MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **64** (2010) 123-128

ISSN 0340 - 6253

More Upper Bounds for the Incidence Energy

Bo Zhou

Department of Mathematics, South China Normal University, Guangzhou 510631, P. R. China
e-mail: `zhoubo@scnu.edu.cn`

(Received April 30, 2009)

Abstract

The incidence energy of a graph is defined as the sum of the singular values of the incidence matrix. We obtain upper bounds for the incidence energy using the first Zagreb index.

1. INTRODUCTION

Let G be a simple graph with n vertices. The eigenvalues of G are the eigenvalues of its adjacency matrix $\mathbf{A}(G)$ [1]. These eigenvalues, arranged in a non-increasing order, will be denoted as $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$. Then the energy of the graph G is defined as [2]

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)| \ .$$

Various properties of graph energy may be found in [3–5].

The singular values of a real matrix $\mathbf{M}$ are the square roots of the eigenvalues of the matrix $\mathbf{M}\mathbf{M}^t$, where $\mathbf{M}^t$ denotes the transpose of $\mathbf{M}$. The energy $E(\mathbf{M})$ of the matrix $\mathbf{M}$ is defined [6] as the sum of its singular values. Then for a graph G, $E(G) = E(\mathbf{A}(G))$.

Let $\mathbf{I}(G)$ be the (vertex–edge) incidence matrix of the graph G. For a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{e_1, e_2, \ldots, e_m\}$, the (i,j)-entry of $\mathbf{I}(G)$ is 0 if v_i is not incident with e_j and 1 if v_i is incident with e_j. Jooyandeh et al. [7] began the study of the energy of the matrix $\mathbf{I}(G)$, which was called the incidence energy of the graph G and was denoted by $IE(G)$ (if G is an empty graph, then $IE(G) = 0$). Some basic properties of this quantity were established in [7–9].

Let $\mathbf{D}(G)$ be the diagonal matrix of order n whose (i,i)-entry is the degree (= number of first neighbors) of the vertex v_i of the graph G. The matrix $\mathbf{L}^+(G) = \mathbf{D}(G) + \mathbf{A}(G)$ is the signless Laplacian matrix, for details see [10]. As well known in graph theory, we have $\mathbf{L}^+(G) = \mathbf{I}(G)\,\mathbf{I}(G)^t$. Denote by $\mu_1, \mu_2, \ldots, \mu_n$ the eigenvalues of the signless Laplacian matrix $\mathbf{L}^+(G)$, arranged in a non-increasing order. Then

$$IE(G) = \sum_{i=1}^{n} \sqrt{\mu_i}\,,$$

which was noted in [8].

We obtain upper bounds for IE using a method invented by Koolen and Moulton [11, 12].

2. UPPER BOUNDS FOR INCIDENCE ENERGY

Let K_n be the complete graph with n vertices. Let $K_{r,s}$ be the complete bipartite graph with r and s vertices in its two partite sets respectively.

If G is a connected non-bipartite graph, then $\mu_i > 0$ for $i = 1, 2, \ldots, n$ [10]. Recall that the matrix $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$ is the Laplacian matrix of the graph G, for details see [13, 14]. If G is bipartite, then $\mu_1, \mu_2, \ldots, \mu_n$ are also the eigenvalues of $\mathbf{L}(G)$, and in particular $\mu_n = 0$ [10].

The first Zagreb index $Zg(G)$ of a graph G is defined as $Zg(G) = \sum_{u \in V(G)} d_u^2$, where d_u denotes the degree of vertex u in G [15, 16].

Let $\mathcal{L}(G)$ be the line graph of the graph G. Note that for a graph G with n vertices and $m \geq 1$ edges, we have $\mathbf{L}^+(G) = \mathbf{I}(G)\,\mathbf{I}(G)^t$ and $\mathbf{I}(G)^t\,\mathbf{I}(G) = 2\,\mathbf{1}_m + \mathbf{A}(\mathcal{L}(G))$, where $\mathbf{1}_m$ stands for the unit matrix of order m. From the two identities it follows that

$$\mu_1 = \lambda_1(\mathcal{L}(G)) + 2\,.$$

Note also that for a graph G, $\lambda_1(G) \geq a(G)$ with equality if and only if G is regular, where $a(G)$ is the average degree of G [1].

Proposition 1. *Let G be a graph with $n \geq 2$ vertices, $m \geq 1$ edges, and the first Zagreb index Zg. Then*

$$IE(G) \leq \sqrt{\frac{Zg}{m}} + \sqrt{(n-1)\left(2m - \frac{Zg}{m}\right)} \tag{1}$$

with equality if and only if $G \cong K_n$, or $n \geq 3$ and $m = 1$.

