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Extending a Theorem by Fiedler and Applications to Graph Energy

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Abstract

We use a lemma due to Fiedler to obtain eigenspaces of some graphs and apply these results to graph energy (= the sum of absolute values of the graph eigenvalues = the sum of singular values of the adjacency matrix). We obtain some new upper and lower bounds for graph energy and find new examples of graphs whose energy exceeds the number of vertices.

1 INTRODUCTION

Let A be an $n \times n$ matrix. The scalars λ and vectors $v \neq 0$ satisfying $Av = \lambda v$ we call eigenvalues and eigenvectors of A, respectively, and any such pair (λ, v) is called an eigenpair for A.

The set of distinct eigenvalues (including multiplicities), denoted by $\sigma(A)$, is called the spectrum of A. The eigenvectors of the adjacency matrix of a graph G, together with the eigenvalues, provide a useful tool in the investigation of the structure of the graph. In this work, having a result presented in [1] as motivation, we obtain, for graphs with a special type of adjacency matrix, a lower bound for the energy of these graphs. In 1974 Fiedler obtained the following result [1] .

Let A, B be $m \times m$ and $n \times n$ symmetric matrices with corresponding eigenpairs (α_i, u_i) , $i = 1, \ldots, m$, (β_ℓ, v_ℓ) , $\ell = 1, \ldots, n$, respectively.

Lemma 1.1. *[1] Let A, B be $m\times m$ and $n\times n$ symmetric matrices with corresponding eigenpairs (α_i, u_i) , $i = 1, \ldots, m$, (β_i, v_i) , $i = 1, \ldots, n$, respectively. Suppose that $\|u_1\| = 1 = \|v_1\|$. Then, for any ρ , the matrix*

$$C = \begin{pmatrix} A & \rho\, u_1\, v_1^T \\ \rho\, v_1\, u_1^T & B \end{pmatrix}$$

has eigenvalues $\alpha_2, \ldots, \alpha_n, \beta_2, \ldots, \beta_m, \gamma_1, \gamma_2$, where γ_1, γ_2 are eigenvalues of

$$\widehat{C} = \begin{pmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{pmatrix} .$$

We now offer a generalization of Fiedler's lemma.

Suppose now that $u_1, \ldots, u_m$ (resp. $v_1, \ldots, v_n$) constitute an orthonormal system of eigenvectors of A (resp. B). Let $(\,u_1|\,, \ldots, |u_m\,)$ and $(\,v_1|\,, \ldots, |v_n\,)$ be the matrices whose columns consist of the cordinates of the eigenvectors u_i and v_i associated to the eigenvalues α_i and β_i, respectively.

In fact,

$$(\,u_1|\,, \ldots, |u_m\,) = \begin{pmatrix} u_{11} & u_{12} & & u_{1m} \\ u_{21} & u_{22} & & u_{2m} \\ \vdots & \vdots & & \vdots \\ u_{m1} & u_{m2} & & u_{mm} \end{pmatrix}$$

where $u_p = \begin{pmatrix} u_{1p} \\ u_{2p} \\ \vdots \\ u_{mp} \end{pmatrix}$ for $p = 1, \ldots, m$, and

$$(\,v_1|\,, \ldots, |v_n\,) = \begin{pmatrix} v_{11} & v_{12} & & v_{1n} \\ v_{21} & v_{22} & & v_{2n} \\ \vdots & \vdots & & \vdots \\ v_{n1} & v_{n2} & & v_{nn} \end{pmatrix}$$

where $v_p = \begin{pmatrix} v_{1p} \\ v_{2p} \\ \vdots \\ v_{np} \end{pmatrix}$ for $p = 1, \ldots, n$.

Lemma 1.2. *Let* $k \leq \min\{m,n\}$ *and* $U = (u_1|,\ldots,|u_k)$, $V = (v_1|,\ldots,|v_k)$. *Then, for any* ρ, *the matrix*

$$C = \begin{pmatrix} A & \rho UV^T \\ \rho VU^T & B \end{pmatrix}$$

has eigenvalues $\alpha_{k+1},\ldots,\alpha_m,\beta_{k+1},\ldots,\beta_n,\gamma_{1j},\gamma_{2j}$, $j = 1,2,\ldots,k$, *where for* $s = 1,2$, γ_{sj} *is an eigenvalue of*

$$\widehat{C}_j = \begin{pmatrix} \alpha_j & \rho \\ \rho & \beta_j \end{pmatrix}.$$

