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Note on the Distance Energy of Graphs

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Abstract

The distance energy of a graph G is defined as the sum of the absolute values of the eigenvalues of the distance matrix of G. In this note, we obtain an upper bound for the distance energy of any connected graph G. Specially, we present upper bounds for the distance energy of connected graphs of diameter 2 with given numbers of vertices and edges, and unicyclic graphs with odd girth. Additionally, we give also a lower bound for the distance energy of unicyclic graphs with odd girth.

INTRODUCTION

Let G be a simple connected graph on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The distance matrix of the graph G is defined as a square matrix $D = D(G) = [d_{ij}]$, where d_{ij} is the distance between the vertices v_i and v_j in G [1, 2]. The diameter of G, denoted by $diam(G)$, is the maximum distance between any two vertices of G.

The girth of G is the length of a shortest cycle in G. The eigenvalues of $D(G)$ are called the D-eigenvalues of G . Since $D(G)$ is a real symmetric matrix, its eigenvalues are real numbers. So we can order so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are D-eigenvalues of G. Properties for D-eigenvalues especially for μ_1 may be found in e.g., [3–7].

The distance energy of the connected graph G is defined in [8] as:

$$E_D = E_D(G) = \sum_{i=1}^{n} |\mu_i| \,.$$

Some results for distance energy have been obtained in [8, 9].

In this note, we obtain an upper bound for the distance energy of any connected graph in terms of the number of vertices, elements of the distance matrix and its determinant, and specially, we present upper bounds for the distance energy of connected graphs of diameter 2 with given numbers of vertices and edges, trees and unicyclic graphs with odd girth. A lower bound for the distance energy of unicyclic graphs with odd girth is also given.

In order to obtain bounds for the distance energy of graphs, we need some lemmas.

Lemma 1. [9] *Let G be a connected n-vertex graph and let $\mu_1, \mu_2, \cdots, \mu_n$ be its D-eigenvalues. Then*

$$\sum_{i=1}^{n} \mu_i^2 = 2\sum_{i<j} (d_{ij})^2.$$

Lemma 2. [10] *Let $a_1, a_2, \ldots, a_n$ be non-negative numbers. Then*

$$\begin{aligned} n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}\right] &\leq n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2 \\ &\leq n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}\right]. \end{aligned}$$

RESULTS

In the following, we establish new upper bounds for the distance energy, which are compared with the known bounds.

In [9] the following results for $E_D(G)$ were obtained:

$$\sqrt{2\sum_{i<j}(d_{ij})^2 + n(n-1)\Delta^{2/n}} \leq E_D(G) \leq \sqrt{2n\sum_{i<j}(d_{ij})^2} \tag{1}$$

where Δ be the absolute value of the determinant of the distance matrix $D(G)$.

Theorem 1. *Let G be a connected n-vertex graph and Δ be the absolute value of the determinant of the distance matrix $D(G)$. Then*

$$\sqrt{2\sum_{i<j}(d_{ij})^2 + n(n-1)\Delta^{2/n}} \leq E_D(G) \leq \sqrt{2(n-1)\sum_{i<j}(d_{ij})^2 + n\Delta^{2/n}}\,. \tag{2}$$

Proof. Let $a_i = \mu_i^2$, $i = 1, 2, \ldots, n$. Then by Lemma 1 and Lemma 2 we obtain

$$K \leq n\sum_{i=1}^{n}\mu_i^2 - \left(\sum_{i=1}^{n}|\mu_i|\right)^2 \leq (n-1)K$$

that is,

$$K \leq 2n\sum_{i<j}(d_{ij})^2 - E_D(G)^2 \leq (n-1)K$$

where

$$\begin{aligned} K &= n\left[\frac{1}{n}\sum_{i=1}^{n}\mu_i^2 - \left(\prod_{i=1}^{n}\mu_i^2\right)^{\frac{1}{n}}\right] \\ &= n\left[\frac{2}{n}\sum_{i<j}(d_{ij})^2 - \left(\prod_{i=1}^{n}|\mu_i|\right)^{\frac{2}{n}}\right] \\ &= 2\sum_{i<j}(d_{ij})^2 - n\Delta^{2/n}. \end{aligned}$$

Hence we get the result. ■

Remark. The lower bound in (2) coincides the lower bound in (1). The upper bound in (2) is always better than the upper bound in (1). This is because by using arithmetic–geometric mean inequality, we have

$$2\sum_{i<j}(d_{ij})^2 \geq n\Delta^{2/n}$$

and bearing in mind the upper bound in (2), we arrive at

$$E_D(G) \leq \sqrt{2n\sum_{i<j}(d_{ij})^2}$$

which is the upper bound in (1).

A graph with n vertices and m edges is said to be an (n, m)-graph.

Lemma 3. [8] *Let G be a connected (n, m)-graph with $diam(G) \leq 2$. Then*

$$\sum_{i=1}^{n} \mu_i^2 = 2(2n^2 - 2n - 3m).$$

For a connected (n, m)-graph G with $diam(G) \leq 2$, the following results were obtained for $E_D(G)$ in in [8]:

$$\sqrt{4n(n-1) - 6m + n(n-1)\Delta^{2/n}} \leq E_D(G) \leq \sqrt{2n(2n^2 - 2n - 3m)}\ . \tag{3}$$

By Theorem 1 and Lemma 3 we get the following result.

Corollary 1. *Let G be a connected (n, m)-graph with $diam(G) \leq 2$. Then*

$$E_D(G) \leq \sqrt{2(n-1)(2n^2 - 2n - 3m) + n\Delta^{2/n}}\ . \tag{4}$$

Remark. The upper bound in (4) is always better than the upper bound in (3).

For any n-vertex tree T [2, 3]

$$\det D(T) = (-1)^{n-1}(n-1)2^{n-2}. \tag{5}$$

For any n-vertex tree T, the following results were obtained for $E_D(T)$ in [9]:

$$\sqrt{2\sum_{i<j}(d_{ij})^2 + n[(n-1)^{n+2}4^{n-2}]^{1/n}} \leq E_D(T) \leq \sqrt{2n\sum_{i<j}(d_{ij})^2}\ . \tag{6}$$

Using the equality (5) and Theorem 1 we give the following result.

Corollary 2. For any n-vertex tree T,

$$E_D(T) \leq \sqrt{2(n-1)\sum_{i<j}(d_{ij})^2 + n[(n-1)^2 4^{n-2}]^{1/n}}\ . \tag{7}$$

Remark. The upper bound in (7) is always better than the upper bound in (6). For a tree T with at least two vertices, we have $\mu_1 > 0 > \mu_2$, see [5]. Thus, for any tree

T, $E_D(T) = 2\mu_1$. We note that (7) is essentially known in [4] as an upper bound for $2\mu_1$.

Let U be an n-vertex unicyclic graph with girth r. If r is odd, then

$$\det D(U) = (-2)^{n-r-2}(2rn - r^2 - 1) \tag{8}$$

and if r is even, then $\det D(U) = 0$ [11]. Using (8) and Theorem 1, we get the following results.

Corollary 3. *For an* n-vertex unicyclic graph U with odd girth r,

$$\begin{aligned} & \sqrt{2\sum_{i<j}(d_{ij})^2 + n(n-1)[4^{n-r-2}(2rn - r^2 - 1)^2]^{1/n}} \\ \leq \quad & E_D(U) \\ \leq \quad & \sqrt{2(n-1)\sum_{i<j}(d_{ij})^2 + n[4^{n-r-2}(2rn - r^2 - 1)^2]^{1/n}} \,. \end{aligned}$$

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