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A Note on D-Equienergetic Graphs*

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Abstract: The distance energy of a graph G is defined as the sum of the absolute values of the eigenvalues of the distance matrix of G. Two graphs with the same distance energy (resp. different distance matrix spectra) are called D-equienergetic (resp. non-D-cospectral). Recently, Indulal et. al. proved that there exists pairs of non-D-cospectral D-equienergetic graphs of order n when $n \equiv 1 (mod\, 3)$ or $n \equiv 0 (mod\, 6)$. In this paper, we prove that there exist pairs of non-D-cospectral D-equienergetic graphs of order n for every $n \geq 6$.

1 Introduction

Let G be a connected undirected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *distance matrix*, denoted by $D(G)$, of G is defined as $D(G) = [d_{ij}]$, where d_{ij} is the distance between the vertices v_i and v_j in G. The eigenvalues $\mu_1, \ldots, \mu_n$ of $D(G)$ are said to form the spectrum of $D(G)$, denoted by $spec_D(G)$. Since D is symmetric, its eigenvalues are all real and can be arranged in non-increasing order: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

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Two graphs G and H are said to be *non-D-cospectral* if $spec_D(G) \neq spec_D(H)$. The characteristic polynomial of $D(G)$ is defined as $\Phi(G,x) = det(xI - D(G))$, where I is the identity matrix. Recently, the *distance energy* $E_D(G)$ of a graph G is defined as [1]

$$E_D(G) = \sum_{i=1}^{n} |\mu_i| \ . \tag{1}$$

The notation of distance energy is put forward in full analogy to the definition of the (ordinary) graph energy $E(G)$ [2], i. e.,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of G. For more recent research on $E(G)$ see [3–6].

Two graphs with the same distance energy are called *D-equienergetic* [1]. We are, of course, interested in D-equienergetic graphs only if these are non-D-cospectral.

The study of the spectrum of a distance matrix or the distance energy pertaining to a graph became popular in past years. Balaban et al. [7] proposed the use of μ_1 as a molecular descriptor. In [7,8], it was successfully used to infer the extent of branching and model boiling points of alkanes. For the application and the background of distance matrix on the chemistry, one can refer to [7–13]. Recently, bounds for the distance energy of graphs of diameter 2 were determined in [1]. Moreover, it has been proven [1] that there exists pairs of non-D-cospectral D-equienergetic graphs of order n when $n \equiv 1 (mod\, 3)$ or $n \equiv 0 (mod\, 6)$. In this paper, we prove that there exist pairs of non-D-cospectral D-equienergetic graphs of order n for each $n \geq 6$.

2 Main results

As usual, $K_{a,b}$ denotes the complete bipartite graph with a vertices in one part and b in the other, and C_n defines a cycle on n vertices.

Lemma 2.1 [1,17] *Let M, N, P, Q be matrices, and let M be invertible. Let*

$$S = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}.$$

Then, $det\, S = det\, M \cdot det\, [Q - PM^{-1}N]$.

By Lemma 2.1 we get:

Lemma 2.2 *Let a, b be two positive integers. The characteristic polynomial of $D(K_{a,b})$ is*

$$\Phi(K_{a,b}, x) = (x+2)^{a+b-2}(x^2 - 2(a+b-2)x + 3ab - 4a - 4b + 4)\ . \tag{2}$$

The next result gives the distance energy for a complete bipartite graph

Theorem 2.1 *Let n, a be two positive integers. If $2 \le a \le \lfloor \frac{n}{2} \rfloor$, then $E_D(K_{a,n-a}) = 4n - 8$.*

Proof. Equality (2) implies that

$$spec_D(K_{a,n-a}) = \begin{pmatrix} n-2+\sqrt{n^2-3a(n-a)} & n-2-\sqrt{n^2-3a(n-a)} & -2 \\ 1 & 1 & n-2 \end{pmatrix}. \tag{3}$$

Thus, by equality (1) we only need to show that $n - 2 \ge \sqrt{n^2 - 3a(n-a)}$, equivalently, we need to prove that $3a(n-a) - 4n + 4 \ge 0$. Once this is proven, we are done.