Proof. By the Cauchy–Schwarz inequality,

$$\sum_{i=2}^{n} \sqrt{\mu_i} \;\leq\; \sqrt{(n-1)\sum_{i=2}^{n} \mu_i} = \sqrt{(n-1)(2m-\mu_1)}$$

with equality if and only if $\mu_2 = \cdots = \mu_n$. Thus,

$$IE(G) \leq f(\mu_1)$$

with $f(x) = \sqrt{x} + \sqrt{(n-1)(2m-x)}$. It is easily seen that $f(x)$ is decreasing for $x \geq \frac{2m}{n}$. Note that the number of edges of $\mathcal{L}(G)$ is equal to $\sum_{u \in V(G)} \binom{d_u}{2} = \frac{1}{2}(Zg - 2m)$ and then $a(\mathcal{L}(G)) = \frac{Zg}{m} - 2$. Thus $\lambda_1(\mathcal{L}(G)) \geq a(\mathcal{L}(G)) = \frac{Zg}{m} - 2$ with equality if and only if $\mathcal{L}(G)$ is regular. By Cauchy–Schwarz inequality again, $Zg \geq \frac{4m^2}{n}$. Thus, $\mu_1 = \lambda_1(\mathcal{L}(G)) + 2 \geq \frac{Zg(G)}{m} > \frac{2m}{n}$. It follows that

$$IE(G) \leq f\left(\frac{Zg}{m}\right) = \sqrt{\frac{Zg}{m}} + \sqrt{(n-1)\left(2m - \frac{Zg}{m}\right)}$$

with equality if and only if $\mathcal{L}(G)$ is regular and $\mu_2 = \cdots = \mu_n$.

Suppose that $\mathcal{L}(G)$ is regular and $\mu_2 = \cdots = \mu_n$. If G is connected, then $G \cong K_n$. This is because the number of distinct signless Laplacian eigenvalues of a connected graph with diameter d is at least $d+1$ [10]. Suppose that G is not connected with $n \geq 3$. Then every component of G is a complete graph. Note that a complete graph with at least two vertices has exactly two distinct signless Laplacian eigenvalues and

$m \geq 1$. Then $\mu_1 > \mu_2$ and thus one component of G is a complete graph on two vertices and every other component is trivial.

Conversely, it is easily seen that $\mathcal{L}(G)$ is regular, $\mu_2 = \cdots = \mu_n$ and then (1) is an equality if $G \cong K_n$, or $n \geq 3$ and $m = 1$. ■

In [9], an upper bound for the incidence energy using the first Zagreb index was given: If G is graph with n vertices, m edges, and the first Zagreb index Zg, then

$$IE(G) \leq \left(2\sqrt{\frac{Zg}{n}}\right)^{1/2} + \sqrt{2(n-1)\left(m - \sqrt{\frac{Zg}{n}}\right)} \tag{2}$$

with equality if and only if $G \cong \overline{K_n}$ (empty graph) or $G \cong K_n$. For a graph G with at least one edge, (1) is better than (2) because the upper bound in (2) is equal to $f\left(2\sqrt{\frac{Zg}{n}}\right)$ and $\mu_1 \geq \frac{Zg}{m} \geq 2\sqrt{\frac{Zg}{n}} > \frac{2m}{n}$.

Note that for the complete bipartite graph $K_{r,s}$ with $s \geq r \geq 1$, we have $\mu_1 = r+s$, $\mu_2 = \cdots = \mu_r = s$, $\mu_{r+1} = \cdots = \mu_{r+s-1} = r$ and $\mu_{r+s} = 0$. For a bipartite graph, we have:

Proposition 2. *Let G be a bipartite graph with $n \geq 2$ vertices, $m \geq 1$ edges, and the first Zagreb index Zg. Then*

$$IE(G) \leq \sqrt{\frac{Zg}{m}} + \sqrt{(n-2)\left(2m - \frac{Zg}{m}\right)} \tag{3}$$

with equality if and only if $G \cong K_{1,n-1}$, or n is even and $G \cong K_{n/2,n/2}$ with $n \geq 4$, or $n \geq 3$ and $m = 1$.

Proof. The case for $n = 2$ is trivial. Suppose that $n \geq 3$. Note that $\mu_n = 0$. By the Cauchy–Schwarz inequality,

$$\sum_{i=2}^{n} \sqrt{\mu_i} \leq \sqrt{(n-2)\sum_{i=2}^{n} \mu_i} = \sqrt{(n-2)(2m - \mu_1)}$$

with equality if and only if $\mu_2 = \cdots = \mu_{n-1}$. Thus,

$$IE(G) \leq h(\mu_1)$$

with $h(x) = \sqrt{x} + \sqrt{(n-2)(2m - x)}$. It is easily seen that $h(x)$ is decreasing for $x \geq \frac{2m}{n-1}$. As above, we have $\mu_1 = \lambda_1(\mathcal{L}(G)) + 2 \geq \frac{Zg(G)}{m} > \frac{2m}{n-1}$. Thus,

$$IE(G) \leq h\left(\frac{Zg}{m}\right) = \sqrt{\frac{Zg}{m}} + \sqrt{(n-2)\left(2m - \frac{Zg}{m}\right)}$$

with equality if and only if $\mathcal{L}(G)$ is regular and $\mu_2 = \cdots = \mu_{n-1}$.