Proof. For $j = 1,2,\ldots,k$, let $(\gamma_{sj},\widehat{w_{sj}})$, $s = 1,2$, be an eigenpair of the matrix

$$\widehat{C}_j = \begin{pmatrix} \alpha_j & \rho \\ \rho & \beta_j \end{pmatrix}$$

where $\widehat{w}_{sj} = \begin{pmatrix} w_{1sj} & w_{2sj} \end{pmatrix}^T$. Then, for $j = 1,2,\ldots,k$, $s = 1,2$, we have,

$$\begin{pmatrix} \alpha_j & \rho \\ \rho & \beta_j \end{pmatrix} \begin{pmatrix} w_{1sj} \\ w_{2sj} \end{pmatrix} = \gamma_{sj} \begin{pmatrix} w_{1sj} \\ w_{2sj} \end{pmatrix}.$$

Let $\begin{pmatrix} w_{1sj}\, u_j \\ w_{2sj}\, v_j \end{pmatrix}$ be an $(m+n)-$vector. Then,

$$\begin{pmatrix} A & \rho UV^T \\ \rho VU^T & B \end{pmatrix} \begin{pmatrix} w_{1sj}\, u_j \\ w_{2sj}\, v_j \end{pmatrix} = \begin{pmatrix} w_{1sj}\, Au_j + \rho w_{2sj}\, UV^T v_j \\ \rho w_{1sj}\, VU^T u_j + w_{2sj}\, Bv_j \end{pmatrix}$$

as

$$w_{1sj}\, Au_j + \rho w_{2sj}\, UV^T v_j = w_{1sj}\, \alpha_j \begin{pmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{mj} \end{pmatrix} + \rho w_{2sj}\, U \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\rho w_{1sj}\, VU^T u_j + w_{2sj} Bv_j = \rho w_{1sj}\, V \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} + w_{2sj}\, \beta_j \begin{pmatrix} v_{1j} \\ v_{2j} \\ \vdots \\ v_{nj} \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} A & \rho UV^T \\ \rho VU^T & B \end{pmatrix} \begin{pmatrix} w_{1sj}\, u_j \\ w_{2sj}\, v_j \end{pmatrix} = \begin{pmatrix} w_{1sj}\, \alpha_j\, u_j + \rho w_{2sj}\, u_j \\ \rho w_{1sj}\, v_j + w_{2sj}\, \beta_j\, v_j \end{pmatrix}$$

$$= \begin{pmatrix} (w_{1sj}\, \alpha_j + \rho w_{2sj}\,)\, u_j \\ (\rho w_{1sj} + w_{2sj}\, \beta_j)\, v_j \end{pmatrix} = \begin{pmatrix} \gamma_{sj}\, w_{1sj}\, u_j \\ \gamma_{sj}\, w_{2sj}\, v_j \end{pmatrix} = \gamma_{sj} \begin{pmatrix} w_{1sj}\, u_j \\ w_{2sj}\, v_j \end{pmatrix}.$$

Recall that $\left(\gamma_{sj}, \widehat{w_{sj}}\right)$, $s=1,2$ is an eigenpair of $\widehat{C}_j = \begin{pmatrix} \alpha_j & \rho \\ \rho & \beta_j \end{pmatrix}$, where $\widehat{w}_{sj} = \begin{pmatrix} w_{1sj} & w_{2sj} \end{pmatrix}^T$. Then, for $s = 1, 2$, we have,

$$\begin{pmatrix} \alpha_j & \rho \\ \rho & \beta_j \end{pmatrix} \begin{pmatrix} w_{1sj} \\ w_{2sj} \end{pmatrix} = \gamma_{sj} \begin{pmatrix} w_{1sj} \\ w_{2sj} \end{pmatrix}.$$

Therefore, for $j = 1, 2, \ldots, k$, $s = 1, 2$, $\left(\gamma_{sj}, \begin{pmatrix} w_{1sj}\, u_j \\ w_{2sj}\, v_j \end{pmatrix}\right)$ are $2k$ eigenpairs for C. For $i = k+1, \ldots, m$, we have

$$\begin{pmatrix} A & \rho U V^T \\ \rho V U^T & B \end{pmatrix} \begin{pmatrix} u_i \\ 0 \end{pmatrix} = \begin{pmatrix} A u_i \\ 0 \end{pmatrix} = \alpha_i \begin{pmatrix} u_i \\ 0 \end{pmatrix}$$

and for $t = k+1, \ldots, n$, we set

$$\begin{pmatrix} A & \rho U V^T \\ \rho V U^T & B \end{pmatrix} \begin{pmatrix} 0 \\ v_t \end{pmatrix} = \begin{pmatrix} 0 \\ B v_t \end{pmatrix} = \beta_t \begin{pmatrix} 0 \\ v_t \end{pmatrix}.$$