Let $f(x) = 3x(n-x) - 4n + 4$, where $2 \le x \le \lfloor \frac{n}{2} \rfloor$. Since $f'(x) = 3n - 6x$, then $f(x)$ is an increasing function for $2 \le x \le \lfloor \frac{n}{2} \rfloor$. It follows that $f(x) \ge f(2) = 2n - 8 \ge 0$ for $2 \le x \le \lfloor \frac{n}{2} \rfloor$. Recall that $2 \le a \le \lfloor \frac{n}{2} \rfloor$, then $3a(n-a) - 4n + 4 = f(a) \ge 0$. This completes the proof. ∎

There only exist two graphs of order three, and only six graphs of order four (for instance, see [17, pp. 270-275]). By an elementary computation, we can conclude that **there doesn't exist any D-equienergetic graphs of order three or/and four**. Note that

$$spec_D(K_{2,3}) = \{3+\sqrt{7}, 3-\sqrt{7}, -2, -2, -2\}$$

and

$$spec_D(C_5) = \{6, \frac{-3+\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}, -\frac{3+\sqrt{5}}{2}, -\frac{3+\sqrt{5}}{2}\} .$$

Then $E_D(K_{2,3}) = 12 = E_D(C_5)$. Therefore, C_5 and $K_{2,3}$ is a pair of non-D-cospectral D-equienergetic graphs of order five. When $n \geq 6$, as the following Theorem shows, there exists a pair of non-D-cospectral D-equienergetic graphs of order n.

Theorem 2.2 *If $n \geq 6$, then $K_{2,n-2}$ and $K_{3,n-3}$ is a pair of non-D-cospectral D-equienergetic graphs.*

Proof. By equality (3), $K_{2,n-2}$ and $K_{3,n-3}$ are non-D-cospectral because $n \geq 6$. Thus, the result immediately follows from Theorem 2.1. ∎

By Theorem 2.1, we also have the next more general result.

Theorem 2.3 *Let a, b, n be three positive integers. If $n - b \geq b > a \geq 2$, then $K_{a,n-a}$ and $K_{b,n-b}$ is a pair of non-D-cospectral D-equienergetic graphs.*

Lemma 2.3 [1,18] *Let*

$$A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$$

be a symmetric 2×2 block matrix. Then the spectrum of A is the union of the spectra of $A_0 + A_1$ and $A_0 - A_1$.

Next we construct another pair of D-equienergetic graphs of order $n = 2t \geq 8$. Let $e = (u, v)$ denote an edge of G, whose end vertices are u and v. The notation $G - e$ indicates the graph obtained from G by deleting the edge e from G. In the following, let $W_{t,t}$ be the graph obtained from $K_{t,t}$ by deleting t independent edges from $K_{t,t}$, i. e., $W_{t,t} = K_{t,t} - (v_1, v_{t+1}) - (v_2, v_{t+2}) - \cdots - (v_t, v_{2t})$, where $\{v_1, \ldots, v_t\}$ is the vertex set in one part of $K_{t,t}$, and $\{v_{t+1}, \ldots, v_{2t}\}$ in the other part.

Theorem 2.4 *Suppose $t \geq 4$, then $K_{t,t}$ and $W_{t,t}$ is a pair of non-D-cospectral D-equienergetic graphs.*

Proof. Let J, I be the all-one square matrix and identity matrix of order t, respectively. Note that the distance matrix of $W_{t,t}$ is

$$D(W_{t,t}) = \begin{pmatrix} 2J-2I & J+2I \\ J+2I & 2J-2I \end{pmatrix}.$$

By Lemma 2.3, we have

$$spec_D(W_{t,t}) = \begin{pmatrix} 3t & t-4 & -4 & 0 \\ 1 & 1 & t-1 & t-1 \end{pmatrix}. \tag{4}$$

Since $t \geq 4$, then $E_D(W_{t,t}) = 8t - 8 = E_D(K_{t,t})$ by equality (4) and Theorem 2.1. ∎

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