis of (33) were incorrect too.

Analogous errors were committed also in the cases of unicyclic and bicyclic graphs.

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