Therefore, $\left(\alpha_i, \begin{pmatrix} u_i \\ 0 \end{pmatrix}\right)$ for $i = k+1, \ldots, m$, and $\left(\beta_t, \begin{pmatrix} 0 \\ v_t \end{pmatrix}\right)$ for $t = k+1, \ldots, n$, are eigenpairs for C. Thus the result is proved. $\square$

2 APPLICATIONS

A simple graph G is a pair of sets (V, E), such that V is a nonempty finite set of n vertices and E is the set of m edges. We say that G is a simple $(n, m)-$ graph. Let $A(G)$ be the adjacency matrix of the graph G. Its eigenvalues $\lambda_1, \ldots, \lambda_n$ form the spectrum of G (cf. [2]).

The notion of energy of an (n, m)-graph G (written $E(G)$) is a nowadays much studied spectral invariant, see the reviews [3, 4] and the recent works [5–9]. This concept is of great interest in a vast range of fields, especially in chemistry since it can be used to approximate the total π-electron energy of a molecule. It is defined as [10]

$$E(G) = \sum_{j=1}^{n} |\lambda_j| \ .$$

Given a complex $m \times n$ matrix C, we index its singular values by $s_1(C), s_2(C), \ldots$. The value

$$E(C) = \sum_j s_j(C)$$

is the energy of C (cf. [11]), thereby extending the concept of graph energy. Consequently, if the matrix $C \in \mathbb{R}^{n\times n}$ is symmetric with eigenvalues $\beta_1(C), \ldots, \beta_n(C)$, then its energy is given by

$$E(C) = \sum_{i=1}^{n} |\beta_i(C)| \ .$$

In this section, using Lemma 1.2, we present an application of the concept of energy to some special kinds of graphs.

We first formulate an auxiliary result.

Let $Q = (q_{ij})$ be an $n_1 \times n_2$ matrix with real-valued elements. Without loss of generality we may assume that $n_1 \le n_2$. Let the singular values of Q be $s_1, s_2, ..., s_{n_1}$. Then $E(Q) = \sum\limits_{i=1}^{n_1} s_i$.

Lemma 2.1.

$$E(Q) \le \sqrt{n_1}\ \|Q\|_F$$

where $\|Q\|_F$ is the Frobenius norm of Q, defined as [12]

$$\|Q\|_F = \sqrt{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} q_{ij}^2} \ .$$

Proof. Let $\xi_1, \xi_2, ..., \xi_p$ be real numbers. Their variance

$$\frac{1}{p}\sum_{i=1}^{p}\xi_i^2 - \left(\frac{1}{p}\sum_{i=1}^{p}\xi_i\right)^2$$

is known to be non-negative. Therefore,

$$\sum_{i=1}^{p}\xi_i \le \sqrt{p}\sqrt{\sum_{i=1}^{p}\xi_i^2} \ .$$

Setting in the above inequality $p = n_1$ and $\xi_i = s_i$, we obtain

$$E(Q) = \sum_{i=1}^{n_1} s_i \le \sqrt{n_1}\sqrt{\sum_{i=1}^{n_1} s_i^2} \ .$$

Now, the singularities of the matrix Q are just the square roots of the eigenvalues of QQ^T. Therefore, $\sum\limits_{i=1}^{n_1} s_i^2$ is equal to the sum of the eigenvalues of QQ^T, which in turn is the trace of QQ^T. Lemma 2.1 follows from

$$Tr(QQ^T) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} q_{ij}\, q_{ji}^T = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} q_{ij}^2 = \|Q\|_F^2 \ .$$

□

Let

$$A(G) = \begin{pmatrix} B & X \\ X^T & C \end{pmatrix}$$

be a partition of the adjacency matrix of a graph G, where B and C have orders $n_1 \times n_1$ and $n_2 \times n_2$, respectively. Thus X has order $n_1 \times n_2$.