Suppose that $\mathcal{L}(G)$ is regular and $\mu_2 = \cdots = \mu_{n-1}$. If G is connected, then G is a complete bipartite graph and thus $G \cong K_{1,n-1}$, or n is even and $G \cong K_{n/2,n/2}$ with $n \geq 4$. This is because the number of distinct signless Laplacian eigenvalues of a connected graph with diameter d is at least $d+1$ [10] and the diameter of G is at least two. Suppose that G is not connected. Then $\mu_2 = \cdots = \mu_{n-1} = 0$. Since $m \geq 1$, G has exactly $n-1$ components, and thus $m = 1$.

Conversely, it is easily seen that (3) is an equality if $G \cong K_{1,n-1}$, or n is even and $G \cong K_{n/2,n/2}$ with $n \geq 4$, or $n \geq 3$ and $m = 1$. ■

Let G be a bipartite graph with $n \geq 2$ vertices, $m \geq 1$ edges, and the first Zagreb index Zg. Note that $\mu_1 \geq \frac{Zg}{m} \geq 2\sqrt{\frac{Zg}{n}}$. As above, we have

$$IE(G) \leq \left(2\sqrt{\frac{Zg}{n}}\right)^{1/2} + \sqrt{(n-2)\left(2m - 2\sqrt{\frac{Zg}{n}}\right)} \tag{4}$$

with equality if and only if n is even and $G \cong K_{n/2,n/2}$. Note that in [17], this upper bound was obtained for connected bipartite graph as a particular case in a inequality involving the sum of powers of the Laplacian eigenvalues. For a bipartite graph G with at least one edge, (3) is better than (4).

Acknowledgement. This work was supported by the Guangdong Provincial Natural Science Foundation of China (no. 8151063101000026).

References

[1] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, Barth, Heidelberg, 1995.

[2] I. Gutman, The energy of a graph, *Ber. Math.–Statist. Sekt. Forschungsz. Graz* **103** (1978) 1–22.

[3] I. Gutman, The energy of a graph: Old and new results, in: A. Betten, A. Kohnert, R. Laue (Eds.), *Algebraic Combinatorics and Applications*, Springer–Verlag, Berlin, 2001, pp. 196–211.

[4] I. Gutman, Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology, *J. Serb. Chem. Soc.* **70** (2005) 441–456.

[5] I. Gutman, X. Li, J. Zhang, Graph energy, in: M. Dehmer, F. Emmert–Streib (Eds.), *Analysis of Complex Networks. From Biology to Linguistics*, Wiley–VCH, Weinheim, 2009, pp. 145–174.

[6] V. Nikiforov, The energy of graphs and matrices, *J. Math. Anal. Appl.* **326** (2007) 1472–1475.

[7] M. Jooyandeh, D. Kiani, M. Mirzakhah, Incidence energy of a graph, *MATCH Commun. Math. Comput. Chem.* **62** (2009) 561–572.

[8] I. Gutman, D. Kiani, M. Mirzakhah, On incidence energy of graphs, *MATCH Commun. Math. Comput. Chem.* **62** (2009) 573–580.

[9] I. Gutman, D. Kiani, M. Mirzakhah, B. Zhou, On incidence energy of a graph, *Lin. Algebra Appl.*, in press.

[10] D. Cvetković, P. Rowlinson, S. Simić, Signless Laplacians of finite graphs, *Lin. Algebra Appl.* **423** (2007) 155–171.

[11] J. Koolen, V. Moulton, Maximal energy graphs, *Adv. Appl. Math.* **26** (2001) 47–52.

[12] J. Koolen, V. Moulton, Maximal energy bipartite graphs, *Graph Combin.* **19** (2003) 131–135.

[13] R. Merris, Laplacian matrices of graphs: A survey, *Lin. Algebra Appl.* **197–198** (1994) 143–176.

[14] R. Merris, A survey of graph Laplacians, *Lin. Multilin. Algebra* **39** (1995) 19–31.

[15] S. Nikolić, G. Kovačević, A. Miličević, N. Trinajstić, The Zagreb indices 30 years after, *Croat. Chem. Acta* **76** (2003) 113–124.

[16] I. Gutman, K. C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.* **50** (2004) 83–92.

[17] B. Zhou, On sum of powers of the Laplacian eigenvalues of graphs, *Lin. Algebra Appl.* **429** (2008) 2239–2246.