Theorem 2.2. *Let G be a graph of order $n = n_1 + n_2$, such that its adjacency matrix is given by $A(G) = \begin{pmatrix} B & X \\ X^T & C \end{pmatrix}$, where B and C represent adjacency matrices of graphs. Moreover, let $k \leq \min\{n_1, n_2\}$ and consider the eigenpairs $\{(\alpha_i, u_i) : 1 \leq i \leq k\}$ of B and $\{(\beta_i, v_i) : 1 \leq i \leq k\}$ of C , such that the sets $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ are orthonormal vectors. Let $U = (u_1|\ldots|u_k)$ and $V = (v_1|\ldots|v_k)$. Then*

$$\begin{aligned} E(G) \;\leq\; & \frac{1}{2}\sum_{i=1}^{k}\left|\alpha_i + \beta_i + \sqrt{(\alpha_i - \beta_i)^2 + 4}\right| + \frac{1}{2}\sum_{i=1}^{k}\left|\alpha_i + \beta_i - \sqrt{(\alpha_i - \beta_i)^2 + 4}\right| \\ + \; & \sqrt{n_1 - k}\sqrt{\|B\|_F^2 - \sum_{i=1}^{k}|\alpha_i|^2} + \sqrt{n_2 - k}\sqrt{\|C\|_F^2 - \sum_{i=1}^{k}|\beta_i|^2} \\ + \; & 2\sqrt{\min\{n_1, n_2\}}\,\left\|X - UV^T\right\|_F \; . \end{aligned}$$

Proof.

$$A(G) = \begin{pmatrix} B & X \\ X^T & C \end{pmatrix} = \begin{pmatrix} B & UV^T \\ VU^T & C \end{pmatrix} + \begin{pmatrix} 0 & X - UV^T \\ X^T - VU^T & 0 \end{pmatrix} = M + N \, .$$

Thus, according by the Ky Fan theorem [13], $E(G) \leq E(M) + E(N)$.

Note that the spectrum of the matrix $\widehat{M_i} = \begin{pmatrix} \alpha_i & 1 \\ 1 & \beta_i \end{pmatrix}$ is

$$\sigma\left(\widehat{M_i}\right) = \left\{\frac{1}{2}\left(\alpha_i + \beta_i \pm \sqrt{(\alpha_i - \beta_i)^2 + 4}\right)\right\}.$$

Then, by applying Lemma 1.2,

$$\begin{aligned} E(M) \;=\; & \frac{1}{2}\sum_{i=1}^{k}\left|\alpha_i + \beta_i + \sqrt{(\alpha_i - \beta_i)^2 + 4}\right| + \frac{1}{2}\sum_{i=1}^{k}\left|\alpha_i + \beta_i - \sqrt{(\alpha_i - \beta_i)^2 + 4}\right| \\ + \; & \sum_{i=k+1}^{n_1}|\alpha_i| + \sum_{i=k+1}^{n_2}|\beta_i| \, . \end{aligned}$$

By Lemma 2.1,

$$\sum_{i=k+1}^{n_1} |\alpha_i| + \sum_{i=k+1}^{n_2} |\beta_i| \leq \sqrt{n_1-k}\sqrt{\sum_{i=k+1}^{n_1} |\alpha_i|^2}$$

$$= \sqrt{n_1-k}\sqrt{\|B\|_F^2 - \sum_{i=1}^{k} |\alpha_i|^2} + \sqrt{n_2-k}\sqrt{\|C\|_F^2 - \sum_{i=1}^{k} |\beta_i|^2} \,.$$

Moreover, also by Lemma 2.1,

$$E(N) = 2E(X - UV^T) \leq 2\sqrt{\min\{n_1, n_2\}}\, \left\|X - UV^T\right\|_F \,.$$

□

Consider the following special case of Theorem 2.2. Let both submatrices B and C be equal, say equal to A, the adjacency matrix of an (n,m)-graph G. Further, let $k = n_1 = n_2 = n$. By setting in Lemma 1.2, $A = B = A(G)$, and considering (λ_i, u_i), $i = 1, \ldots, n$, such that $U = (u_1|, \ldots, |u_n)$ is an orthonormal matrix and $\rho = 1$, the matrix

$$C = \begin{pmatrix} A & \rho UU^T \\ \rho UU^T & A \end{pmatrix} = \begin{pmatrix} A & I_n \\ I_n & A \end{pmatrix}$$

has eigenvalues $\beta_{s1}, \beta_{s2}, \ldots, \beta_{sn}$, $s = 1, 2$, where for $j = 1, 2, \ldots, n$, $s = 1, 2$, β_{sj} is an eigenvalue of

$$\widehat{C}_j = \begin{pmatrix} \lambda_j & 1 \\ 1 & \lambda_j \end{pmatrix} \,.$$

Thus $\beta_{1j} = \lambda_j - 1$, $\beta_{2j} = \lambda_j + 1$, and for the symmetric matrix C, we have

$$E(C) = \sum_{i=1}^{n} |\lambda_i - 1| + \sum_{i=1}^{n} |\lambda_i + 1| \geq \left|\sum_{i=1}^{n} (\lambda_i - 1)\right| + \left|\sum_{i=1}^{n} (\lambda_i + 1)\right| = n + n = 2n \,.$$

Hence $E(C) > 2n$ holds, except if for all $i = 1, 2, \ldots, n$, the value of the terms $|\lambda_i - 1|$ and $|\lambda_1 + 1|$ is either zero or equal to some constant $\gamma > 0$. This can happen only if either for all $i = 1, 2, \ldots, n$, $\lambda_i = 0$ or $\lambda_i \in \{-1, +1\}$, i. e., if either (a) the underlying graph G is without edges, or (b) all components of G are isomorphic to K_2. In these two cases, $E(C)$ is equal to the number of vertices of the graph whose adjacency matrix is C.

In conclusion, except in the two "pathological" cases (a) and (b), the energy of the graph whose adjacency matrix is C is greater than its number of vertices.

At this point it is worth noting that the graph whose adjacency matrix is C is just the sum of the graphs G and K_2 , denoted by $G+K_2$, where K_2 is the complete graph on two vertices.

The graph operation marked by + is described in detail elsewhere [2, 14]. Let G_1 and G_2 be two graphs, with (disjoint) vertex sets $V(G_1)$ and $V(G_2)$. Then the vertex set of G_1+G_2 is $V(G_1) \times V(G_2)$. The vertices (x_1, x_2) and (y_1, y_2) of G_1+G_2 are adjacent if and only if $x_1 = y_1$ and (x_2, y_2) is an edge of G_2 or if $x_2 = y_2$ and (x_1, y_1) is an edge of G_1.

For the present consideration it is important that the spectrum of G_1+G_2 consists of the sums of eigenvalues of G_1 and G_2 [2]. In view of this, the finding that $\beta_{1j} = \lambda_j - 1$, $\beta_{2j} = \lambda_j + 1$ is an immediate consequence of the fact that the spectrum of K_2 consists of the numbers $+1$ and -1.

The result $E(C) \geq 2n$ can now be generalized as follows.

Let $\alpha_1, \ldots, \alpha_{n_1}$ and $\beta_1, ..., \beta_{n_2}$ be, respectively, the eigenvalues of G_1 and G_2. Then the eigenvalues of G_1+G_2 are of the form $\alpha_i + \beta_j$, $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$, and

$$E(G_1+G_2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} |\alpha_i + \beta_j| \ .$$

We thus have,

$$E(G_1+G_2) \geq \sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} (\alpha_i + \beta_j) \right| = \sum_{i=1}^{n_1} |n_2\, \alpha_i| = n_2 \sum_{i=1}^{n_1} |\alpha_i| = n_2\, E(G_1)$$

because of

$$\sum_{j=1}^{n_2} \alpha_i = n_2\, \alpha_i \qquad \text{and} \qquad \sum_{j=1}^{n_2} \beta_j = 0 \ .$$

In an analogous manner it can be shown that $E(G_1+G_2) \geq n_1\, E(G_2)$.

Same as in the above example, equality $E(G_1+G_2) = n_2\, E(G_1)$ occurs if (a') G_2 is without edges, whereas equality $E(G_1+G_2) = n_1\, E(G_2)$ occurs if (a") G_1 is without edges. Both equalities $E(G_1+G_2) = n_2\, E(G_1)$ and $E(G_1+G_2) = n_1\, E(G_2)$ occur if (b') all components of both G_1 and G_2 are isomorphic to K_2.

The graph G_1+G_2 has $n_1\, n_2$ vertices. Bearing this in mind we arrive at:

Theorem 2.3. *With the exception of the (above specified) "pathological" cases (a'), (a"), and (b'), if either the energy of G_1 exceeds or is equal to the number of vertices of G_1, or the energy of G_2 exceeds or is equal to the number of vertices of G_2, then the energy of $G_1 + G_2$ exceeds the number of vertices of $G_1 + G_2$.*

In connection with Theorem 2.3 it is worth noting that the problem of constructing and characterizing graphs whose energy exceeds the number of vertices was first considered in [15] and thereafter in [16–19]. A closely related problem is the construction and characterization of *hypoenergetic* graphs, namely (connected) graphs whose energy is less than the number of vertices [13,20–25]. Also graphs whose energy is equal to the number of vertices were recently studied [26]. From this point of view, Theorem 2.3 provides an additional possibility to obtain (infinitely many) non-hypoenergetic graphs.

Let A, B, C, X, U and V be the same matrices as in the formulation and proof of Theorem 2.2. Let, as before

$$M = \begin{pmatrix} B & UV^T \\ VU^T & C \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} 0 & UV^T \\ VU^T & 0 \end{pmatrix} .$$

Theorem 2.4. $E(A) \geq E(M) - E(Q)$.

Proof. Let

$$\widehat{A} = \begin{pmatrix} B & -X \\ -X^T & C \end{pmatrix} .$$

Then, as well known, $E(A) = E(\widehat{A})$. We have $A = M + P$ and $\widehat{A} = M + \widehat{P}$, where

$$P = \begin{pmatrix} 0 & X - UV^T \\ X^T - VU^T & 0 \end{pmatrix} \quad \text{and} \quad \widehat{P} = \begin{pmatrix} 0 & -X - UV^T \\ -X^T - VU^T & 0 \end{pmatrix} .$$

Therefore

$$A + \widehat{A} = 2M - 2Q .$$

By Ky Fan theorem [13],

$$2E(M) \leq E(A) + E(\widehat{A}) + 2E(Q)$$

implying the theorem. □

As a special case of Theorem 2.4, for $k = 1$,

$$M = \begin{pmatrix} B & u_1\, v_1^T \\ v_1\, u_1^T & C \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} 0 & u_1\, v_1^T \\ v_1\, u_1^T & 0 \end{pmatrix} .$$

Hence, $E(Q) = 2\left|v_1^T u_1\right|$ and $E(A) \geq E(M) - 2\left|v_1^T u_1\right|$.

By Fiedler's Lemma 1.1,

$$\begin{aligned} E(M) &= E(B) + E(C) + \frac{1}{2}\left|\alpha_1 + \beta_1 + \sqrt{(\alpha_1 - \beta_1)^2 + 4}\right| \\ &+ \frac{1}{2}\left|\alpha_1 + \beta_1 - \sqrt{(\alpha_1 - \beta_1)^2 + 4}\right| - (|\alpha_1| + |\beta_1|) . \end{aligned}$$

Therefore, $E(A) \geq E(B) + E(C) + \varepsilon$, where

$$\begin{aligned} \varepsilon &= \frac{1}{2}\left|\alpha_1 + \beta_1 + \sqrt{(\alpha_1 - \beta_1)^2 + 4}\right| + \frac{1}{2}\left|\alpha_1 + \beta_1 - \sqrt{(\alpha_1 - \beta_1)^2 + 4}\right| \\ &- (|\alpha_1| + |\beta_1|) - 2\left|v_1^T u_1\right| . \end{aligned}$$

The inequality $E(A) \geq E(B) + E(C)$ has been reported earlier [27]. Therefore our result will be an improvement of that inequality only if $\varepsilon > 0$. If $\alpha_1 > 0$, $\beta_1 > 0$, and $\alpha_1\, \beta_1 \geq 1$, then

$$\frac{1}{2}\left|\alpha_1 + \beta_1 + \sqrt{(\alpha_1 - \beta_1)^2 + 4}\right| + \frac{1}{2}\left|\alpha_1 + \beta_1 - \sqrt{(\alpha_1 - \beta_1)^2 + 4}\right| - (|\alpha_1| + |\beta_1|) = 0$$

and therefore $\varepsilon < 0$. Thus, in order to get an improvement, we must choose α_1 and β_1 such that $\alpha_1\, \beta_1 < 1$. For instance, for $\alpha_1 = \beta_1 = 0$ we get $\varepsilon = 2 - 2\left|v_1^T u_1\right|$, which is positive because of $\left|v_1^T u_1\right| \leq \|v_1\|\, \|u_1\| = 1 \cdot 1 = 1$.

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